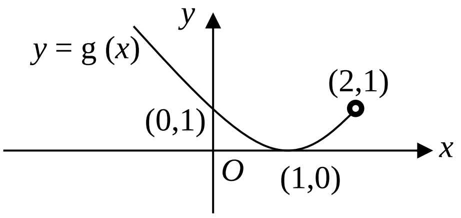
H2 Mathematics (9758)
Chapter 3 Functions
Extra Practice Solutions

Q1	2018/NYJC Promo/7
(i)	$g(0.5)=g(1.5)=\frac{1}{4}$. Since g is not one-one, g does not have an inverse.

Alternative:



Since the horizontal line $y=0.5$ cuts the graph of $y=g(x)$ at 2 different points, f is not one-one. Thus g^{-1} does not exist.

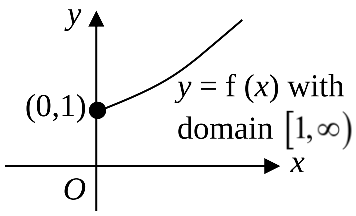
(ii)

$R_g=[0,\infty)\subseteq D_f=[0,\infty)$

Since $R_g\subseteq D_f$, $\therefore$ fg exists

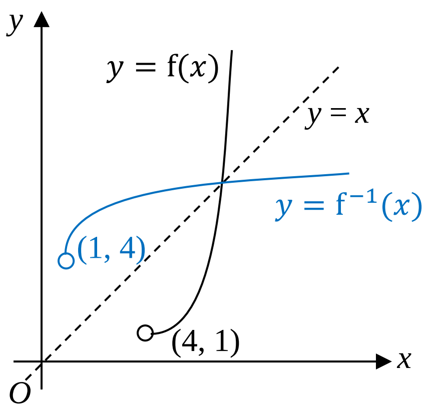
(iii)

$D_{fg}=D_g=(-\infty,2)\xrightarrow{g}R_g=[0,\infty)\xrightarrow{f}R_{fg}=[1,\infty)$



$\because R_g=D_f$,
$\therefore R_{fg}=R_f$

Q2	2008(9740)/II/4
(i), (iii)	



Sketch based on the domain given.

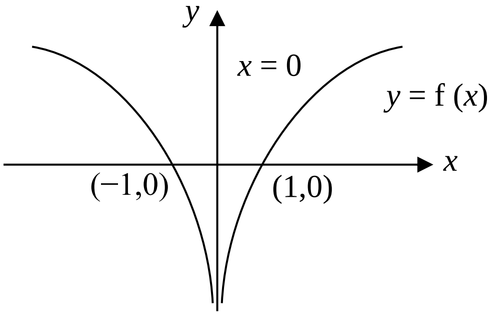
Use open circle for exclusion of points.

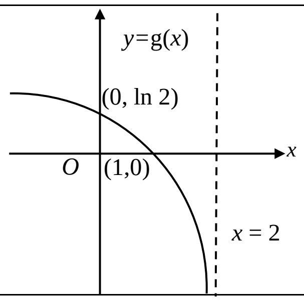
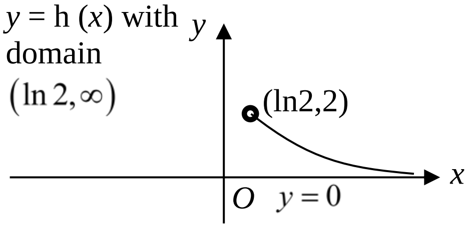
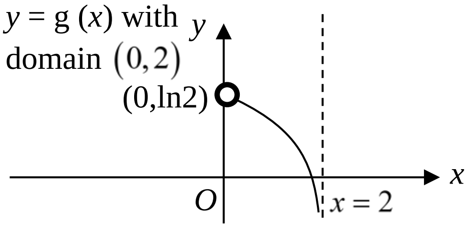
Graphs should be symmetrical about the line $y=x$ (Use same scale for x- and y-axis)

Graphs must pass vertical and horizontal line test.

Be careful with the curvature, ensure that the graph of $f^{-1}(x)$ does not curve downwards!

