1 An athlete on a special diet would like to have battered fish fillet, coleslaw and fries for lunch. He must ensure that his intake (in grams) of protein, carbohydrates and fat per meal is 55g, 130g, and 70g respectively. The table below shows the nutritional breakdown for one serving of each item.

	Protein (in grams)	Carbohydrates (in grams)	Fat (in grams)
Battered fish fillet	20	20	15
Coleslaw	4	15	8
Fries	3	45	16

Calculate the number of servings of battered fish fillet, coleslaw and fries that the athlete should take for his lunch. [3]

Suggested Solutions
Let x, y and z be the number of servings of battered fish fillet, coleslaw and fries respectively. $20x+4y+3z=55$ $20x+15y+45z=130$ $15x+8y+16z=70$ By GC, $x = 2$, $y = 3$, $z = 1$ Therefore, the runner should take 2 servings of battered fish fillet, 3 servings of coleslaw and 1 serving of fries for his lunch.

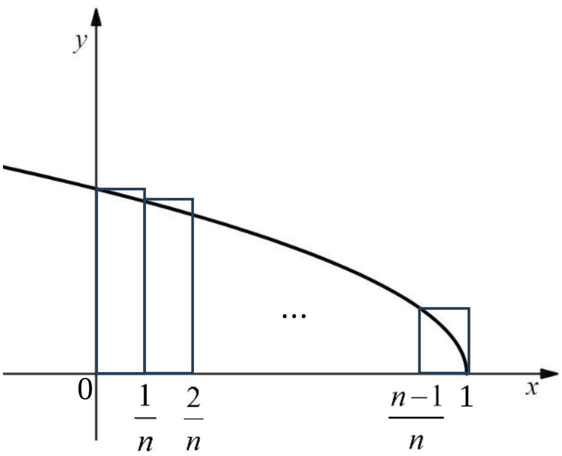
2 **(a)** Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, find $\sum_{r=1}^{n}\left[\left(\frac{1}{2}\right)^r - r\left(r+\frac{1}{3}\right)\right]$, in terms of n. [4]

	Suggested Solutions
(a)	$\sum_{r=1}^{n}\left[\left(\frac{1}{2}\right)^r - r\left(r+\frac{1}{3}\right)\right]$ $=\sum_{r=1}^{n}\left(\frac{1}{2}\right)^r - \sum_{r=1}^{n} r^2 - \frac{1}{3}\sum_{r=1}^{n} r$ $=\frac{\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} - \frac{n}{6}(n+1)(2n+1) - \frac{n}{6}(1+n)$ $=1-\left(\frac{1}{2}\right)^n - \frac{n}{6}(n+1)(2n+1+1)$ $=1-\left(\frac{1}{2}\right)^n - \frac{n}{3}(n+1)^2$

(b) Explain why $\sum_{r=1}^{n}\left[\left(\frac{1}{2}\right)^r - r\left(r+\frac{1}{3}\right)\right] < 1$ for all positive integers n. [1]

	Suggested Solutions
(b)	Since $n \in \mathbb{Z}^+$, $\left(\frac{1}{2}\right)^n > 0$ and $\frac{n}{3}(n+1)^2 > 0$. Thus $\sum_{r=1}^{n}\left[\left(\frac{1}{2}\right)^r - r\left(r+\frac{1}{3}\right)\right] < 1$.

3 The diagram below shows a curve with equation $y=\sqrt{1-x}$. Let A be the area bounded by the curve and the axes. A total of n rectangles, each of equal width, are constructed to approximate the value of A.



(a) Show that the total area of n rectangles is given by

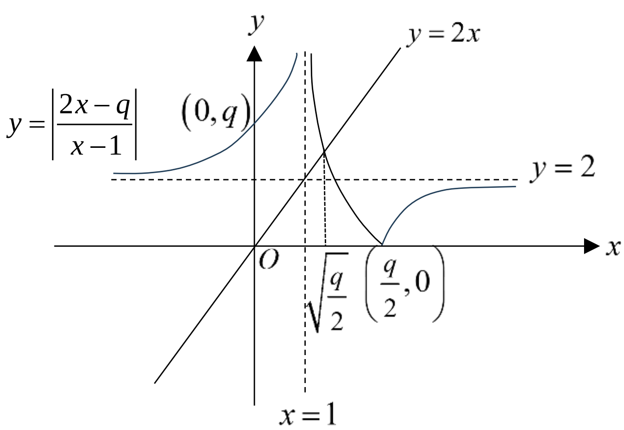
$$\frac{1}{n\sqrt{n}}\sum_{r=0}^{n-1}\sqrt{n-r}\,.$$ [2]

	Suggested Solutions
(a)	Area of n rectangles $=\frac{1}{n}\left(\sqrt{1-\frac{0}{n}}\right)+\frac{1}{n}\left(\sqrt{1-\frac{1}{n}}\right)+\ldots+\frac{1}{n}\left(\sqrt{1-\frac{n-1}{n}}\right)$ $=\frac{1}{n}\left[\sqrt{\frac{n-0}{n}}+\sqrt{\frac{n-1}{n}}+\ldots+\sqrt{\frac{n-(n-1)}{n}}\right]$ $=\frac{1}{n}\sum_{r=0}^{n-1}\sqrt{\frac{n-r}{n}}$ $\frac{1}{n\sqrt{n}}\sum_{r=0}^{n-1}\sqrt{n-r}$

