**National Junior College**
**2016 – 2017 H2 Further Mathematics**
**Topic F8: Further Complex Numbers**

*Key Questions to Answer*:

**Complex Numbers in Polar & Exponential Form**

1. How do we interpret geometrically the effects of conjugating a complex number, and adding/subtracting/multiplying/dividing two complex numbers?
   - *What is the geometrical effect of multiplication by* i?

2. What is de Moivre's Theorem?
   - *How do we apply de Moivre's Theorem to find the powers and nth roots of a complex number, and to derive trigonometric identities?*

**Complex Loci**

How do we sketch the loci of simple equations and inequalities involving a complex variable in the Argand diagram such as

$$|z-c| \le r,$$
$$|z-a| = |z-b| \text{ and}$$
$$\arg(z-a) = \alpha \ ?$$

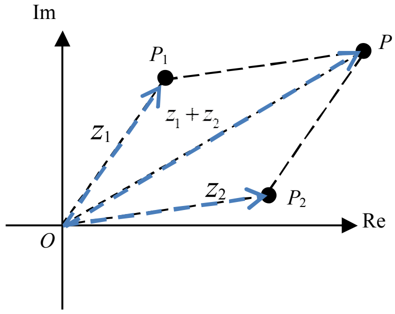
## §1 Geometrical Effects of Operations on Complex Numbers

### 1.1 Addition and Subtraction of Complex Numbers

Recall that complex numbers can alternatively be represented by position vectors, i.e., a complex number $z = x + \mathrm{i}y$ can be represented as a position vector $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$ in the *x*-*y* plane.

Techniques and operations used in coordinate geometry and vectors can be applied to adding and subtracting complex numbers, corresponding to the **parallelogram law of vector addition and subtraction** respectively.

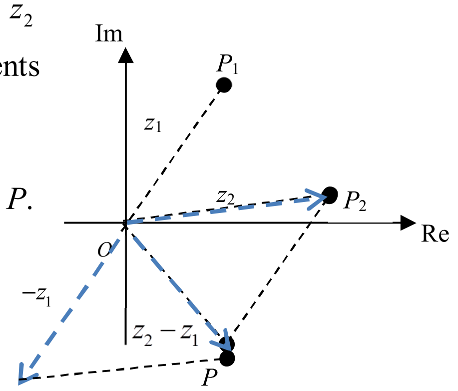
Let $z_1 = a + \mathrm{i}b$ be represented by $P_1$.
$z_2 = c + \mathrm{i}d$ be represented by $P_2$.
$z = z_1 + z_2 = (a + c) + \mathrm{i}(b + d)$ be represented by $P$.




In terms of vectors,

$$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{OP_2}.$$

The subtraction $z_2 - z_1$ is the same as the addition of $z_2$ and $z_1$. Thus, let $\overrightarrow{P_2P}$ represents $-z_1$. Then $\overrightarrow{OP}$ represents $z_2 - z_1$.




$$z = z_2 - z_1 = (c - a) + \mathrm{i}\,(d - b) \text{ is represented by } P.$$

In terms of vectors,

$$\overrightarrow{OP} = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}.$$

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What are some possible limitations of considering complex numbers as vectors on the Argand Diagram?


*Every complex number can only be represented by position vectors and not displacement vectors. So for example when finding the difference between two complex numbers (e.g. $z_2 - z_1$), the vector representing the complex number $z_2 - z_1$ must be translated to start from the origin first before the real and imaginary parts of the complex number can be obtained. Finding differences of vectors on the other hand, has no such restriction.*

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## 1.2 Conjugation of Complex Numbers

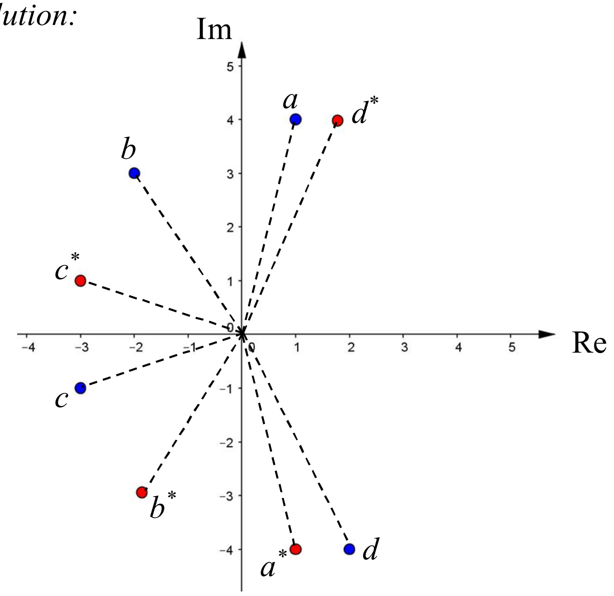
### Example 1.2.1

Represent the following complex numbers and their corresponding conjugates in an Argand diagram:

$$a = 1 + 4\mathrm{i},\ b = -2 + 3\mathrm{i},\ c = -3 - \mathrm{i},\ d = 2 - 4\mathrm{i}$$

What is the geometrical effect of conjugating a complex number?

*Solution:*




From the Argand diagram, the conjugate is obtained by reflecting the corresponding complex number about the real axis.

*Can you give an explanation of the effect by considering the modulus and argument of the conjugate?*

## 1.3 Multiplication and Division of Complex Numbers

*We first consider the special case of multiplication of a complex number by* i.

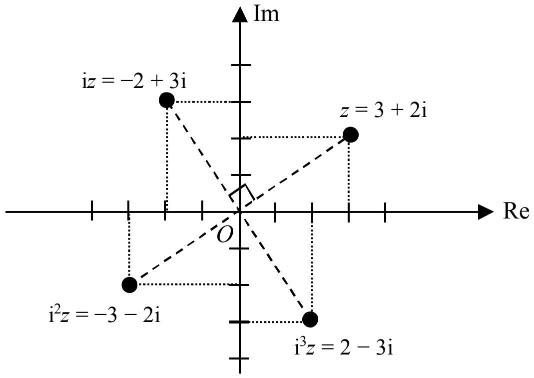
Let $z = 3 + 2\text{i}$.

$\text{i}z = \text{i}(3 + 2\text{i}) = -2 + 3\text{i}$

$\text{i}^2 z = -(3 + 2\text{i}) = -3 - 2\text{i} = -z$

$\text{i}^3 z = -\text{i}(3 + 2\text{i}) = 2 - 3\text{i}$

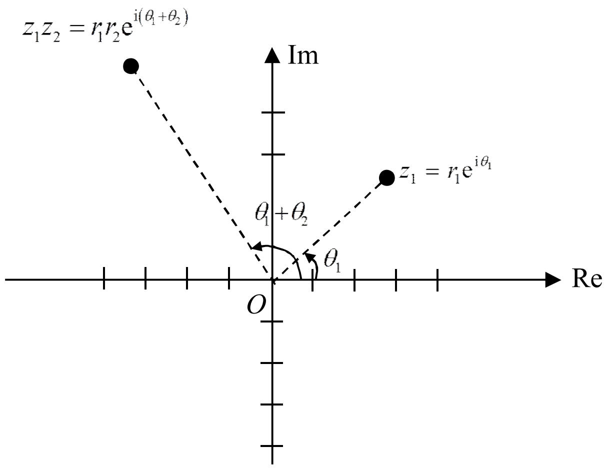
$\text{i}^4 z = 3 + 2\text{i} = z$




Using the same scale on both axes on the Argand diagram, we observe that if a point $P$ represents a complex number $z$, then the point representing $\text{i}z$ is obtained by **rotating** $OP$ **90° anti-clockwise about the origin**.

