Sequence & Series Part I: Additional Practice Questions

1. Given that θ is sufficiently small, show that $\tan\left(\frac{\pi}{3}+\theta\right) \approx \sqrt{3}+4\theta$.

$$\tan\left(\frac{\pi}{3}+\theta\right)$$
$$=\frac{\tan\frac{\pi}{3}+\tan\theta}{1-\tan\frac{\pi}{3}\tan\theta}$$
$$=\frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta}$$
$$\approx\frac{\sqrt{3}+\theta}{1-\sqrt{3}\theta}$$
$$=\left(\sqrt{3}+\theta\right)\left(1-\sqrt{3}\theta\right)^{-1}$$
$$\approx\left(\sqrt{3}+\theta\right)\left(1+(-1)\left(-\sqrt{3}\theta\right)\right)$$
$$=\left(\sqrt{3}+\theta\right)\left(1+\sqrt{3}\theta\right)$$
$$=\sqrt{3}+3\theta+\theta+\sqrt{3}\theta^2$$
$$\approx\sqrt{3}+4\theta \text{ (shown)}$$

2. **[MI/2020/Promo/PU2/P1/Q6(b)]**

Find the expansion of $\frac{1+3x}{\sqrt{9-x^2}}$ in ascending powers of x, up to and including the term in x^3.

State the set of values of x for which the expansion is valid. **[4]**

	$\frac{1+3x}{\sqrt{9-x^2}}=(1+3x)\left(9-x^2\right)^{-\frac{1}{2}}$ $=(1+3x)\left[9\left(1-\frac{x^2}{9}\right)\right]^{-\frac{1}{2}}$ $=(1+3x)\left[9^{-\frac{1}{2}}\left(1-\frac{x^2}{9}\right)^{-\frac{1}{2}}\right]$ $=\frac{1}{3}(1+3x)\left(1-\frac{x^2}{9}\right)^{-\frac{1}{2}}$

	$=\frac{1}{3}(1+3x)\left[1+\left(-\frac{1}{2}\right)\left(-\frac{1}{9}x^2\right)+...\right]$ $=\frac{1}{3}(1+3x)\left(1+\frac{1}{18}x^2+...\right)$ $=\frac{1}{3}\left(1+3x+\frac{1}{18}x^2+\frac{1}{6}x^3+...\right)$ $=\frac{1}{3}+x+\frac{1}{54}x^2+\frac{1}{18}x^3+...$ For expansion to be valid, $\left\|-\frac{x^2}{9}\right\|<1$ $\left\|x^2\right\|<9$ $\therefore -3<x<3$

3. **[TJC/2020/J1MYE/1]**

Expand $\left(\frac{1-2x}{x-2}\right)^2$ in ascending powers of x up to and including the term in x^2 and state the range of values of x for which this expansion is valid. **[4]**

	$\left(\frac{1-2x}{x-2}\right)^2=(1-2x)^2(x-2)^{-2}$ $=\left(1-4x+4x^2\right)(-2)^{-2}\left(1-\frac{x}{2}\right)^{-2}$ $=\frac{1}{4}\left(1-4x+4x^2\right)\left(1+x+3\left(\frac{x}{2}\right)^2+\cdots\right)$ $=\frac{1}{4}\left(1+x+\frac{3}{4}x^2-4x-4x^2+4x^2+\cdots\right)$ $=\frac{1}{4}-\frac{3}{4}x+\frac{3}{16}x^2+\cdots$ Expansion is valid for $\left\|\frac{x}{2}\right\|<1$, i.e. $-2<x<2$

Sequence & Series Part II: Additional Practice Questions

1. An arithmetic progression has 100 terms and first term a. If the sum of the last 50 terms is 1875 more than the sum of the first 50 terms, find the common difference. If $a = 2$, find the least value of n such that the sum of the first n terms of the arithmetic progression exceeds 122.