(b) Without using a calculator, evaluate $\lim_{n\to\infty}\frac{1}{n\sqrt{n}}\sum_{r=0}^{n-1}\sqrt{n-r}$. [3]

	Suggested Solutions
(b)	$\lim_{n\to\infty}\frac{1}{n\sqrt{n}}\sum_{r=0}^{n-1}\sqrt{n-r}$ = Area of infinitely many rectangles = Area under the curve from $x=0$ to $x=1$ $=\int_0^1\sqrt{1-x}\ \mathrm{d}x$ $=-\int_0^1-\sqrt{1-x}\ \mathrm{d}x$ $=\left[-\frac{2}{3}\sqrt{(1-x)^3}\right]_0^1$ $=\frac{2}{3}$

4 **(a)** Sketch the curve with equation $y=\left|\frac{2x-q}{x-1}\right|$, where $q>2$, stating the equations of the asymptotes and the coordinates of the points where the curve meets the axes. [3]

	Suggested Solutions
(a)	

(b) Hence, by adding a suitable line on the same diagram in part **(a)**, solve the inequality $\left|\dfrac{2x-q}{x-1}\right|-2x>0,$ giving your answer in terms of q. [3]

	Suggested Solutions
(b)	$\left\|\dfrac{2x-q}{x-1}\right\|>2x$ To find the *x*-coordinate of point of intersection: $-\dfrac{2x-q}{x-1}=2x$ $-2x+q=2x^2-2x$ $2x^2=q$ $x=\pm\sqrt{\dfrac{q}{2}}$ Since $x>0$, $x=\sqrt{\dfrac{q}{2}}$ For $\left\|\dfrac{2x-q}{x-1}\right\|-2x>0,$ $x<1$ or $1<x<\sqrt{\dfrac{q}{2}}$

5 Show that the distance L between any point (x, y) on the curve $y = \mathrm{e}^{2x} - 3$ and the fixed point $A(2, -1)$ satisfies the equation $L^2 = (x-2)^2 + \left(\mathrm{e}^{2x} - 2\right)^2$. [1]

Using differentiation, find the x-coordinate of the point P on the curve that is closest to the point A, leaving your answer in 4 decimal places. You do not need to show that P is closest to A. [3]

A variable point Q moves along the curve $y = \mathrm{e}^{2x} - 3$ such that its x-coordinate is increasing at a rate of 0.5 units per second. Find the exact rate of change in L when Q is on the y-axis. [3]

Suggested Solutions

$$L = \sqrt{(x-2)^2 + \left(\mathrm{e}^{2x} - 3 - (-1)\right)^2}$$

$$L^2 = (x-2)^2 + \left(\mathrm{e}^{2x} - 2\right)^2 \quad \text{(Shown)}$$

<u>Method 1</u>

$$L^2 = (x-2)^2 + \left(\mathrm{e}^{2x} - 2\right)^2$$

$$2L\frac{\mathrm{d}L}{\mathrm{d}x} = 2(x-2) + 2\left(\mathrm{e}^{2x} - 2\right)\left(2\mathrm{e}^{2x}\right)$$

$$L\frac{\mathrm{d}L}{\mathrm{d}x} = (x-2) + \left(\mathrm{e}^{2x} - 2\right)\left(2\mathrm{e}^{2x}\right)$$

$$= x - 2 + 2\mathrm{e}^{4x} - 4\mathrm{e}^{2x}$$

$$= 2\mathrm{e}^{4x} - 4\mathrm{e}^{2x} + x - 2$$

<u>Method 2</u>

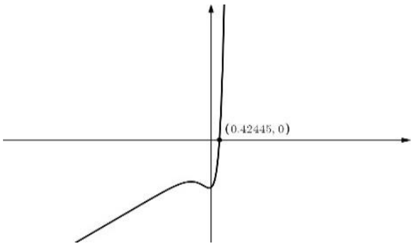
$$L = \sqrt{(x-2)^2 + \left(\mathrm{e}^{2x} - 2\right)^2}$$

$$\frac{\mathrm{d}L}{\mathrm{d}x} = \frac{2(x-2) + 2\left(\mathrm{e}^{2x} - 2\right)\left(2\mathrm{e}^{2x}\right)}{2\sqrt{(x-2)^2 + \left(\mathrm{e}^{2x} - 2\right)^2}}$$

$$\frac{\mathrm{d}L}{\mathrm{d}x} = \frac{2\mathrm{e}^{2x} - 4\mathrm{e}^{4x} + x - 2}{\sqrt{(x-2)^2 + \left(\mathrm{e}^{2x} - 2\right)^2}}$$