Similar geometrical effect applies to the points representing $\text{i}^2 z, \text{i}^3 z$ and $\text{i}^4 z$.

In general, let us consider multiplying a complex number $z_2 = r_2 \text{e}^{\text{i}\theta_2}$ to another complex number $z_1 = r_1 \text{e}^{\text{i}\theta_1}$. The result is $z_1 \cdot z_2 = r_1 r_2 \text{e}^{\text{i}(\theta_1 + \theta_2)}$.
Geometrically, the length of $z_1 \cdot z_2$ equals to the product of the lengths of $z_1$ and $z_2$ i.e. $|z_1 \cdot z_2| = r_1 r_2$, and the argument of $z_1 \cdot z_2$ equals to the sum of the arguments of $z_1$ and $z_2$ i.e. $\arg(z_1 \cdot z_2) = \theta_1 + \theta_2$. It is observed that $z_1$ has been **rotated** by $\theta_2$ radians anti-clockwise (assuming $\theta_2$ is positive) and its length scaled by factor $r_2$ to give $z_1 \cdot z_2$.




Now, let us consider dividing a complex number $z_1 = r_1 \text{e}^{\text{i}\theta_1}$ by another complex number $z_2 = r_2 \text{e}^{\text{i}\theta_2}$.
The result is $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \text{e}^{\text{i}(\theta_1 - \theta_2)}$.
How would you illustrate this division geometrically?

- $z_1$ has been **rotated** by $\theta_2$ radians **clockwise** (assuming $\theta_2$ is positive) and its length scaled by factor $\dfrac{1}{r_2}$ to give $\dfrac{z_1}{z_2}$.

## §2 Powers and *n*th Roots of a Complex Number

### 2.1 de Moivre's Theorem

If $z$ is a complex number with $|z| = r$ and $\arg(z) = \theta$, then
$z^n = \left[ r\left(\cos\theta + \mathrm{i}\sin\theta\right)\right]^n = r^n\left(\cos n\theta + \mathrm{i}\sin n\theta\right)$ for all ***real*** values of $n$.


Try using mathematical induction to prove the de Moivre's Theorem for positive integer values of n.

**Example 2.1.1 (Power of a Complex Number)**

Find the exact value of $\left(1+\sqrt{3}\mathrm{i}\right)^6$.

*Solution:*

$$\begin{aligned}\left(1+\sqrt{3}\mathrm{i}\right)^6 &= \left[2\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)\right]^6 \\ &= 2^6\left(\cos 2\pi+\mathrm{i}\sin 2\pi\right)\left(\text{by De Moivre's Theorem}\right) \\ &= 64\end{aligned}$$


What other methods can we apply to answer the above question?

**Example 2.1.2**

Prove that $1+(\cos\theta+\mathrm{i}\sin\theta)^4 = 2\cos 2\theta\left(\cos 2\theta+\mathrm{i}\sin 2\theta\right)$, and that, provided the denominator is not zero, $\dfrac{1+(\cos\theta+\mathrm{i}\sin\theta)^4}{1+(\cos\theta-\mathrm{i}\sin\theta)^4} = \cos 4\theta+\mathrm{i}\sin 4\theta$.

*Solution:*

$$\begin{aligned}1+(\cos\theta+\mathrm{i}\sin\theta)^4 &= 1+\cos 4\theta+\mathrm{i}\sin 4\theta \\ &= 1+\left(2\cos^2 2\theta-1\right)+\mathrm{i}\left(2\sin 2\theta\cos 2\theta\right) \\ &= 2\cos 2\theta\left(\cos 2\theta+\mathrm{i}\sin 2\theta\right).\end{aligned}$$

$$\begin{aligned}\frac{1+(\cos\theta+\mathrm{i}\sin\theta)^4}{1+(\cos\theta-\mathrm{i}\sin\theta)^4} &= \frac{2\cos 2\theta\left(\cos 2\theta+\mathrm{i}\sin 2\theta\right)}{2\cos 2\theta\left(\cos 2\theta-\mathrm{i}\sin 2\theta\right)} \\ &= \frac{\cos 2\theta+\mathrm{i}\sin 2\theta}{\cos 2\theta-\mathrm{i}\sin 2\theta}\times\frac{\cos 2\theta+\mathrm{i}\sin 2\theta}{\cos 2\theta+\mathrm{i}\sin 2\theta} \\ &= \frac{\left(\cos 2\theta+\mathrm{i}\sin 2\theta\right)^2}{\cos^2 2\theta+\sin^2 2\theta} \\ &= \cos 4\theta+\mathrm{i}\sin 4\theta\left(\text{since } \cos^2 2\theta+\sin^2 2\theta=1\right).\end{aligned}$$

**Example 2.1.3**

Using de Moivre's Theorem, prove that

$$\sin 4\theta - \sin 8\theta + \sin 12\theta - ..... + \sin 36\theta = \frac{\sin 20\theta \cos 18\theta}{\cos 2\theta},$$

provided $\cos 2\theta \neq 0$.

*Solution:*

$$\sin 4\theta - \sin 8\theta + \sin 12\theta - ..... + \sin 36\theta$$
$$= \mathrm{Im}\left[\mathrm{e}^{\mathrm{i}(4\theta)} - \mathrm{e}^{\mathrm{i}(8\theta)} + \mathrm{e}^{\mathrm{i}(12\theta)} - ..... + \mathrm{e}^{\mathrm{i}(36\theta)}\right]$$
$$= \mathrm{Im}\left\{\frac{\mathrm{e}^{\mathrm{i}(4\theta)}\left[\left(-\mathrm{e}^{\mathrm{i}(4\theta)}\right)^9 - 1\right]}{\left(-\mathrm{e}^{\mathrm{i}(4\theta)}\right) - 1}\right\}$$
$$= \mathrm{Im}\left\{\frac{\mathrm{e}^{\mathrm{i}(4\theta)}\left[\mathrm{e}^{\mathrm{i}(36\theta)} + 1\right]}{\mathrm{e}^{\mathrm{i}(4\theta)} + 1}\right\}$$
$$= \mathrm{Im}\left\{\frac{\mathrm{e}^{\mathrm{i}(4\theta)} \cdot \mathrm{e}^{\mathrm{i}(18\theta)}\left[\mathrm{e}^{\mathrm{i}(18\theta)} + \mathrm{e}^{-\mathrm{i}(18\theta)}\right]}{\mathrm{e}^{\mathrm{i}(2\theta)}\left[\mathrm{e}^{\mathrm{i}(2\theta)} + \mathrm{e}^{-\mathrm{i}(2\theta)}\right]}\right\}$$
$$= \mathrm{Im}\left[\frac{\mathrm{e}^{\mathrm{i}(20\theta)} \cdot 2\cos 18\theta}{2\cos 2\theta}\right] = \mathrm{Im}\left[\frac{(\cos 20\theta + \mathrm{i}\sin 20\theta) \cdot 2\cos 18\theta}{2\cos 2\theta}\right] = \frac{\sin 20\theta \cos 18\theta}{\cos 2\theta}.$$

## 2.2 $n^{\text{th}}$ Roots of a Complex Number

**Learning Task I**

**(i)** Write down the exponential form of $-32$.

**(ii)** State the number of complex roots of the equation $z^5 + 32 = 0$.

**(iii)** Hence solve the equation $z^5 + 32 = 0$.

Below are the steps to finding the roots of the equation $z^n = p$.