$$S_{100} - S_{50} - S_{50} = 1875$$

$$\frac{100}{2}[2a + 99d] - 2\left(\frac{50}{2}\right)(2a + 49d) = 1875$$

$$100a + 4950d - 100a - 2450d = 1875$$

$$d = \frac{3}{4}$$

If $a = 2$:

$$S_n > 122$$

$$\Rightarrow \quad \frac{n}{2}\left[2(2) + (n-1)\left(\frac{3}{4}\right)\right] > 122$$

$$\Rightarrow \quad 16n + 3n(n-1) > 976$$

$$\Rightarrow \quad 3n^2 + 13n - 976 > 0$$

$$\Rightarrow \quad n = \frac{-13 \pm \sqrt{169 - 4(3)(-976)}}{2(3)}$$

$$= \frac{-13 \pm 109}{6}$$

$$= 16 \text{ or } -20\tfrac{1}{3}$$

$$\Rightarrow \quad n < -20\tfrac{1}{3} \text{ or } n > 16$$

$$\Rightarrow \quad n = 17$$

2. **[N2009/I/8]**

Two musical instruments, *A* and *B*, consist of metal bars of decreasing lengths.

(i) The first bar of instrument *A* has length 20cm and the lengths of the bars form a geometric progression. The 25th bar has length 5cm. Show that the total length of all the bars must be less than 357cm, no matter how many bars there are. [4]

Instrument *B* consists of only 25 bars which are identical to the first 25 bars of instrument *A*.

(ii) Find the total length, *L* cm, of all the bars of instrument *B* and the length of the 13th bar. [3]

(iii) Unfortunately the manufacturer misunderstands the instructions and constructs instrument *B* wrongly, so that the lengths of the bars are in arithmetic progression with common difference *d* cm. If the total length of the 25 bars is still *L* cm and the length of the 25th bar is still 5cm, find the value of *d* and the length of the longest bar. [4]

(i)	$ar^{24} = 20r^{24} = 5$ $r^{24} = \frac{1}{4}$ $r = \sqrt[12]{\frac{1}{2}}$ Total length of all bars $< S_\infty$ $= \dfrac{20}{1 - \sqrt[12]{\frac{1}{2}}}$ $= 356.343 < 357$ (shown)
(ii)	$L = 20\dfrac{1 - \left(\sqrt[12]{\frac{1}{2}}\right)^{25}}{1 - \sqrt[12]{\frac{1}{2}}} = 272.2573 \approx 272$ Length of 13th bar $= 20\left(\sqrt[12]{\frac{1}{2}}\right)^{12} = 20 \times \frac{1}{2} = 10$ cm
(iii)	Let b = length of first bar of instrument *B*. $\frac{25}{2}[\,2b + 24d\,] = L$ $b + 12d = \frac{L}{25}$ -----(1) Also $b + (25-1)d = 5 \Rightarrow b + 24d = 5$ -----(2) (2) – (1): So $12d = 5 - \frac{L}{25} = 5 - \frac{272.2573}{25}$ $\therefore\ d = -0.49086 \approx -0.491$ Length of longest bar $= b$ $= 5 - 24(-0.49086)$ $= 16.8$ cm

3. The positive multiples of 3 are grouped into sets as $A_1=\{3\}$, $A_2=\{6,9\}$, $A_3=\{12,15,18\}$ and $A_4=\{21,24,27,30\}$,... where the set A_n contains n elements.

(i) Find the total number of elements in the first n sets. Show that the last element of the set A_n is given by $\frac{3}{2}n(n+1)$.

(ii) Hence, or otherwise, find the sum of all the elements in the set A_n in terms of n.

(i) Tt. no. of elements in the first n sets

$= 1+2+3+\dots+n$

$= \frac{n(n+1)}{2}$ //.

Last element of set A_n

$= 3+\left[\frac{n(n+1)}{2}-1\right](3)$ $T_n = a+(n-1)d$

$= \frac{3}{2}n(n+1)$ //.