For shortest distance, $\dfrac{\mathrm{d}L}{\mathrm{d}x} = 0$

$2e^{4x} - 4e^{2x} + x - 2 = 0$



By GC, $x = 0.42445 \approx 0.4245$ (to 4 dp)

Method 3

To minimise L, we can minimise L^2 ----- (*)

$\frac{dL^2}{dx} = 2(x-2) + 2(e^{2x} - 2)(2e^{2x})$

$0 = 2(x-2) + 2(e^{2x} - 2)(2e^{2x})$

By GC, $x = 0.42445 \approx 0.4245$ (to 4 dp)

Method 1

When $x = 0$, $L = \sqrt{(-2)^2 + (1-2)^2} = \sqrt{5}$

$(\sqrt{5})\frac{dL}{dx} = 2(1) - 4(1) - 2$

$\frac{dL}{dx} = -\frac{4}{\sqrt{5}}$

$\frac{dL}{dt} = \frac{dL}{dx} \times \frac{dx}{dt}$

$= -\frac{4}{\sqrt{5}} \times 0.5$

$= -\frac{2}{\sqrt{5}}$

Method 2

When $x = 0$, $\frac{dL}{dx} = \frac{2(1) - 4(1) - 2}{\sqrt{(-2)^2 + (1-2)^2}} = -\frac{4}{\sqrt{5}}$

$\frac{dL}{dt} = \frac{dL}{dx} \times \frac{dx}{dt}$

$= -\frac{4}{\sqrt{5}} \times 0.5$

$= -\frac{2}{\sqrt{5}}$

<u>Method 3</u>

$$L^2 = (x-2)^2 + \left(e^{2x}-2\right)^2$$

$$2L\frac{dL}{dt} = 2(x-2)\frac{dx}{dt} + 2\left(e^{2x}-2\right)\left(2e^{2x}\right)\frac{dx}{dt}$$

$$L\frac{dL}{dt} = \left[(x-2)+\left(e^{2x}-2\right)\left(2e^{2x}\right)\right]\frac{dx}{dt}$$

When $x = 0$, $L = \sqrt{(-2)^2 + (1-2)^2} = \sqrt{5}$

Substitute $x = 0$, $L = \sqrt{5}$ and $\frac{dx}{dt} = 0.5$:

$$\left(\sqrt{5}\right)\frac{dL}{dt} = \left[(-2)+\left(e^{0}-2\right)\left(2e^{0}\right)\right](0.5)$$

$$\frac{dL}{dt} = -\frac{2}{\sqrt{5}}$$

L is decreasing at a rate of $\frac{2}{\sqrt{5}}$ unit per second.

6 The curve C has equation

$$\ln y^2 = x^2 y, \text{ where } x \in \mathbb{R}, -1 \le y < 0.$$

It is given that C has only one turning point.

(a) Show that $\frac{dy}{dx}\left(2 - x^2 y\right) = 2xy^2$. [3]

	Suggested Solutions
(a)	$\frac{1}{y^2} \times 2y\frac{dy}{dx} = 2xy + x^2\frac{dy}{dx}$ $\frac{dy}{dx}\left(\frac{2}{y} - x^2\right) = 2xy$ $\frac{dy}{dx}\left(2 - x^2 y\right) = 2xy^2$

(b) Find the coordinates of the turning point. [2]

	Suggested Solutions
(b)	For turning point, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$. $\therefore 2xy^2=0$ Since $y<0$, $\therefore x=0$. When $x=0$, $\ln y^2=0$ $\Rightarrow y=\pm 1$ Since $-1\le y<0$, the turning point is $(0,-1)$.

(c) Calculate the value of $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ at the turning point and determine whether the turning point is a maximum or a minimum. [3]

	Suggested Solutions
(c)	$\frac{\mathrm{d}y}{\mathrm{d}x}\left(2-x^2y\right)=2xy^2$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(2-x^2y\right)+\frac{\mathrm{d}y}{\mathrm{d}x}\left(-x^2\frac{\mathrm{d}y}{\mathrm{d}x}-2xy\right)=4xy\frac{\mathrm{d}y}{\mathrm{d}x}+2y^2$ Substitute $x=0$, $y=-1$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=0$: $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(2)=2(-1)^2$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=1>0$ Therefore $(0,-1)$ is a minimum point.

7 **(a)** Describe a sequence of two transformations that will transform the graph of $y=\mathrm{f}(x)$ to the graph of $y=\mathrm{f}(\alpha x+\beta)$, where α and β are positive constants. [2]

	Suggested Solution
(a)	Translate β units in the negative x-direction. Scale parallel to x-axis by a scale factor of $\frac{1}{\alpha}$. OR Scale parallel to x-axis by a scale factor of $\frac{1}{\alpha}$.

	Translate $\dfrac{\beta}{\alpha}$ units in the negative *x*-direction.