(ii)	Let $y=(x-4)^2+1$ $y-1=(x-4)^2$ $(x-4)=\pm\sqrt{y-1}$ $x=4\pm\sqrt{y-1}$ Since $x>4$, $\therefore x=4+\sqrt{y-1}$ $\therefore \mathrm{f}^{-1}(x)=4+\sqrt{x-1}$ $\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{f}}=(1,\infty)$ $\therefore \mathrm{f}^{-1}: x \mapsto 4+\sqrt{x-1}, x\in\mathbb{R}, x>1$ Use the domain of f to decide which expression to pick. Find R_{f} by sketching graph of f and find range of possible y values. (i.e. minimum and maximum y values)
(iv)	Reflect graph of $y=\mathrm{f}(x)$ in the line $y=x$ to obtain graph of $y=\mathrm{f}^{-1}(x)$. At intersection, $y=\mathrm{f}(x)=\mathrm{f}^{-1}(x)=x$, Equating $y=x$ and $y=\mathrm{f}(x)$ we have: $x=(x-4)^2+1$ $x^2-9x+17=0$ $x=\dfrac{9\pm\sqrt{9^2-4(17)}}{2}=\dfrac{9\pm\sqrt{13}}{2}$ Since $x>4$, $\therefore x=\dfrac{9+\sqrt{13}}{2}$.

Q3	**2009/TPJC/I/9**
(i)	 From graph, $\mathrm{R}_{\mathrm{f}}=\mathbb{R}$
(ii)	$\mathrm{R}_{\mathrm{g}}=[2,\infty)$, $\mathrm{D}_{\mathrm{f}}=\mathbb{R}\setminus\{0\}$ Since $\mathrm{R}_{\mathrm{g}}\subseteq\mathrm{D}_{\mathrm{f}}$, fg exists.
(iii)	Since the horizontal line $y=1$ cuts the graph of $y=\mathrm{f}(x)$ at 2 different points, f is not one-one. Thus f^{-1} does not exist.
(iv)	Greatest value of $a=0$. $y=\ln(x^2),\quad x<0$ $x^2=\mathrm{e}^y$ $x=\pm\sqrt{\mathrm{e}^y}$ $x=-\mathrm{e}^{\frac{y}{2}}$ since $x<0$ $\mathrm{D}_{\mathrm{f}^{-1}}=\mathbb{R}$ $\mathrm{f}^{-1}: x\to -\mathrm{e}^{\frac{x}{2}},\quad x\in\mathbb{R}$

Q4	2018/SAJC Promo/6
(i)	Let $y=\frac{1}{(1+x)(3-x)}$ $3+2x-x^2=\frac{1}{y}$ $(x-1)^2-1-3=-\frac{1}{y}$ $x-1=\pm\sqrt{4-\frac{1}{y}}$ $x=1\pm\sqrt{4-\frac{1}{y}}$ $x-1=-\sqrt{4-\frac{1}{y}}$ since $x<1$ $\therefore\ x=1-\sqrt{4-\frac{1}{y}}$ $\therefore \mathrm{f}^{-1}(x)=1-\sqrt{4-\frac{1}{x}}$
(ii)	$\mathrm{R_g}=(-\infty,\infty)$ and $\mathrm{D_h}=(\ln 2,\infty)$ Since $\mathrm{R_g}\not\subseteq\mathrm{D_h}$, $\therefore$ hg does not exist. 
(iii)	$\mathrm{gh}(x)=\mathrm{g}(\mathrm{h}(x))$ $=\mathrm{g}(4e^{-x})$ $=\ln(2-4e^{-x})$   $\mathrm{D_h}=(\ln 2,\infty)\to\mathrm{R_h}=(0,2)\to\mathrm{R_{gh}}=(-\infty,\ln 2)$
(iv)	Let $(\mathrm{gh})^{-1}\left(\ln\frac{4}{3}\right)=a$ $\Rightarrow \mathrm{gh}(a)=\ln\frac{4}{3}$

	$\ln\left(2-4e^{-a}\right)=\ln\frac{4}{3}$ $2-4e^{-a}=\frac{4}{3}$ $e^{-a}=\frac{1}{6}$ $e^{a}=6$ $a=\ln 6 \quad \left(\text{accept } -\ln\frac{1}{6}\right)$