(ii) First element of set A_n

$=(\text{last element of set } A_{n-1})+3$

$= \frac{3}{2}(n-1)n+3$

$\therefore$ Sum of all the elements in set A_n

$= \frac{n}{2}\left[\left(\frac{3}{2}(n-1)n+3\right)+\left(\frac{3}{2}n(n+1)\right)\right]$

$= \frac{n}{2}\left[\frac{3}{2}n(n-1+n+1)+3\right]$

$= \frac{n}{2}(3n^2+3)$

$= \frac{3}{2}n(n^2+1)$ //.

4. **[N2011/I/9]**

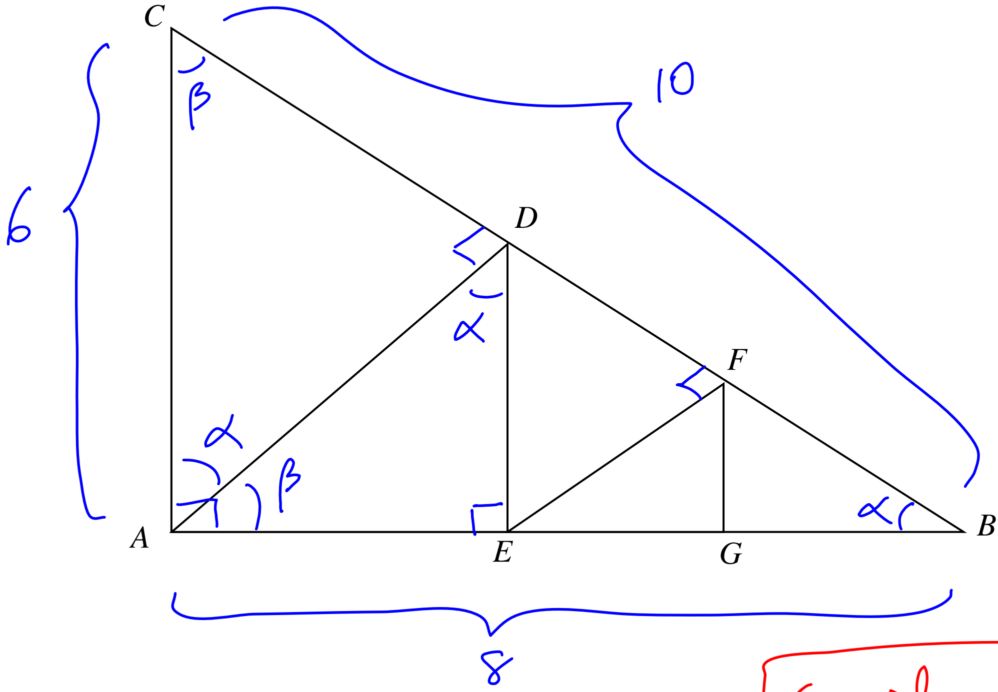
(i) A company is drilling for oil. Using machine *A*, the depth drilled on the first day is 256 metres. On each subsequent day, the depth drilled is 7 metres less than on the previous day. Drilling continues daily up to an including the day when a depth of less than 10 metres is drilled. What depth is drilled on the 10th day, and what is the total depth when drilling is completed? [6]

(ii) Using machine *B*, the depth drilled on the first day is also 256 metres. On each subsequent day, the depth drilled is $\frac{8}{9}$ of the depth drilled on the previous day. How many days does it take for the depth drilled to exceed 99% of the theoretical maximum total depth? [4]

Solutions:

(i)	$a=256,\ d=-7$ Depth drilled on 10th day $=T_{10}=a+(10-1)d$ $=256+(9)(-7)$ $=193$ metres Want $T_n=a+(n-1)d<10$ $\Rightarrow 256+(n-1)(-7)<10$ $\Rightarrow 256-7n+7<10$ $\Rightarrow 7n>253$ $\Rightarrow n>\frac{253}{7}=36\frac{1}{7}$ So it takes 37 days for drilling to complete. Total depth $S_{37}=\frac{37}{2}\left[2(256)+(37-1)(-7)\right]=4810$ metres
(ii)	$a=256,\ r=\frac{8}{9}$, Theoretical maximum total depth $S_\infty=\frac{a}{1-r}$ Want $S_n=\frac{a\left(1-r^n\right)}{1-r}>0.99S_\infty$ $\Rightarrow \frac{a\left(1-r^n\right)}{1-r}>0.99\left(\frac{a}{1-r}\right)$ $\Rightarrow 1-r^n>0.99$ $\Rightarrow \left(\frac{8}{9}\right)^n<0.01$ $\Rightarrow n>\frac{\ln 0.01}{\ln\frac{8}{9}}\Rightarrow n>39.09875612$ Thus, it will take 40 days

5. *In the triangle ABC, $AB = 8$, $BC = 10$ and $CA = 6$. AD is perpendicular to BC, DE to AB, EF to BD, FG to EB and so on.
Find the sum to infinity of $CA + AD + DE + EF + FG + \ldots$



$S_\infty = CA + AD + DE + EF + \ldots$

$= 6 + \left(\frac{4}{5}\right)6 + \left(\frac{4}{5}\right)^2 6 + \left(\frac{4}{5}\right)^3 6 + \ldots$

$= \dfrac{6}{1 - \frac{4}{5}}$

$= 30$ //.

Consider similar Δs ΔCDA & ΔCAB

$\dfrac{AD}{AC} = \dfrac{BA}{BC} = \dfrac{8}{10}$

$\Rightarrow AD = \frac{4}{5}(6)$

Consider similar Δs ΔAED & ΔCAB

$\dfrac{DE}{AD} = \dfrac{BA}{CB} = \dfrac{8}{10}$

$\Rightarrow DE = \frac{4}{5}AD$

$= \left(\frac{4}{5}\right)^2(6)$

6. **[HJC03/1/1]**

By writing $\dfrac{3r^2-2r-3}{(r+1)!}$ in the form $\dfrac{A}{(r-1)!}+\dfrac{B}{r!}+\dfrac{C}{(r+1)!}$, find $\displaystyle\sum_{r=1}^{n}\frac{3r^2-2r-3}{(r+1)!}$. [4]

$$\frac{3r^2-2r-3}{(r+1)!} = \frac{A}{(r-1)!}+\frac{B}{r!}+\frac{C}{(r+1)!}$$

$$= \frac{Ar(r+1)+B(r+1)+C}{(r+1)!}$$

$$= \frac{Ar^2+(A+B)r+(B+C)}{(r+1)!}$$

Comparing coeffs, $A=3$

$A+B=-2 \Rightarrow B=-5$

$B+C=-3 \Rightarrow C=2$

$$\therefore \sum_{r=1}^{n}\frac{3r^2-2r-3}{(r+1)!} = \sum_{r=1}^{n}\left[\frac{3}{(r-1)!}-\frac{5}{r!}+\frac{2}{(r+1)!}\right]$$

$$\begin{aligned}
= & \frac{3}{0!}-\frac{5}{1!}+\frac{2}{2!} \\
& +\frac{3}{1!}-\frac{5}{2!}+\frac{2}{3!} \\
& +\frac{3}{2!}-\frac{5}{3!}+\frac{2}{4!} \\
& \vdots \\
& +\frac{3}{(n-2)!}-\frac{5}{(n-1)!}+\frac{2}{n!} \\
& +\frac{3}{(n-1)!}-\frac{5}{n!}+\frac{2}{(n+1)!}
\end{aligned}$$

$$= 3-5+3+\frac{2}{n!}-\frac{5}{n!}+\frac{2}{(n+1)!}$$

$$= 1-\frac{3}{n!}+\frac{2}{(n+1)!} \quad //.$$

7.* **[SAJC03/1/4 (modified)]**
Given the series

$$2\log\left(\frac{2}{1}\right)+4\log\left(\frac{3}{2}\right)+6\log\left(\frac{4}{3}\right)+8\log\left(\frac{5}{4}\right)+\ldots\ .$$

Let S_n denote the sum of the first n terms of the above series.