Diagram 1 and Diagram 2 show the graphs of $y=\mathrm{f}\left(x\right)$ and $y=\mathrm{f}\left(\alpha x+\beta\right)$ respectively. The turning points with coordinates $(0,0)$ and $(2,4)$ on $y=\mathrm{f}\left(x\right)$ correspond to the points with coordinates $(-2,0)$ and $(2,4)$ respectively on $y=\mathrm{f}\left(\alpha x+\beta\right)$. The asymptotes $x=1$ and $y=x+\lambda$ on $y=\mathrm{f}\left(x\right)$ correspond to the asymptotes $x=0$ and $y=\alpha x+\beta+\lambda$ respectively on $y=\mathrm{f}\left(\alpha x+\beta\right)$.

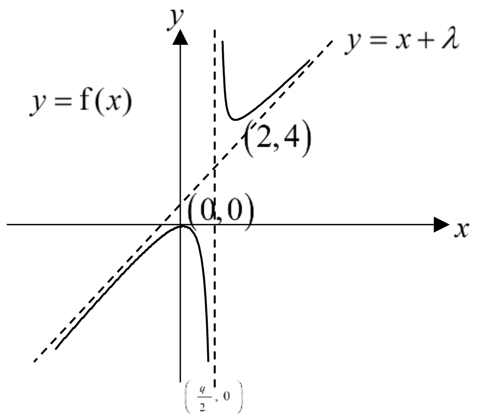


Diagram 1

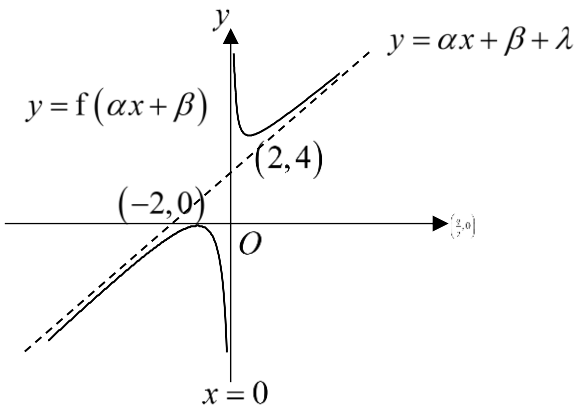
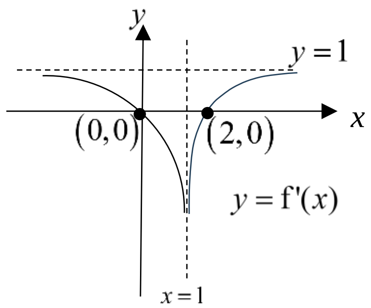


Diagram 2

(b) Find the value of α and of β. [2]

	Suggested Solution
(b)	<u>Method 1</u> By observation, $\alpha=\dfrac{1}{2}, \quad \beta=1$ <u>Method 2</u> $\dfrac{0-\beta}{\alpha}=-2 \quad \Rightarrow \quad \beta=2\alpha$ -----(I) $\dfrac{2-\beta}{\alpha}=2 \quad \Rightarrow \quad 2-\beta=2\alpha$ -----(II) Solving (I) and (II), $\alpha=\dfrac{1}{2}, \quad \beta=1.$

(c) On a separate diagram, sketch the graph of $y = \mathrm{f}'(x)$, giving the coordinates of the points where the graph crosses the axes and the equations of the asymptotes, if any. [3]

	Suggested Solution
(c)	

8 The function f is defined by

$$\mathrm{f}(x) = \begin{cases} 2 - \ln(x + \mathrm{e}) & \text{for } -\mathrm{e} < x < 0, \\ \cos\left(\frac{\pi}{2}x\right) & \text{for } 0 \le x \le 1. \end{cases}$$

(a) Given that the inverse function f^{-1} exists, find the exact value of m if $\mathrm{f}^{-1}(m) = -1$. [2]

	Suggested Solutions
(a)	$\mathrm{f}^{-1}(m) = -1$ $m = \mathrm{f}(-1)$ $= 2 - \ln(\mathrm{e} - 1)$

Another function g is defined by

$$\mathrm{g}(x) = \sin x, \quad \text{for } 0 \le x \le \frac{\pi}{4}.$$

(b) Explain why the composite function fg exists. [2]

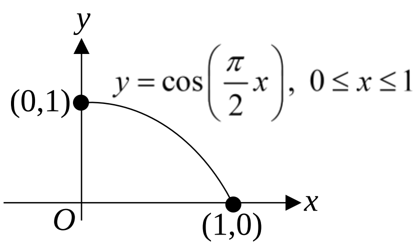
	Suggested Solutions
(b)	$\mathrm{R_g} = \left[0, \frac{1}{\sqrt{2}}\right]$ $\mathrm{D_f} = (-\mathrm{e}, 1]$ Since $\mathrm{R_g} \subseteq \mathrm{D_f}$. Hence, fg exists.

(c) Find the exact value of $\mathrm{fg}\left(\frac{\pi}{6}\right)$. [2]

	Suggested Solutions
(c)	$\mathrm{fg}\left(\frac{\pi}{6}\right)=\mathrm{f}\left(\frac{1}{2}\right)$ $=\cos\left(\frac{\pi}{4}\right)$ $=\frac{1}{\sqrt{2}}$

The domain of f is further restricted to $0 \le x \le 1$.