Suppose $p = x + \mathrm{i}y$. Then $z^n = p = x + \mathrm{i}y$, where $r = |x + \mathrm{i}y|$ and $\theta = \arg(x + \mathrm{i}y)$.

**Step 1**: Convert *p* to the exponential form.

$$z^n = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}$$

**Step 2**: Add $2k\pi$ to the argument.

$$z^n = r\mathrm{e}^{\mathrm{i}(\theta + 2k\pi)}, \ k \in \mathbb{Z}$$

**Step 3**: Raise both sides by power $\frac{1}{n}$.

$$z = r^{\frac{1}{n}}\mathrm{e}^{\mathrm{i}\frac{(\theta+2k\pi)}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + \mathrm{i}\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

**Step 4**: Obtain the distinct roots by substituting suitable integer values of *k*.

The solutions to $z^n = p$ are known as the **$n^{\text{th}}$ roots of the complex number *p***.

**Note:** When *p* = 1, we obtain the special case $z^n = 1$.
Such solutions are called the **$n^{\text{th}}$ roots of unity**.

Let us now reflect on the steps to finding the roots of $z^n = p$.

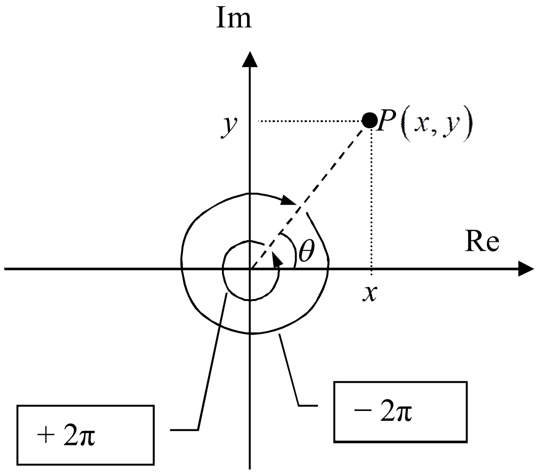

1. *In **Step 1**, why do we want to convert p to exponential form?*

**Ans**: Notice that in Step 3, we need to raise both sides to power $\frac{1}{n}$. It is more efficient to work in exponential form when dealing with powers of functions. (What about polar form?)

2. *In **Step 2**, why do we add 2kπ to the argument?*

**Ans**: Suppose we have $z^3 = \mathrm{i} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$. Then we raise both sides to power $\frac{1}{3}$. We will have $z = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ as the answer. However, according to the Fundamental Theorem of Algebra, the equation should have three roots.

Recall that the argument of a complex number is not unique, although we always write down the argument within the principal range.




In an Argand diagram, we can obtain the same complex number by rotating the argument by 2π, 4π, 6π, etc. (integer multiples of 2π) either clockwise or anti-clockwise about the origin.

In general, we can add/subtract integer multiples of 2π to the argument of a complex number without changing its final value.

This is actually a concept we have acquired when we solve trigonometric problems, e.g. $\sin 2\theta = \frac{1}{2}$, for $0 \le \theta \le 2\pi$ .

3. In **Step 3**, we are actually applying de Moivre's Theorem when we raise the power by $\frac{1}{n}$.

4. *In **Step 4**, how many values of k do we need to obtain all the distinct roots?*

**Ans:** The number of values of $k$ to take, depends on the number of distinct roots of the equation. In general, if the equation has $n$ distinct roots, we will take $n$ **consecutive** integer values of $k$. The choice of values of $k$ determines whether we obtain roots with arguments within the principal range.


***Question**: Can we interchange Steps 2 and 3?*

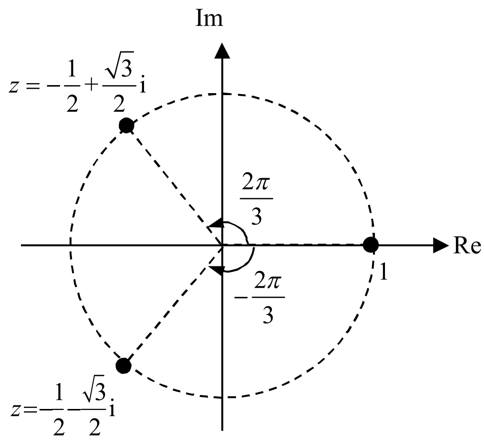
**Example 2.2.1**

Find the cube roots of unity. Display the roots on an Argand diagram.

*Solution:*

$z^3 = 1 \Rightarrow z^3 = e^{i2k\pi}$, $k \in \mathbb{Z}$ .

$z = e^{i\frac{2k\pi}{3}}$, $k = 0, \pm 1$




Taking $k = -1, 0, 1$ to find the *distinct* solutions, we get the following:

$k = -1$:

$$z = e^{i\left(-\frac{2\pi}{3}\right)} = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$k = 0$: $z = 1$

$k = 1$: $z = e^{i\frac{2\pi}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

**Note:**

1. While we could have chosen any 3 consecutive integer values of $k$ (0, 1, 2 or −3,−2, −1), we often choose the ones which result in principal arguments for the roots. If the argument is not in the principal range, we can easily convert to the principal argument. (Do you remember how to do so?)

   Example: $k=2,\ z=\mathrm{e}^{\mathrm{i}\frac{4\pi}{3}}=\cos\frac{4\pi}{3}+\mathrm{i}\sin\frac{4\pi}{3}$

   In terms of principal argument, $z=\cos\left(-\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{2\pi}{3}\right)=-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$

   This is equivalent to the case when we take $k=-1$.

2. Observe that the roots of the equation lie along the circumference of a circle of radius 1 unit and the angle between each root on the Argand diagram is equal.

**Example 2.2.2**

Find, in exact form, all complex numbers $z$ such that $z^5=\frac{5+\mathrm{i}}{2+3\mathrm{i}}$. Display the roots on an Argand diagram.

*Solution:*

$$z^5=\frac{5+\mathrm{i}}{2+3\mathrm{i}}=1-\mathrm{i}=\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}+2k\pi\right)}\Rightarrow z=\sqrt{2}^{\frac{1}{5}}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{20}+\frac{2k\pi}{5}\right)},\ k\in\mathbb{Z}$$

We take $k=-2,-1,0,1,2$ (or any 5 consecutive integers) to get *distinct* solutions.