Q5	**2009(9740)/II/3**
(i)	Let $\mathrm{f}\left(x\right)=y,$ $y=\frac{ax}{bx-a}$ $bxy-ay=ax \Rightarrow bxy-ax=ay$ $\therefore x=\frac{ay}{by-a}$ Note: $\mathrm{R_f}=\mathbb{R}\setminus\left\{\frac{a}{b}\right\}=\mathrm{D_f}$, that's why f^2 exists. $\mathrm{f}^{-1}: x \mapsto \frac{ax}{bx-a}, x\in\mathbb{R}, x\neq\frac{a}{b}$ Since $\mathrm{f}\left(x\right)=\mathrm{f}^{-1}\left(x\right)$, $\mathrm{f}^2\left(x\right)=\mathrm{ff}\left(x\right)=\mathrm{f}^{-1}\mathrm{f}\left(x\right)=x$. $\mathrm{D_f}=\mathbb{R}\setminus\left\{\frac{a}{b}\right\}\rightarrow\mathrm{R_f}=\mathbb{R}\setminus\left\{\frac{a}{b}\right\}\rightarrow\mathrm{R_{f^2}}=\mathbb{R}\setminus\left\{\frac{a}{b}\right\}$ $\mathrm{f}^2: x\mapsto x, x\in\mathbb{R}, x\neq\frac{a}{b}$ $\mathrm{R_{f^2}}=\mathbb{R}\setminus\left\{\frac{a}{b}\right\}$
(ii)	Since $\mathrm{f}^2\left(x\right)=x$, $\mathrm{f}^{2023}(x)=\mathrm{ff}^{2022}(x)=\mathrm{f}(x)=\frac{ax}{bx-a}$ $\mathrm{f}^{2023}(1)=\frac{a}{b-a}$
(iii)	$\mathrm{g}\left(x\right)=\frac{1}{x}, x\in\mathbb{R}, x\neq 0$ $\mathrm{R_g}=\mathbb{R}\setminus\left\{0\right\}$, $\mathrm{D_f}=\mathbb{R}\setminus\left\{\frac{a}{b}\right\}$ Observe that $\frac{a}{b}\in\mathrm{R_g}$ but $\frac{a}{b}\notin\mathrm{D_f}$. Since $\mathrm{R_g}\not\subseteq\mathrm{D_f}$, the composite function fg does not exist.

(iv)

$f^{-1}(x) = x$

$\dfrac{ax}{bx-a} = x$

$ax = bx^2 - ax$

$bx^2 - 2ax = 0$

$x(bx - 2a) = 0$

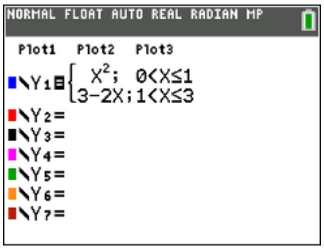
$x = 0$ or $x = \dfrac{2a}{b}$

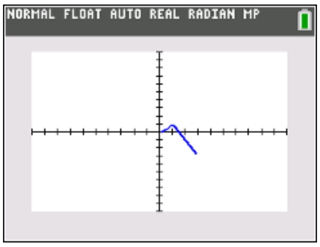
Do **NOT** "cancel" x, always do factorisation and conclude from there.

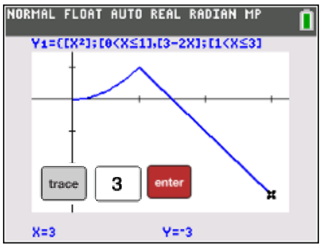
Q 6	Suggested Solutions

Check for continuous/disjoint at $x=1$:
When $x=1$, $f(x) = 1^2 = 1$
When $x=1$, $f(x) = 3-2(1) = 1$
Thus, the graph is continuous at $x=1$.