(d) By considering $\mathrm{f}^2(1)$ and $\mathrm{f}^3(1)$, deduce the value(s) of $\mathrm{f}^n(1)$ where n is a positive integer. [2]

	Suggested Solutions
(d)	 $\mathrm{f}^2(1)=\mathrm{f}\left[\mathrm{f}(1)\right]=\mathrm{f}(0)=1$ $\mathrm{f}^3(1)=\mathrm{f}\left[\mathrm{f}^2(1)\right]=\mathrm{f}(1)=0$ $\mathrm{f}^n(1)=\begin{cases}0 & \text{if } n \text{ is odd}\\ 1 & \text{if } n \text{ is even}\end{cases}$

9 Referred to the origin O, a variable point R has position vector $\mathbf{r}$ and the fixed points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors.

(a) If $\mathbf{a}$ and $\mathbf{b}$ are parallel, find $\mathbf{a}\times\mathbf{b}$. [1]

	Suggested Solutions
(a)	Given $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel, $\underset{\sim}{a} = k\underset{\sim}{b}$, $k \in \mathbb{R}, k \neq 0$. $\therefore\ \underset{\sim}{a}\times\underset{\sim}{b} = k(\underset{\sim}{b}\times\underset{\sim}{b}) = \underset{\sim}{0}$

(b) If $\mathbf{a}$ and $\mathbf{b}$ are non-parallel, interpret geometrically the following vector equations:

(i) $\mathbf{r}\times(\mathbf{b}-\mathbf{a})=\mathbf{0}$, [2]

(ii) $\mathbf{r}\cdot(\mathbf{b}-\mathbf{a})=0$. [2]

	Suggested Solutions
(b)(i)	For $\underset{\sim}{r}\times(\underset{\sim}{b}-\underset{\sim}{a})=\underset{\sim}{0}$, $\overrightarrow{OR} \,//\, \overrightarrow{AB}$. $\overrightarrow{OR}=\lambda\overrightarrow{AB},\ \lambda\in\mathbb{R}$ $\therefore\ \underset{\sim}{r}\times(\underset{\sim}{b}-\underset{\sim}{a})=\underset{\sim}{0}$ describes the equation of a line passing through origin O and parallel to $\overrightarrow{AB}$.
(b)(ii)	For $\underset{\sim}{r}\cdot(\underset{\sim}{b}-\underset{\sim}{a})=0$, $\overrightarrow{OR}\perp\overrightarrow{AB}$. $\therefore\ \underset{\sim}{r}\cdot(\underset{\sim}{b}-\underset{\sim}{a})=0$ describes the equation of a plane containing the origin O with normal $\overrightarrow{AB}$.

It is now given that the coordinates of A and B are (−1, 2, 2) and (6, −2, −4) respectively.

(c) Use a vector product to find the area of the triangle OAB. [3]

	Suggested Solutions
(c)	Method 1 Area of triangle OAB $= \frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|$ $= \frac{1}{2}\left\|\begin{pmatrix}-1\\2\\2\end{pmatrix}\times\begin{pmatrix}6\\-2\\-4\end{pmatrix}\right\|$ $= \frac{1}{2}(2)\left\|\begin{pmatrix}-1\\2\\2\end{pmatrix}\times\begin{pmatrix}3\\-1\\-2\end{pmatrix}\right\|$ $= \left\|\begin{pmatrix}-2\\4\\-5\end{pmatrix}\right\|$ $= \sqrt{(-2)^2+4^2+(-5)^2}$ $= 3\sqrt{5}$ units^2 Method 2 Area of triangle OAB $= \frac{1}{2}\left\|\overrightarrow{AO}\times\overrightarrow{AB}\right\|$ $= \frac{1}{2}\left\|\begin{pmatrix}1\\-2\\-2\end{pmatrix}\times\begin{pmatrix}7\\-4\\-6\end{pmatrix}\right\|$ $= \frac{1}{2}\left\|\begin{pmatrix}4\\-8\\10\end{pmatrix}\right\|$ $= \left\|\begin{pmatrix}2\\-4\\5\end{pmatrix}\right\|$ $= \sqrt{2^2+(-4)^2+5^2}$ $= 3\sqrt{5}$ units^2

10 It is given that $I_n = \int \frac{x^n}{\sqrt{1-9x^4}} \, dx$.

(a) Find I_3. [2]

	Suggested Solutions
(a)	$I_3 = \int \frac{x^3}{\sqrt{1-9x^4}} \, dx$ $= -\frac{1}{36} \int \frac{1}{\sqrt{1-9x^4}} \cdot \left(-36x^3\right) \, dx$ $= -\frac{1}{36} \left[2\sqrt{1-9x^4} \right] + C$ $= -\frac{1}{18} \sqrt{1-9x^4} + C$

(b) By using the substitution $u = x^2$, find I_1. [4]