Write down the values of S_n for $n = 1, 2$ and 3.

Suggest an expression for S_n for all positive integers n.

$S_1 = 2\log 2$

$S_2 = 2\log 2 + 4\log\left(\frac{3}{2}\right)$

$= \log\left[2^2\cdot\left(\frac{3}{2}\right)^4\right] = \log\left(\frac{3^4}{2^2}\right)$

$S_3 = 2\log 2 + 4\log\left(\frac{3}{2}\right) + 6\log\left(\frac{4}{3}\right)$

$= \log\left[2^2\cdot\left(\frac{3}{2}\right)^4\cdot\left(\frac{4}{3}\right)^6\right]$

$= \log\left[\frac{4^6}{2^2\cdot 3^2}\right]$

$\vdots$

$S_n = \log\left[\frac{(n+1)^{2n}}{2^2\cdot 3^2\cdot 4^2\cdots n^2}\right]$

$= \log\left[\frac{(n+1)^{2n}}{(n!)^2}\right]$

$= 2\log\left[\frac{(n+1)^n}{n!}\right]$ //

8. (a) Find the sum of the series $3\cdot 5+5\cdot 7+7\cdot 9+...$ n terms.
   (b) Find the sum of $S=1^2-2^2+3^2-4^2+....+100^2$.
   (c) Given that $\sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2$ show that

$$\sum_{r=n}^{2n}(3r^3-n-1)=\frac{1}{4}(n+1)\left[\ 9n^2(5n+1)\ -\ 4(n+1)\ \right]$$

(a)

$$3\cdot 5+5\cdot 7+7\cdot 9+...+(2n+1)(2n+3)$$

$$= \sum_{r=1}^{n}(2r+1)(2r+3)$$

$$= \sum_{r=1}^{n}(4r^2+8r+3)$$

$$= 4\sum_{r=1}^{n}r^2 + 8\sum_{r=1}^{n}r + \sum_{r=1}^{n}3$$

$$= 4.\frac{n}{6}(n+1)(2n+1) + 8.\frac{n}{2}(n+1) + 3n$$

$$= \frac{2}{3}n(n+1)\ [\ (2n+1) + 6\ ] + 3n$$

$$= \frac{2}{3}n(n+1)(2n+7) + 3n$$

$$= \frac{1}{3}n\left[\ 2(n+1)(2n+7) + 9\ \right]$$

$$=\frac{1}{3}n(4n^2+18n+23)$$

(b)

$$S=\left(1^2+3^2+5^2+...+99^2\right)-\left(2^2+4^2+6^2+...+100^2\right)$$

$$= \sum_{r=1}^{50}(2r-1)^2 - \sum_{r=1}^{50}(2r)^2$$

$$= \sum_{r=1}^{50}(4r^2-4r+1) - 4\sum_{r=1}^{50}r^2$$

$$= 4\sum_{r=1}^{50}r^2-4\sum_{r=1}^{50}r+\sum_{r=1}^{50}1 - 4\sum_{r=1}^{50}r^2$$

$$= -4.\frac{50}{2}(50+1) + 50$$

$$= -\ 5050$$

(c)