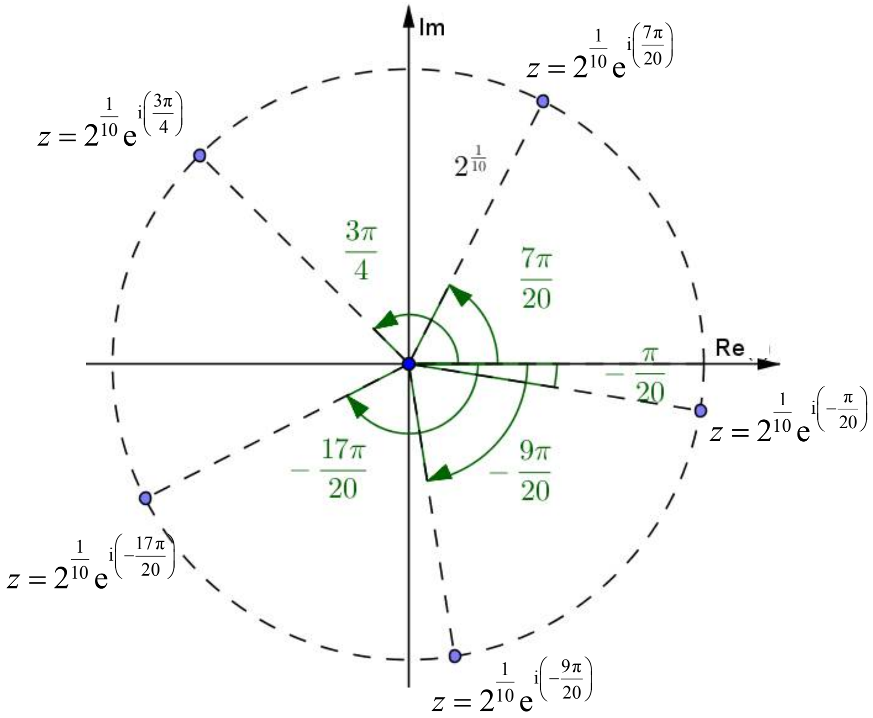
$$k=-2:\ z=\sqrt{2}^{\frac{1}{5}}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{20}+\frac{2(-2)\pi}{5}\right)}=2^{\frac{1}{10}}\mathrm{e}^{\mathrm{i}\left(-\frac{17\pi}{20}\right)}=2^{\frac{1}{10}}\left[\cos\left(-\frac{17\pi}{20}\right)+\mathrm{i}\sin\left(-\frac{17\pi}{20}\right)\right]$$

$$k=-1:\ z=\sqrt{2}^{\frac{1}{5}}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{20}+\frac{2(-1)\pi}{5}\right)}=2^{\frac{1}{10}}\mathrm{e}^{\mathrm{i}\left(-\frac{9\pi}{20}\right)}=2^{\frac{1}{10}}\left[\cos\left(-\frac{9\pi}{20}\right)+\mathrm{i}\sin\left(-\frac{9\pi}{20}\right)\right]$$

$$k=0:\ z=\sqrt{2}^{\frac{1}{5}}\mathrm{e}^{\mathrm{i}\left(\frac{2(0)\pi}{5}-\frac{\pi}{20}\right)}=2^{\frac{1}{10}}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{20}\right)}=2^{\frac{1}{10}}\left[\cos\left(-\frac{\pi}{20}\right)+\mathrm{i}\sin\left(-\frac{\pi}{20}\right)\right]$$

$$k=1:\ z=\sqrt{2}^{\frac{1}{5}}\mathrm{e}^{\mathrm{i}\left(\frac{2(1)\pi}{5}-\frac{\pi}{20}\right)}=2^{\frac{1}{10}}\mathrm{e}^{\mathrm{i}\left(\frac{7\pi}{20}\right)}=2^{\frac{1}{10}}\left[\cos\left(\frac{7\pi}{20}\right)+\mathrm{i}\sin\left(\frac{7\pi}{20}\right)\right]$$

$$k=2:\ z=\sqrt{2}^{\frac{1}{5}}\mathrm{e}^{\mathrm{i}\left(\frac{2(2)\pi}{5}-\frac{\pi}{20}\right)}=2^{\frac{1}{10}}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}=2^{\frac{1}{10}}\left[\cos\left(\frac{3\pi}{4}\right)+\mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right]$$




**Note:** The roots lie along the circumference of a circle of radius $2^{\frac{1}{10}}$ units, and the angle between each root on the Argand diagram is equal.

**Example 2.2.3**

Write down all the $8^{\text{th}}$ roots of unity.
Show that $\left(z-\mathrm{e}^{\mathrm{i}\theta}\right)\left(z-\mathrm{e}^{-\mathrm{i}\theta}\right)\equiv z^2-\left(2\cos\theta\right)z+1$.
Hence express $z^8-1$ as the product of two linear factors and three quadratic factors, where all the coefficients are real and expressed in a non-trigonometrical form.

*Solution:*

$z^8=1\Rightarrow z^8=\mathrm{e}^{\mathrm{i}2k\pi},\ k\in\mathbb{Z}$.

$z=\mathrm{e}^{\mathrm{i}\frac{2k\pi}{8}}=\mathrm{e}^{\mathrm{i}\frac{k\pi}{4}},\ k=0,\pm1,\pm2,\pm3,4$

$$\begin{aligned}\left(z-\mathrm{e}^{\mathrm{i}\theta}\right)\left(z-\mathrm{e}^{-\mathrm{i}\theta}\right)&\equiv z^2-(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta})z+1\\&\equiv z^2-\left[\cos\theta+\mathrm{i}\sin\theta+\cos(-\theta)+\mathrm{i}\sin\left(-\theta\right)\right]z+1\\&\equiv z^2-\left(2\cos\theta\right)z+1\end{aligned}$$

For $k=\pm1$, $\left(z-\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)\left(z-\mathrm{e}^{\mathrm{i}\frac{-\pi}{4}}\right)=z^2-\left(2\cos\frac{\pi}{4}\right)z+1=z^2-\sqrt{2}z+1$

For $k=\pm2$, $\left(z-\mathrm{e}^{\mathrm{i}\frac{2\pi}{4}}\right)\left(z-\mathrm{e}^{\mathrm{i}\frac{-2\pi}{4}}\right)=z^2-\left(2\cos\frac{2\pi}{4}\right)z+1=z^2+1$

For $k=\pm3$, $\left(z-\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}\right)\left(z-\mathrm{e}^{\mathrm{i}\frac{-3\pi}{4}}\right)=z^2-\left(2\cos\frac{3\pi}{4}\right)z+1=z^2+\sqrt{2}z+1$

Hence,

$$\begin{aligned}&z^8-1\\&=\left(z-\mathrm{e}^{\mathrm{i}(0)}\right)\left(z-\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)\left(z-\mathrm{e}^{\mathrm{i}\frac{-\pi}{4}}\right)\left(z-\mathrm{e}^{\mathrm{i}\frac{2\pi}{4}}\right)\left(z-\mathrm{e}^{\mathrm{i}\frac{-2\pi}{4}}\right)\left(z-\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}\right)\left(z-\mathrm{e}^{\mathrm{i}\frac{-3\pi}{4}}\right)\left(z-\mathrm{e}^{\mathrm{i}(\pi)}\right)\\&=\left(z-1\right)\left(z^2-\sqrt{2}z+1\right)\left(z^2+1\right)\left(z^2+\sqrt{2}z+1\right)\left(z+1\right)\end{aligned}$$

## §3 Loci in the Complex Plane

In the Argand diagram, we let $z = x + \mathrm{i}y$ be represented by point $P(x, y)$. If the values of $x$ and $y$ vary according to some given condition, the set of all possible points in the Argand diagram will describe some line or curve, known as the **locus** of $z$.