	Suggested Solutions
(b)	$I_1 = \int \frac{x}{\sqrt{1-9x^4}} \, dx$ Let $u = x^2$, $\frac{du}{dx} = 2x$ $I_1 = \int \frac{1}{\sqrt{1-9u^2}} \cdot \left(\frac{1}{2}\right) \, du$ $= \frac{1}{2} \int \frac{1}{\sqrt{1-(3u)^2}} \, du$ $= \frac{1}{2(3)} \sin^{-1}(3u) + C$ $= \frac{1}{6} \sin^{-1}\left(3x^2\right) + C$

(c) Using integration by parts and the result in part **(a)**, find the exact value of

$$\int_0^{\frac{1}{\sqrt{3}}} x\sin^{-1}\left(3x^2\right)\,\mathrm{d}x.$$ [4]

	Suggested Solutions
(c)	$I=\int_0^{\frac{1}{\sqrt{3}}} x\sin^{-1}\left(3x^2\right)\,\mathrm{d}x$ $u=\sin^{-1}\left(3x^2\right), \quad v'=x$ $u'=\dfrac{6x}{\sqrt{1-9x^4}}, \quad v=\dfrac{1}{2}x^2$ $I=\left[\dfrac{1}{2}x^2\sin^{-1}\left(3x^2\right)\right]_0^{\frac{1}{\sqrt{3}}}-\int_0^{\frac{1}{\sqrt{3}}}\dfrac{3x^3}{\sqrt{1-9x^4}}\,\mathrm{d}x$ $=\left[\dfrac{1}{2}x^2\sin^{-1}\left(3x^2\right)-3\left(-\dfrac{1}{18}\sqrt{1-9x^4}\right)\right]_0^{\frac{1}{\sqrt{3}}}$ (from (a)) $I=\left[\dfrac{1}{2}x^2\sin^{-1}\left(3x^2\right)+\dfrac{1}{6}\sqrt{1-9x^4}\right]_0^{\frac{1}{\sqrt{3}}}$ $=\left[\dfrac{1}{2}\left(\dfrac{1}{3}\right)\sin^{-1}(1)+\dfrac{1}{6}\sqrt{1-9\left(\dfrac{1}{9}\right)}\right]-\left[\dfrac{1}{2}(0)+\dfrac{1}{6}(1)\right]$ $=\left[\dfrac{1}{6}\left(\dfrac{\pi}{2}\right)+\dfrac{1}{6}(0)\right]-\left[\dfrac{1}{6}\right]$ $=\dfrac{\pi}{12}-\dfrac{1}{6}$

11 A construction company is installing cables on the side of a building. Points (x, y, z) are defined relative to a main anchor point O located at $(0,0,0)$, where units are in metres. Cables are laid in straight lines and the widths of the cables can be neglected.

An existing cable L starts at the main anchor point O and goes in the direction of $\begin{pmatrix}1\\2\\4\end{pmatrix}$. A new cable M passing through points $P(1,-1,\ 1)$ and $Q(3,0,0)$ is installed.

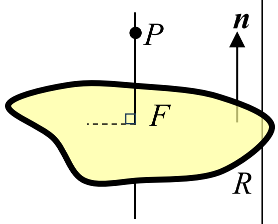
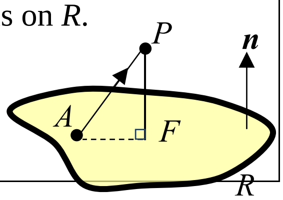
(a) Using a non-calculator method, show that the cables L and M do not meet. [4]

	Suggested Solutions
(a)	$L:\ \underset{\sim}{r} = \lambda\begin{pmatrix}1\\2\\4\end{pmatrix},\ \ \lambda \in \mathbb{R}$ $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}3\\0\\0\end{pmatrix} - \begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix}$ $M:\ \underset{\sim}{r} = \begin{pmatrix}3\\0\\0\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-1\end{pmatrix},\ \ \mu \in \mathbb{R}$ If L and M meet, $\lambda\begin{pmatrix}1\\2\\4\end{pmatrix} = \begin{pmatrix}3\\0\\0\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$ $\begin{cases}\lambda = 3+2\mu & \text{------------- (1)}\\ 2\lambda = \mu & \text{------------- (2)}\\ 4\lambda = -\mu & \text{------------- (3)}\end{cases}$ From (2) and (3), $\lambda = 0$ and $\mu = 0$. Substitute $\lambda = 0$ and $\mu = 0$ into (1): LHS = 0 RHS = 3 $\neq$ LHS OR Substituting (2) into (1): $\lambda = 3+2(2\lambda)$ $-3\lambda = 3$ $\lambda = -1$

	From (2), $\mu = -2$. Substitute $\lambda = -1$ and $\mu = -2$ into (3): LHS = −4 RHS = 2 ≠ LHS OR Substituting (1) into (3): $4(3+2\mu) = -\mu$ $9\mu = -12$ $\mu = -\frac{4}{3}$ From (3), $\lambda = \frac{1}{3}$. Substitute $\lambda = \frac{1}{3}$ and $\mu = -\frac{4}{3}$ into (2): LHS $= \frac{2}{3}$ RHS $= -\frac{4}{3} \neq$ LHS $\therefore$ there are no solutions. $\therefore$ cables *L* and *M* do not meet. (shown)

The construction company wishes to connect the point P to an inclined rectangular roof extension that is modelled by the plane with equation $2x+3y+z=28$. It was found that the best way is to use a cable to connect the point P to the point F on the rectangular roof extension that is closest to P.