Axial intercept:
$3-2x=0$
$\Rightarrow x = \dfrac{3}{2}$





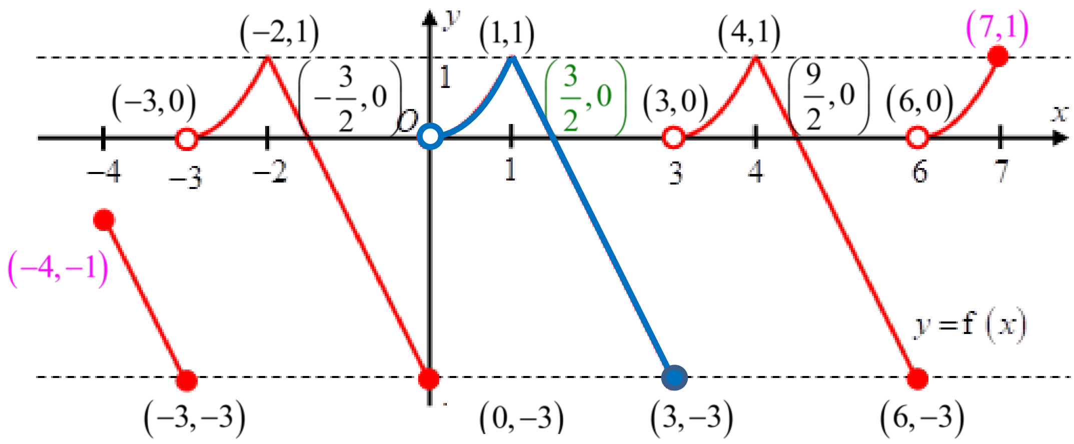


$f(x+3) = f(x)$
$\Rightarrow$ The period of the graph is 3 units. The graph repeat itself every 3 units.

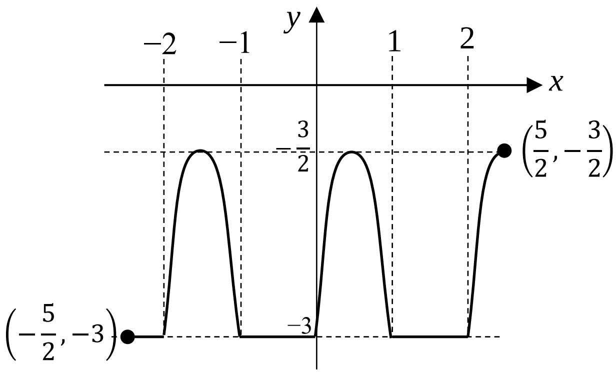
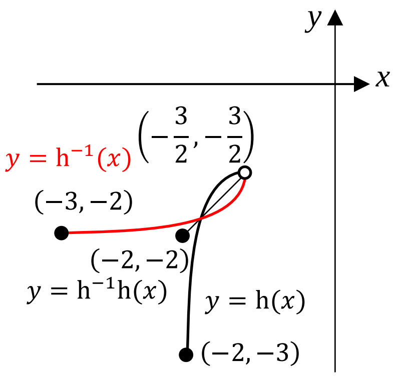
Calculating the coordinates of end-points for $-4 \le x \le 7$:
$f(7) = f(4) = f(1) = 1^2 = 1 \Rightarrow$ closed circle
$f(-4) = \ldots = f(2) = 3-2(2) = -1 \Rightarrow$ closed circle

<u>**Things to note:**</u>

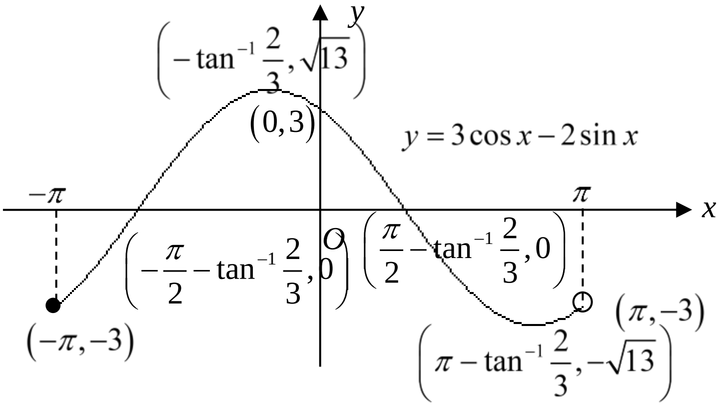
- Proper scale for x-axis (e.g. 1cm denote 1 unit)
- **Open circles for excluded endpoints** and **closed circles for included endpoints**
- Draw dotted reference lines for critical y-values to ensure you always end at the same height