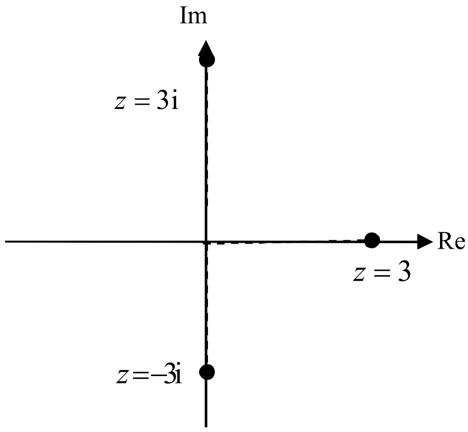
### 3.1 Locus of $|z - a| = r$

For a start, let us consider the following questions:

What is the distance between the origin and each of the points representing 3, 3i and $-3\mathrm{i}$?

What other points also have the same distance from the origin?

What shape does such collection of points result in?

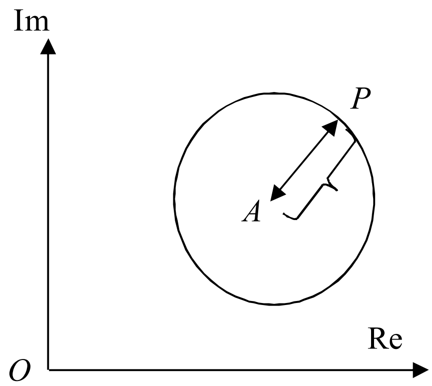



***So what does $|z - a| = r$ describe?***

Recall: The **modulus** of the difference of two complex numbers $|z_1 - z_2|$ is the distance between the two points representing $z_1$ and $z_2$.

Let variable point $P$ and fixed point $A$ represent the complex variable $z$ and fixed complex number $a$ respectively.

If $|z - a| = r$ i.e. the distance between $P$ and $A$ is equal to $r$ (a fixed real number), then $P$ lies in a circle centred at the fixed point $A$ and with radius $r$.




r

| |
|---|
| **The locus of $z$ expressed in the form: $|z - a| = r$ is** <br> **a circle with centre at the fixed point $A$ and radius $r$.** |

Let $z = x + \mathrm{i}y$ and $a = \alpha + \mathrm{i}\beta$,

Then $|(x-\alpha) + (y-\beta)\mathrm{i}| = r \Rightarrow \sqrt{(x-\alpha)^2 + (y-\beta)^2} = r \Rightarrow (x-\alpha)^2 + (y-\beta)^2 = r^2$

which is the Cartesian equation of a circle centred at $(\alpha, \beta)$ with radius $r$.

**Example 3.1.1**

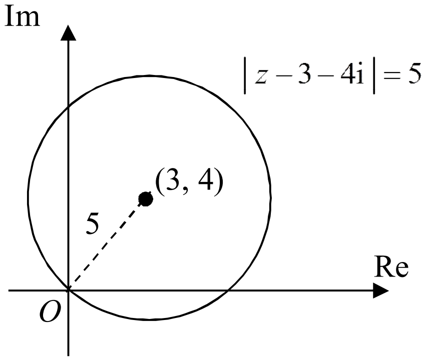
Describe and sketch the locus of $z$ for the following:

**(a)** $|z-3-4\mathrm{i}|=5$ **(b)** $|2-\mathrm{i}-2z|=1$

*Solution:*

**(a)** $|z-3-4\mathrm{i}|=|z-(3+4\mathrm{i})|=5$

The locus of $z$ is a circle centred at (3, 4) with radius 5.





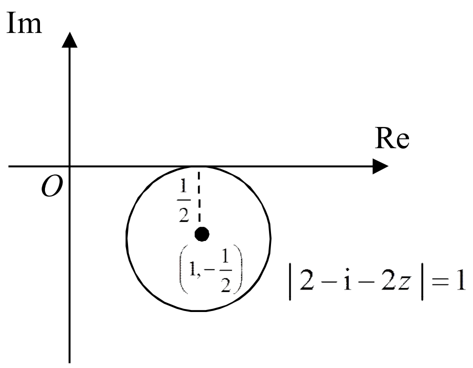
How do we know (without drawing to scale) if the circle passes through the origin?

*Method 1: By substituting z = 0 (the origin) into the equation, we have* $|0-3-4\mathrm{i}|=5$*. Since the equation is satisfied, the origin lies on the circle.*

*Method 2: By using Pythagoras Theorem to find the distance from the origin to the centre of the circle, we notice that the distance* $\left(\sqrt{3^2+4^2}=5\right)$ *is equal to the radius of the circle. Thus, the circle passes through the origin. (What if the calculated distance is less than or greater than the radius?)*

*In general, both methods could be used to check if a circle passes through a point* $(a,b)$ *representing the complex number* $z=a+b\mathrm{i}$.

**(b)** $|2-\mathrm{i}-2z|=\left|-2\left(z-1+\frac{\mathrm{i}}{2}\right)\right|=\left|-2\left(z-\left(1-\frac{\mathrm{i}}{2}\right)\right)\right|=2\left|z-\left(1-\frac{\mathrm{i}}{2}\right)\right|=1\Rightarrow\left|z-\left(1-\frac{\mathrm{i}}{2}\right)\right|=\frac{1}{2}$ The locus of $z$ is a circle centred at $\left(1,-\frac{1}{2}\right)$ with radius $\frac{1}{2}$.




How do we know (without drawing to scale) if the circle touches the positive real axis?

*Method 1: By substituting* $z = 1$ *into the equation, we have* $|2-\mathrm{i}-2|=|-\mathrm{i}|=1$. *Since the equation is satisfied, the circle passes through the point* $(1,0)$.

*Method 2: By observing that the perpendicular distance from the centre of the circle to the real axis is* $\frac{1}{2}$ *which is the radius of the circle.*

$|z-a|=r$ represents the set of points on the circumference of a circle with centre at the fixed point $A$ and radius $r$. What happens if we change '=' to an inequality sign?

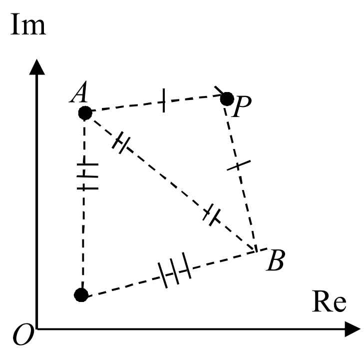

### 3.2 Locus of $|z-a|=|z-b|$

For a start, we consider the following questions:

Given fixed points $A$ and $B$ representing fixed complex numbers $a$ and $b$ respectively, what are some points that are equidistant from points $A$ and $B$?