(b) Find the coordinates of the point F. [3]

	Suggested Solutions
(b)	$R: 2x+3y+z=28 \Rightarrow \underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 28$ <u>Method 1</u> Since F is the foot of the perpendicular from P to R, $l_{PF}: \underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix},\quad t \in \mathbb{R}$  Since F is the point of intersection between l_{PF} and R, $\begin{pmatrix} 1+2t \\ -1+3t \\ 1+t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 28$ $2+4t-3+9t+1+t=28$ $14t=28$ $t=2$ $\therefore\ \overrightarrow{OF} = \begin{pmatrix} 1+2(2) \\ -1+3(2) \\ 1+(2) \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix}$ $\therefore\ F = (5, 5, 3)$ <u>Method 2</u> Let A be the point (14, 0, 0) that is on R. $\overrightarrow{AP} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 14 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -13 \\ -1 \\ 1 \end{pmatrix}$ 

	$\overrightarrow{FP} = \left(\overrightarrow{AP} \cdot \hat{\underset{\sim}{n}}\right)\hat{\underset{\sim}{n}}$ $= \left[\begin{pmatrix} -13 \\ -1 \\ 1 \end{pmatrix} \cdot \frac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}{\sqrt{14}}\right] \frac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}{\sqrt{14}}$ $= \frac{-26-3+1}{14}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ $= -2\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ $\overrightarrow{OF} = 2\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix}$ $\therefore$ F = (5, 5, 3)

(c) Hence find the exact length, in metres, of the cable needed to connect point P to the rectangular roof extension. [2]

	Suggested Solutions
(c)	Minimum length = PF $= \sqrt{(5-1)^2 + (5-(-1))^2 + (3-1)^2}$ $= \sqrt{16+36+4}$ $= \sqrt{56}$ $= 2\sqrt{14}$ m

12 The parametric equations of the curve C are

$$x = \tan\theta - 1 \text{ and } y = 2\sec\theta, \text{ where } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

(a) Find $\frac{dy}{dx}$ in terms of θ, leaving your answer in a single trigonometric function. [2]

	Suggested Solutions
(a)	$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ $= \frac{2\sec\theta\tan\theta}{\sec^2\theta}$ $= 2\sin\theta$

(b) Find the equation of the tangent to C at the point where it cuts the y-axis, leaving your answer in exact form. [3]

	Suggested Solutions
(b)	At the y-intercept, $x = 0$: $\tan\theta - 1 = 0$ $\tan\theta = 1$ $\theta = \frac{\pi}{4}$ since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ At $\theta = \frac{\pi}{4}$, $y = \frac{2}{\cos\frac{\pi}{4}} = 2\sqrt{2}$ and $\frac{dy}{dx} = 2\sin\frac{\pi}{4} = \sqrt{2}$ Equation of tangent: $y = \sqrt{2}x + 2\sqrt{2}$

(c) Find the Cartesian equation of C. [1]

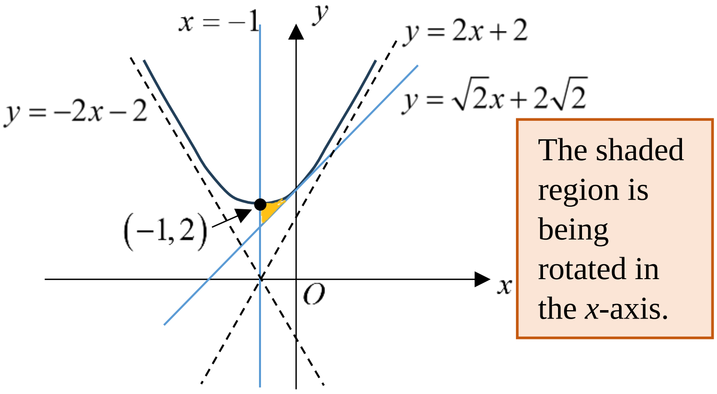
	Suggested Solutions
(c)	$\tan^2\theta+1=\sec^2\theta$ $(x+1)^2+1=\left(\frac{y}{2}\right)^2$ For $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, $0<\cos\theta\le 1\Rightarrow\sec\theta\ge 1$ $\therefore (x+1)^2+1=\left(\frac{y}{2}\right)^2, y\ge 2$.

(d) Find the equations of the asymptotes and state the coordinates of any turning point(s).

[3]

	Suggested Solutions
(d)	To find asymptotes: $\left(\frac{y}{2}\right)^2=(x+1)^2$ $\frac{y}{2}=\pm(x+1)$ $y=2x+2$ or $y=-2x-2$ Since the centre of the hyperbola is $(-1,0)$ and $y\ge 2$, the turning point is $(-1,2)$.