$$\sum_{r=n}^{2n}(3r^3-n-1) = \sum_{r=n}^{2n}[3r^3-(n+1)]$$

$$= 3\sum_{r=n}^{2n}r^3-(n+1)\sum_{r=n}^{2n}1$$

$$= 3\left[\sum_{r=1}^{2n} r^3 - \sum_{r=1}^{n-1} r^3\right] - (n+1)(2n-n+1)$$

$$= 3\left\{\frac{(2n)^2}{4}(2n+1)^2 - \frac{(n-1)^2}{4}[(n-1)+1]^2\right\} - (n+1)^2$$

$$= \frac{3}{4}n^2(2n+1)^2 - \frac{3}{4}n^2(n-1)^2 - (n+1)^2$$

$$= \frac{3}{4}n^2\left[(2n+1)^2 - (n-1)^2\right] - (n+1)^2$$

$$= \frac{3}{4}n^2\left(15n^2 + 18n + 3\right) - (n+1)^2$$

$$= \frac{9}{4}n^2(5n+1)(n+1) - (n+1)^2$$

$$= \frac{1}{4}(n+1)\left[\ 9n^2(5n+1) - 4(n+1)\ \right]$$

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9. [YIJC/2020/Promo/13]

A mask production factory produces 5000 reusable masks in the first week. In each subsequent week, the number of reusable masks produced is 200 more than the previous week.

(i) Find the number of reusable masks produced in the 10th week. [2]

(ii) Find the minimum number of weeks required by the factory to produce a total number of at least 100 000 reusable masks. [3]

Due to a certain flu pandemic, the demand for reusable masks has increased drastically. If the demand for reusable masks in the preceding week is M, it would become $100 + aM$ this week.

(iii) Given that the demand for reusable masks in the first week is 500, show that the demand for reusable masks in the Nth week can be written as
$$\frac{100}{a-1}\left(5a^N - 4a^{N-1} - 1\right).$$ [3]

It is now given that $a = 1.15$.

To meet the growing demand of reusable masks, the factory adjusts its production capacity such that the number of reusable masks produced from the 10th week onwards is k more than the previous week.

(iv) Find the least value of k required by the factory in order to meet the total demand for the first 25 weeks. [4]

Solution:

13(i)	AP : first term = 5000, common difference = 200 $u_{10} = 5000 + (10-1)200$ $= 6800$

(ii)

$S_n \geq 100000$

$\frac{n}{2}\left[2(5000)+(n-1)200\right] \geq 100000$

$5000n+100n^2-100n \geq 100000$

$n^2+49n-1000 \geq 0$

Method 1

$n \leq -64.5$ or $n \geq 15.5$

Hence, minimum number of weeks is 16.

Method 2

Using GC,

n	$n^2+49n-1000$
15	– 40 (<0)
16	40 (>0)
17	122 (>0)

Hence, minimum number of weeks is 16.

(iii)

Week	Demand
1	500
2	100 + 500a
3	$100+\left(100+500a\right)a$ $=100+100a+500a^2$
4	$100+\left(100+100a+500a^2\right)a$ $=100+100a+100a^2+500a^3$
$\vdots$	$\vdots$
N	$100+100a+\cdots+100a^{N-2}+500a^{N-1}$

The first $N-1$ terms follows a GP with first term = 100 and common ratio = a.

Demand for reusable masks in week N

$=100\left(\frac{a^{N-1}-1}{a-1}\right)+500a^{N-1}$

$=\frac{100}{a-1}\left(a^{N-1}-1+5a^{N-1}(a-1)\right)$

$=\frac{100}{a-1}\left(5a^N-4a^{N-1}-1\right)$

(iv)

Total number of reusable masks produced in the first 25 weeks

$=\frac{8}{2}\left(2(5000)+(8-1)200\right)+\frac{17}{2}\left(2(6800-200)+(17-1)k\right)$

$=45600+112200+136k$

$=157800+136k$

	Total demand of reusable masks in the first 25 weeks $=\sum_{N=1}^{25}\left[\frac{100}{1.15-1}\left(5\left(1.15^{N}\right)-4\left(1.15^{N-1}\right)-1\right)\right]$ $=231591.8537$ $157800+136k\geq 231591.8537$ For $k\geq 542.59$ Hence, least value of k = 543