What other points also have equal distance from points $A$ and $B$?

What shape does such collection of points result in?



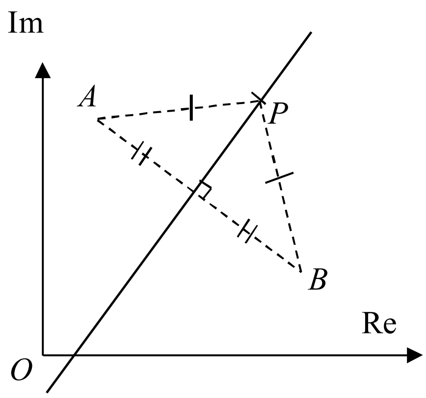

***So what does*** $|z-a|=|z-b|$ ***describe?***

Let variable point $P$ and fixed points $A$ and $B$ represent the complex variable $z$ and fixed complex numbers $a$ and $b$ respectively.

We know that

$$|z-a|=|z-b|$$

means the distance between $P$ and $A$ and that of $P$ and $B$ are the same. Hence, $P$ lies on a perpendicular bisector of line joining $A$ and $B$.




**The locus of $z$ expressed in the form $|z-a|=|z-b|$ is the perpendicular bisector of the line joining the fixed points $A$ and $B$.**

**Example 3.2.1**

Describe and sketch the locus of $z$ for the following:

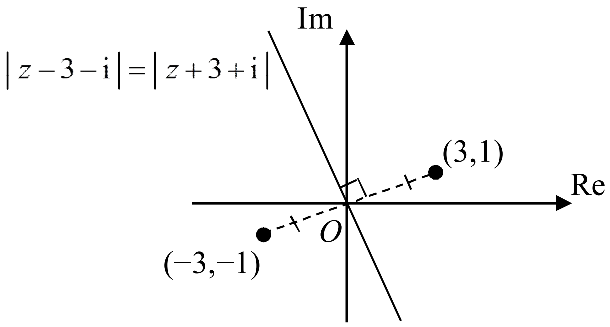
**(a)** $|z-3-\mathrm{i}|=|z+3+\mathrm{i}|$ **(b)** $\left|\frac{\mathrm{i}+z}{\mathrm{i}-z}\right|=1$.

*Solution:*

**(a)**

$|z-3-\mathrm{i}|=|z+3+\mathrm{i}| \Rightarrow |z-(3+\mathrm{i})|=|z-(-3-\mathrm{i})|$

The locus of $z$ is the perpendicular bisector of the line joining (3,1) and (−3,−1).





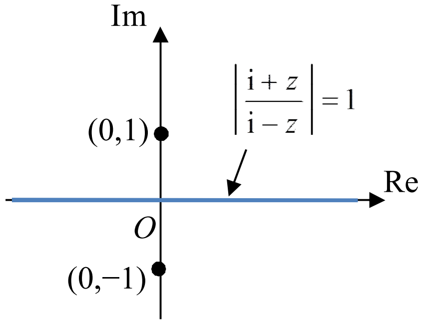
How do we know (without drawing to scale) if the perpendicular bisector passes through the origin?

*By substituting z = 0 (the origin) into the equation, we have* $|0-3-\mathrm{i}|=\sqrt{10}=|0+3+\mathrm{i}|$. *Since the equation is satisfied (i.e. the distance from the origin to the point* (3,1) *equals the distance from the origin to the point* (−3,−1)*), the origin lies on the perpendicular bisector.*

**(b)**

$\left|\frac{\mathrm{i}+z}{\mathrm{i}-z}\right|=1 \Rightarrow |\mathrm{i}+z|=|\mathrm{i}-z| \Rightarrow |z-(-\mathrm{i})|=|-(z-\mathrm{i})|=|z-\mathrm{i}|$

The locus of $z$ is the perpendicular bisector of the line joining (0, −1) and (0, 1).



$|z-a|=|z-b|$ represents the set of points on the perpendicular bisector of the line joining the fixed points $A$ and $B$. What happens if we change the '=' to an inequality sign?


For example, consider the inequality $|z-a|\leq|z-b|$. Referring to the diagram above, do we shade the side containing $A$ or the side containing $B$?

$|z-a|\leq|z-b|$ actually means that if we have a point representing $z$ on the Argand diagram, then the line formed by joining the point representing $z$ and the point representing $a$ is actually shorter or equal in length to the line formed by joining the point representing $z$ and the point representing $b$. Hence we will actually shade the side containing $A$ (including the perpendicular bisector).

### 3.3 Locus of $\arg(z-a)=\alpha$

For a start, we consider the following questions:

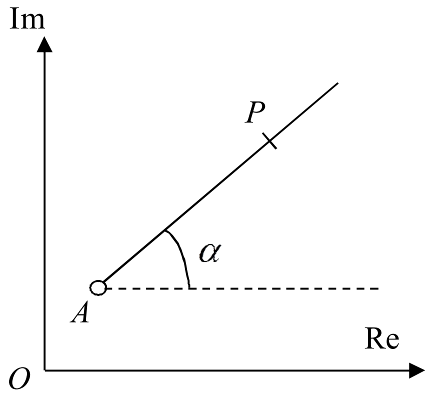
What are the arguments of $1+\mathrm{i}$ and $2+2\mathrm{i}$?

What other complex numbers can you think of with the same argument?

What shape does such collection of points (representing the complex numbers with the same argument) result in?

***So what does* arg *(z – a) = α describe?***

Recall: $\arg(z)$ is the angle between $OP$ (where $P$ is the point representing $z$) and the positive real axis. In particular, we make reference to the origin (the point representing 0) when finding the angle. i.e. $\arg(z-0)$




Let variable point $P$ and fixed point $A$ represent complex variable $z$ and fixed complex number $a$ respectively.

Then $\arg(z-a)=\alpha$ is the angle that $AP$ makes with the horizontal in the positive real axis direction. In particular, we make reference to $A$ (the point representing $a$) when finding the angle.

Thus $P$ moves along the **half-line** that makes an angle $\alpha$ with the horizontal in the positive real axis direction.

**The locus of $z$ expressed in the form $\arg(z-a)=\alpha$ is**
**the half–line from the fixed point $A$ (excluding the point $A$) that makes an angle $\alpha$ with the horizontal in the positive real axis direction.**


$\arg(z-a)=\alpha$ represents the set of points on the half–line from the fixed point $A$ that makes an angle $\alpha$ with the horizontal in the positive real axis direction. What happens if we change the '=' to an inequality sign?