(e) The region bounded by the tangent, the curve C and the line $x=-1$ is rotated completely about the x-axis. Calculate the volume obtained. [3]

	Suggested Solutions
(e)	 $\frac{y^2}{4}=(x+1)^2+1$ $y^2=4(x+1)^2+4$ $\text{Volume}=\pi\int_{-1}^{0}4\left((x+1)^2+1\right)-\left(\sqrt{2}x+2\sqrt{2}\right)^2\ \mathrm{d}x$ $=2.09\ \text{unit}^3$

13 At the start of January 2014, Leon took a home loan of \$500 000 from a bank that offers a fixed monthly interest rate of 0.3%, to buy an apartment unit in a condominium. The interest was added to the amount owed on the 15th of every month, with the first interest amount added on 15 January 2014. Leon made a monthly repayment of \$$x$ on the 20th of every month, with the first repayment on 20 January 2014.

(a) Show that the amount (in dollars) that Leon owed at the end of n months was

$$500\ 000(1.003)^n - \frac{1000x}{3}\left(1.003^n - 1\right). \qquad [3]$$

	Suggested Solutions
(a)	

n	Amount owed at end of n months
1	$500\ 000(1.003) - x$
2	$\left[500\ 000(1.003) - x\right](1.003) - x$ $= 500\ 000(1.003)^2 - 1.003(x) - x$
3	$\left[500\ 000(1.003)^2 - 1.003(x) - x\right](1.003) - x$ $= 500\ 000(1.003)^3 - (1.003)^2(x) - 1.003(x) - x$
	$\vdots$
n	$500\ 000(1.003)^n - (1.003)^{n-1}x - \ldots$ $-(1.003)^2 x - 1.003x - x$ $= 500\ 000(1.003)^n$ $- x\left(1 + 1.003 + 1.003^2 + \ldots + 1.003^{n-1}\right)$ $= 500\ 000(1.003)^n - \dfrac{x\left(1.003^n - 1\right)}{1.003 - 1}$ $= 500\ 000(1.003)^n - \dfrac{1000x}{3}\left(1.003^n - 1\right)$ (Shown)

Leon planned to repay the loan fully in 360 monthly repayments.

(b) Find the value of x, giving your answer to the nearest cents. [2]

	Suggested Solutions
(b)	$500\ 000(1.003)^{360} - \dfrac{1000x}{3}\left(1.003^{360} - 1\right) \le 0$ $\dfrac{1000x}{3}\left(1.003^{360} - 1\right) \ge 500\ 000(1.003)^{360}$

	$x \geq \dfrac{500\ 000(1.003)^{360}}{\frac{1000}{3}\left(1.003^{360}-1\right)}$ $x \geq 2273.226751$ The monthly repayment amount that Leon needs to make is \$2273.23 (to the nearest cents).

The estate management collects a monthly maintenance fee from each apartment unit within the condominium at the start of every month. The monthly maintenance fee on 1 January 2014 was \$400. To cope with the rising cost of maintenance over time, the estate management increased the monthly maintenance fee by \$$k$ every 2 years such that the monthly maintenance fee increased to $\$(400+k)$ on 1 January 2016 and to $\$(400+2k)$ on 1 January 2018 and so on.

(c) Show that the total maintenance fee (in dollars) that Leon paid at the end of 10 years was of the form $s+tk$, where s and t are positive constants to be determined. [3]

	Suggested Solutions
(c)	Total maintenance fee (in dollars) at the end of 10 years $=24(400)+24(400+k)+...+24(400+4k)$ $=24\left[\frac{5}{2}(400+400+4k)\right]$ $=48\ 000+240k$

The value of Leon's apartment unit increased to \$750 000 at the end of December 2023. It was also known that $k=50$.

(d) By considering the amount Leon owed, along with the repayment and maintenance fee that he paid over the 10 years, explain, with justification, whether Leon should sell his apartment unit at the end of December 2023 if he received an offer of \$750 000 from an interested buyer. [4]

	Suggested Solutions
(d)	Total maintenance fee at the end of 10 years $=\$(48\ 000+240\times 50)$ $=\$60\ 000$ Total repayment paid at the end of 10 years (i.e. 120 months)

	$= \$2273.23(120)$ $= \$272\ 787.60$ Amount owed at the end of 10 years $= 500\ 000(1.003)^{120} - \frac{1000(2273.23)}{3}(1.003^{120} - 1)$ $= \$388\ 511.27\ (\text{to 2 dp})$ <u>Method 1</u> Total outlay over 10 years = \$272 787.60 + \$60 000 + \$388 511.27 = \$721 298.87 < \$750 000 Since the total outlay over the past 10 years was less than \$750 000, Leon should sell his apartment unit at the end of 10 years. <u>Method 2</u> Potential Profit = \$750 000 – (\$272 787.60 + \$60 000 + \$388 511.27) = \$28 701.13 Since the potential profit is \$28 701.13, Leon should sell his apartment unit at the end of 10 years.