10. [NYJC/2020/Promo/6]

(i) Express $\dfrac{1}{2r(r+1)}$ in partial fractions and hence find $\displaystyle\sum_{r=1}^{n}\frac{1}{2r(r+1)}$ in terms of n. [3]

(ii) By using **(i)**, express $\displaystyle\sum_{r=6}^{2N}\frac{1}{r(r-1)}$ in terms of N. [4]

(iii) Deduce the value of $\displaystyle\sum_{r=6}^{\infty}\frac{1}{r(r-1)}$. [1]

Solution:

(i)	$\dfrac{1}{2r(r+1)}=\dfrac{1}{2r}-\dfrac{1}{2(r+1)}$ $\displaystyle\sum_{r=1}^{n}\frac{1}{2r(r+1)}=\sum_{r=1}^{n}\left(\frac{1}{2r}-\frac{1}{2(r+1)}\right)$ $=\dfrac{1}{2\cdot 1}-\dfrac{1}{2\cdot 2}$ $+\dfrac{1}{2\cdot 2}-\dfrac{1}{2\cdot 3}$ $+\ldots$ $+\dfrac{1}{2n}-\dfrac{1}{2(n+1)}$ $=\dfrac{1}{2}-\dfrac{1}{2(n+1)}$
(ii)	$\displaystyle\sum_{r=6}^{2N}\frac{1}{r(r-1)}=\sum_{k=5}^{2N-1}\frac{1}{k(k+1)}=2\sum_{k=5}^{2N-1}\frac{1}{2k(k+1)}$ $\displaystyle=2\left[\sum_{k=1}^{2N-1}\frac{1}{2k(k+1)}-\sum_{k=1}^{4}\frac{1}{2k(k+1)}\right]$ $=2\left[\dfrac{1}{2}-\dfrac{1}{4N}-\left(\dfrac{1}{2}-\dfrac{1}{2\cdot 5}\right)\right]$ $=\dfrac{1}{5}-\dfrac{1}{2N}$
(iii)	$\displaystyle\sum_{r=6}^{\infty}\frac{1}{r(r-1)}=\frac{1}{5}$

11. MI Promo 2021/PU2/01/Q10

Joe is an amateur cyclist and he wishes to cycle 115 km around Singapore to celebrate his birthday. In order to prepare adequately for the long ride, he has thought of two plans: a training plan as well as an execution plan for the actual long ride itself.

(a) After conducting some research on training, Joe finds out that he needs to increase the distance he cycles weekly gradually. He plans to cycle 30 km in the first week and then increase his weekly distance by a fixed x km every week thereafter. Joe has 24 weeks to train before his actual long ride.

(i) Find the least value of x such that the distance that Joe cycles in his last week of training is at least 85% of the distance of his actual long ride. [2]

(ii) Suggest a possible reason why Joe decides to set the target for his weekly increment to be higher than the value found in part **(i)**. [1]

(iii) Using the value of x found in part **(i)**, find the total distance that Joe cycles during the 24 weeks of training, giving your answer to the nearest kilometre. [2]

Joe considers an alternative training plan in which he starts with a distance of 30 km in the first week. However, he will only increase his weekly distance by a fixed y km every alternate week. Thus in Weeks 1 and 2 he will cycle 30 km each, in Weeks 3 and 4 he will cycle $(30+y)$ km each, and so on. Joe still wants to cycle the same distance in his last week of training for both training plans.

(iv) Find y in terms of x. [2]

(b) Next, Joe plans how he will execute the actual 115 km ride. He predicts that due to fatigue, the distance he cycles for every half hour will be 94% of the distance he cycled in the preceding half hour.