**Example 3.3.1**

Describe and sketch the locus of $z$ if

**(a)** $\arg(z)=\frac{\pi}{3}$

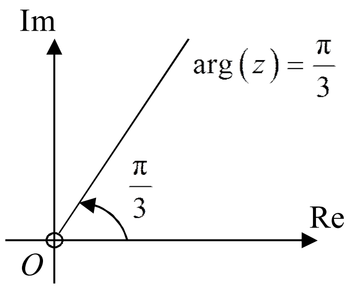
**(b)** $\arg(z+2+3\mathrm{i})=\frac{2\pi}{3}$

**(c)** $\arg(\mathrm{i}z-1)=\pi$ ;

**(d)** $\arg\left(\frac{z-1+2\mathrm{i}}{2+\mathrm{i}}\right)=\frac{\pi}{2}$

*Solution:*

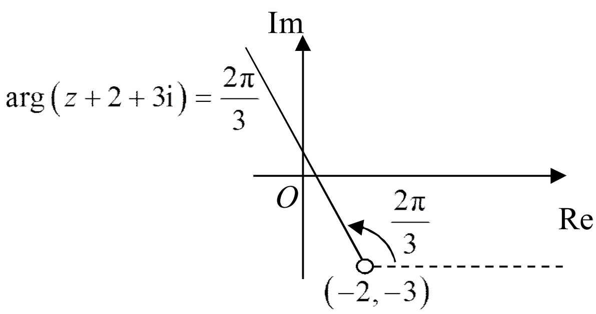
**(a)** The locus of $z$ is a half-line from the origin, making an angle of $\frac{\pi}{3}$ with the positive real axis.




**Note:** The origin (in this question) is not part of the locus and hence, an unshaded circle is drawn at this point.

**(b)** $\arg(z+2+3\mathrm{i})=\arg(z-(-2-3\mathrm{i}))=\frac{2\pi}{3}$

The locus of $z$ is a half-line from (−2, −3), making an angle of $\frac{2\pi}{3}$ with the horizontal in the positive real axis direction.




How do we know (without drawing to scale) if the half-line passes through the origin?

*By substituting $z = 0$ (the origin) into the equation, we have*

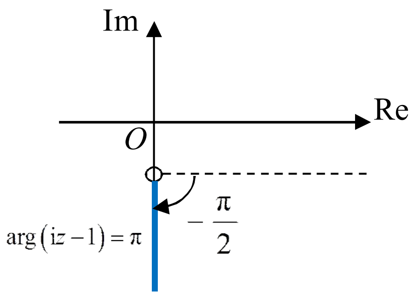
$$\arg(0+2+3\text{i}) \approx 0.983 < \frac{2\pi}{3}.$$

*Since the equation is not satisfied, the origin does not lie on the half-line. In fact, the half-line passes through the positive real axis as it makes a smaller angle with the horizontal dotted line compared to one that passes through the origin.*

**(c)** $\arg(1-\text{i}z) = \arg\left(-\text{i}\left(z-\frac{1}{\text{i}}\right)\right) = \arg(-\text{i}(z+\text{i})) = \arg(-\text{i}) + \arg(z+\text{i}) = \pi$

$\Rightarrow \quad \arg(z+\text{i}) = \pi + \frac{\pi}{2} = \frac{3\pi}{2} \Rightarrow \arg(z+\text{i}) = -\frac{\pi}{2}$

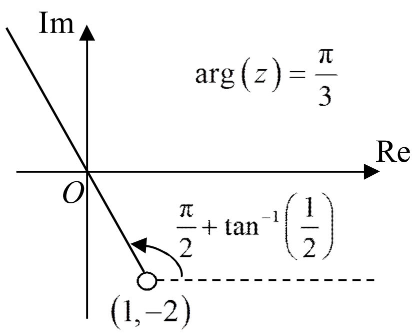
The locus of $z$ is a half-line from (0, −1), making an angle of $-\frac{\pi}{2}$ with the horizontal in the positive real axis direction.




**(d)** $\arg\left(\frac{z-1+2\text{i}}{2+\text{i}}\right) = \arg(z-1+2\text{i}) - \arg(2+\text{i}) = \frac{\pi}{2}$

$\Rightarrow \arg[z-(1-2\text{i})] - \tan^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2} \Rightarrow \arg[z-(1-2\text{i})] = \frac{\pi}{2} + \tan^{-1}\left(\frac{1}{2}\right)$

The locus of $z$ is a half-line from $(1,-2)$, making an angle of $\frac{\pi}{2} + \tan^{-1}\left(\frac{1}{2}\right)$ with the horizontal in the positive real axis direction.




**Note:** Observe that the half-line in (d) passes through the origin. Why?

**Note:**

1. We are able to describe the loci of a circle, perpendicular bisector and half-line if they fall into one of the following categories:

$$|z-a|=r; \quad |z-a|=|z-b|; \quad \arg(z-a)=\theta,$$

where $a$ and $b$ are fixed complex numbers and $r$, $\theta$ are fixed real numbers.

2. The coefficient of $z$ is or must be 1 in the standard equations of the loci.

3. If the equation given is not one of the above, we will either try to change it to one of the above by algebraic manipulation, or try to obtain the Cartesian equation and then describe the locus. A useful result to use is $z \cdot z^* = |z|^2 = x^2 + y^2$.

**Example 3.3.2**

Describe the locus of
**(a)** $|z-2+5\text{i}|=|3+4\text{i}|$ **(b)** $|z-1|=|2z+1|$.

*Solution:*

**(a)** Despite the equation looking like the form $|z-a|=|z-b|$, there is no $z$ in the modulus on the right hand side. This is actually an equation of the form $|z-a|=r$.

$|3+4\text{i}|=\sqrt{3^2+4^2}=5$
Hence, $|z-(2-5\text{i})|=5$.

Thus, the locus is a circle centred at (2, −5) with radius 5.

**(b)** At first glance, the equation looks like the form representing a perpendicular bisector. However, note that the coefficient of $z$ in the modulus on the right hand side is not equal to 1. Thus we can identify the locus by looking at its Cartesian form through substituting $z$ with $x+\text{i}y$.

Let $z=x+\text{i}y$. Then,

$$|z-1|=|2z+1|$$
$$|x+\text{i}y-1|=|2(x+\text{i}y)+1|$$
$$\sqrt{(x-1)^2+y^2}=\sqrt{(2x+1)^2+(2y)^2}$$
$$(x-1)^2+y^2=(2x+1)^2+(2y)^2$$
$$3x^2+3y^2+6x=0$$
$$x^2+y^2+2x=0, \quad \text{i.e. } (x+1)^2+(y-0)^2=1$$

Thus, the locus is a circle centred at (−1, 0) with radius 1.

## 3.4 Problems Involving Inequalities and Maximum/Minimum

### Example 3.4.1

Given that $|z-3+3\mathrm{i}|=2$, find the greatest and least values of

**(a)** $|z+1|$,

**(b)** $\arg(z+1)$.

*Solution:*

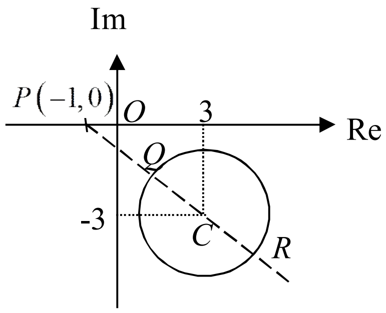
$|z-3+3\mathrm{i}|=|z-(3-3\mathrm{i})|=2$

The locus is a circle centred at $C(3, -3)$ with radius 2.

**(a)**

Let $Z$ be the point representing $z$ in the Argand diagram.