	
	$f(26)=f(8(3)+2)=f(2)=3-2(2)=-1$

Q7	2009/NYJC/I/2
(i)	$D_f=(-\infty,\infty)$ and $R_g=(2,\infty)$ Since $R_g \subseteq D_f \Rightarrow$ function fg exists. $fg(x)=f\left(\sqrt{x+4}\right)=\left(\sqrt{x+4}\right)^2-1$ $fg: x \mapsto x+3, \quad x\in\mathbb{R}^+$
(ii)	$R_{fg}=(3,\infty)$ $D_g=(0,\infty)$ Since $R_{fg}\subseteq D_g \Rightarrow$ function gfg exists $h(x)=gfg(x)=g[fg(x)]=\sqrt{(x+3)+4}=\sqrt{x+7}$
(iii)	$h(x)=g^{-1}g(x), \quad x>0$ $g[fg(x)]=g^{-1}g(x)$ $\sqrt{x+7}=x, \quad x\in\mathbb{R}^+$ $x+7=x^2$ $x^2-x-7=0$ $x=\frac{1\pm\sqrt{1-4(1)(-7)}}{2(1)}=\frac{1}{2}\left(1\pm\sqrt{29}\right)$ $x=\frac{1}{2}\left(1+\sqrt{29}\right)$ since $x\in\mathbb{R}^+$

Q8	**IJC/Prelim2013/P1/Q13 [modified]**
(i)	
(ii)	$\mathrm{f}(-3)+\mathrm{f}(4.5)=-3+\left(-\frac{3}{2}\right)=-4.5$
(iii)	Greatest value of $a=-\frac{3}{2}$
(iv)	
(v)	Since the curves intersect at the line $y=x$, solving $\mathrm{h}(x)=\mathrm{h}^{-1}(x)$ is equivalent to solving $\mathrm{h}(x)=x$. $\frac{3}{(2x+3)^2-2}=x$ $3=x\left(4x^2+12x+9-2\right)$ $4x^3+12x^2+7x-3=0$ Using GC, $x=-1.781$ or $x=-1.5$ (rej, since $x<\frac{3}{2}$) or $x=0.2808$ (rej, since $x<\frac{3}{2}$)

Q9	**2016/H2 Specimen Paper/II/1**
(i)	Let $3\cos x-2\sin x=R\cos\left(x+\alpha\right)$ $=R\cos x\cos\alpha-R\sin x\sin\alpha$ $R\cos\alpha=3$ ---(1), $R\sin\alpha=2$ ---(2) $(1)^2+(2)^2: R^2=13$ $\Rightarrow R=\sqrt{13}$ $\frac{(2)}{(1)}:\tan\alpha=\frac{2}{3}$ $\Rightarrow\alpha=\tan^{-1}\frac{2}{3}$

	$\therefore\ 3\cos x-2\sin x=\sqrt{13}\cos\left(x+\tan^{-1}\frac{2}{3}\right)$ $\Rightarrow \mathrm{R_f}=\left[-\sqrt{13},\sqrt{13}\right]$ $\cos(x+\alpha)=0$ $x+\alpha=\frac{\pi}{2}$ or $x+\alpha=-\frac{\pi}{2}$ $x=\frac{\pi}{2}-\alpha$ or $x=-\frac{\pi}{2}-\alpha$ 
(ii)	Largest value of $b=\pi-\tan^{-1}\frac{2}{3}$ $y=\sqrt{13}\cos\left(x+\tan^{-1}\frac{2}{3}\right)$ $x+\tan^{-1}\frac{2}{3}=\cos^{-1}\frac{y}{\sqrt{13}}$ $x=\cos^{-1}\frac{y}{\sqrt{13}}-\tan^{-1}\frac{2}{3}$ $\therefore \mathrm{f}^{-1}(x)=\cos^{-1}\frac{x}{\sqrt{13}}-\tan^{-1}\frac{2}{3}$ $\mathrm{f}^{-1}: x\mapsto\cos^{-1}\frac{x}{\sqrt{13}}-\tan^{-1}\frac{2}{3}, -\sqrt{13}\le x\le\sqrt{13}$