(i) If Joe cycles 6.5km in the first half hour, explain why he will not reach his target distance of 115 km. [2]

Joe intends to start cycling at 7 am in the morning. He factors in 2 meal breaks during the day, with a duration of 1.5 hours each and decides to cycle 9 km in the first half hour. Assume that the distance cycled every subsequent half hour follows Joe's prediction and that the meal breaks do not affect the distance he cycles.

(ii) What time does Joe expect to finish cycling? Give your answer to the nearest minute. [4]

Qn	Solution
(a)(i)	Cycling distance for Week 24 $=30+(24-1)x$ $=30+23x$ $30+23x\ge 0.85\times 115$ $30+23x\ge 97.75$
	<u>Method 1: Algebraic</u> $23x\ge 67.75$ $x\ge 2.9457$ (5 s.f.) $x\ge 2.95$ (3 s.f.)

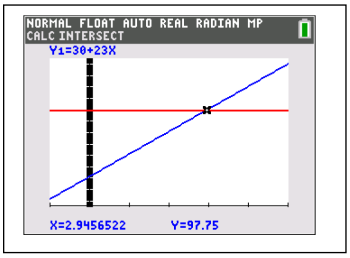
Least value of x is 2.95.

Method 2: GC

From GC, $x \geq 2.9457$ (5 s.f.)

$x \geq 2.95$ (3 s.f.)

Least value of x is 2.95.



(a)(ii)

In case of unforeseen circumstances such as illness, work or commitments that will prevent Joe from meeting the weekly distance for certain weeks, he has a better chance of still being able to reach a distance of at least 85% of his actual long ride for the last week.

OR

To give a buffer for himself to make slight adjustments along the way and still be able to reach a distance of at least 85% of his actual long ride for the last week.

(a)(iii)

Method 1

$$\text{Total distance} = \frac{24}{2}\left[2(30)+(24-1)\times 2.9457\right]$$
$$= 1533.0132$$
$$= 1533 \text{ (nearest kilometre)}$$

Method 2

$$\text{Cycling distance for Week 24} = 30+23(2.9457)$$
$$= 97.7511$$

$$\text{Total distance} = \frac{24}{2}\left[30+97.7511\right]$$
$$= 1533.0132$$
$$= 1533 \text{ (nearest kilometre)}$$

(a)(iv)

Odd-numbered terms form an AP with first term 30, common difference y and consisting of 12 terms.

$$\text{Cycling distance for Week 24} = 30+(12-1)y$$
$$= 30+11y$$

$$\therefore 30+11y = 30+23x$$

$$\Rightarrow y = \frac{23}{11}x$$

(b)(i)

Distance covered every half hour forms a GP with first term $a = 6.5$ and common ratio $r = 0.94$.

Since $|r|<1$, Sum to infinity $=\dfrac{6.5}{1-0.94}$

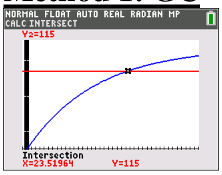
$=108.33$ (5 s.f.)

Joe will not be able to complete 115 km eventually / no matter how much time he has.

(b)(ii) Sum of distance after n half hours $=115$

$$\frac{9\left(1-0.94^n\right)}{1-0.94}=115$$

<u>Method 1: GC</u>



By GC, $n=23.520\ (5\text{ s.f.})$

<u>Method 2: Algebraic</u>

$$\frac{9\left(1-0.94^n\right)}{1-0.94}=115$$

$$0.94^n=1-\frac{115\times 0.06}{9}$$

$$n=\frac{\ln\frac{7}{30}}{\ln 0.94}$$

$n=23.520\ (5\text{ s.f.})$

23.520 half hours $=23.520\times 0.5$ hours

$=11.76$ hours

$=11$ hours and 0.76×60 minutes

$=11$ hours 45.6 minutes

11 hours 45.6 minutes $+\,2\times 1.5$ hours

$=14$ hours 45.6 minutes

Joe expects to finish cycling at 9.46 pm.