Then $|z-(-1)|=$ distance between $Z$ and $P(-1, 0)$.




$$CP=\sqrt{[3-(-1)]^2+(3-0)^2}=5$$

From the Argand diagram,

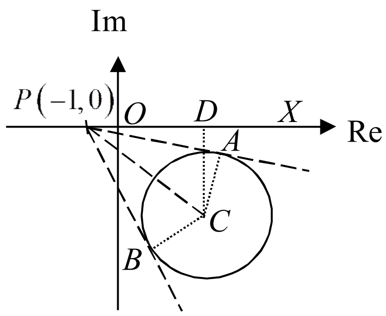
the least value of $|z+1|=PQ=CP-CQ=5-2=3$,

the greatest value of $|z+1|=PR=CP+CR=5+2=7$.

**(b)**

$\arg[z-(-1)]$

= the angle, measured in the anti-clockwise direction, that the line $PZ$ makes with the positive real axis.




In the right-angle triangles $APC$ and $BPC$,

$$\angle APC=\angle BPC=\sin^{-1}\frac{AC}{CP}=\sin^{-1}\frac{2}{5}\approx 0.41152$$

In the right-angle triangle $CDP$,

$$\angle CPD=\tan^{-1}\frac{CD}{DP}=\tan^{-1}\frac{3}{4}\approx 0.64350$$

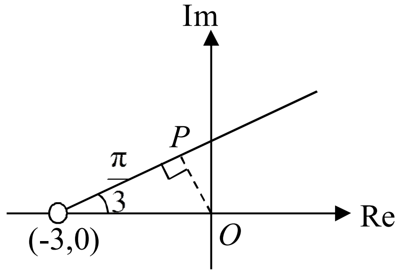
$$\begin{aligned}\text{The least value of } \arg(z+1) &= -\angle BPX \\ &= -(\angle CPD+\angle BPC) \\ &\approx -(0.64350+0.41152) \\ &= -1.06 \quad (\text{to 3 s.f.})\end{aligned}$$

$$\begin{aligned}\text{The greatest value of } \arg(z+1) &= -\angle APX \\ &= -(\angle CPD-\angle APC) \\ &\approx -(0.64350-0.41152) \\ &= -0.232 \quad (\text{to 3 s.f.})\end{aligned}$$

**Example 3.4.2**

If $\arg(z+3)=\frac{\pi}{3}$, find the least value of $|z|$.

*Solution:*




The least value of $|z|$ is the shortest distance between $P$ (a point on the half-line) and the origin.

Least value of $|z|=OP=3\sin\frac{\pi}{3}=\frac{3\sqrt{3}}{2}$

**Example 3.4.3**

Shade the region represented on an Argand diagram by

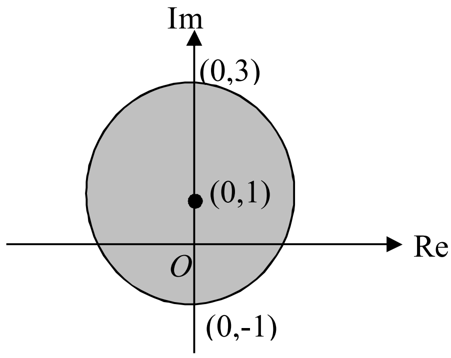
**(a)** $|z-\mathrm{i}|\le 2$,

**(b)** $0<\arg(z+2+\mathrm{i})<\frac{\pi}{6}$,

**(c)** $\frac{\pi}{4}<\arg z<\frac{\pi}{2}$ and $|z|\le|z-2-2\mathrm{i}|$.

*Solution:*

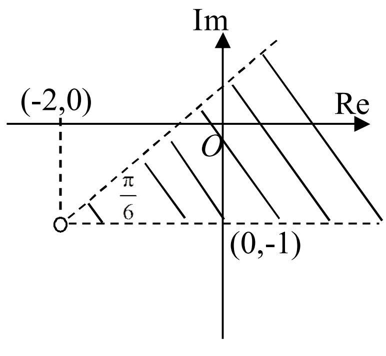
**(a)** The locus of $z$ is the interior of the circle centred at (0, 1) with radius 2, including the circumference.




**(b)** $0<\arg(z+2+\mathrm{i})<\frac{\pi}{6}$

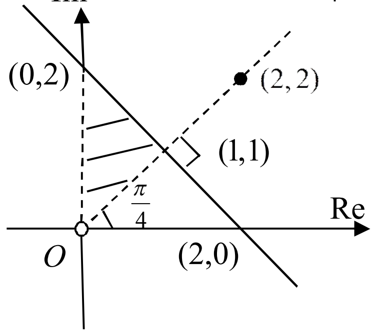
$\Rightarrow 0<\arg[z-(-2-\mathrm{i})]<\frac{\pi}{6}$

**Note**: Observe that the half-line representing $\arg(z+2+\mathrm{i})=\frac{\pi}{6}$ passes through the positive imaginary axis, above the origin. Why?




**(c)** $\frac{\pi}{4}<\arg z<\frac{\pi}{2}$ and $|z|\le|z-2-2\mathrm{i}|$

$\Rightarrow \frac{\pi}{4}<\arg z<\frac{\pi}{2}$ and $|z|\le|z-(2+2\mathrm{i})|$



**Example 3.4.4**

On a single Argand diagram, sketch the loci given by

**(i)** $|z-3|=4$,

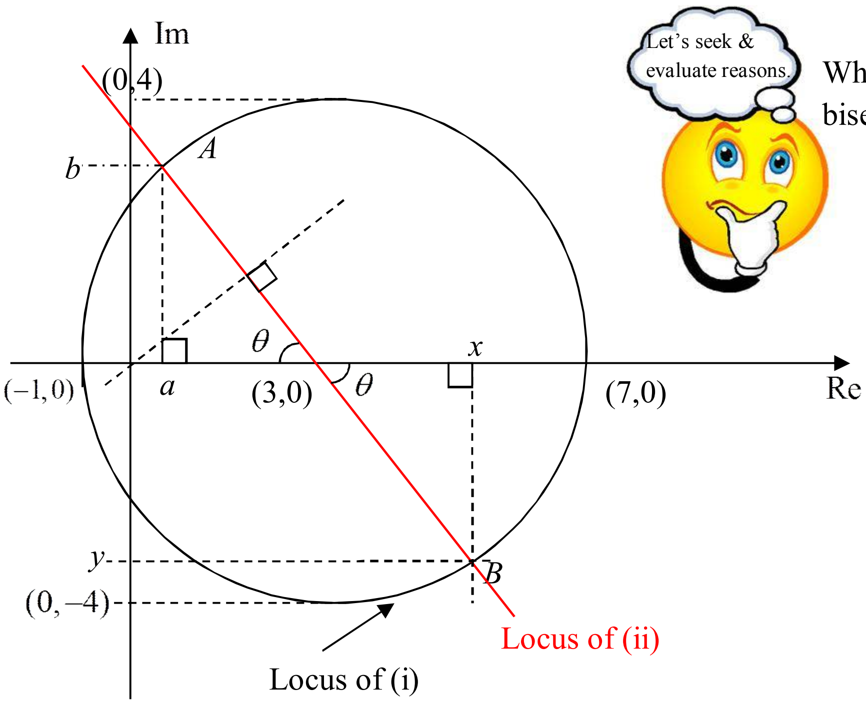
**(ii)** $|z-3-3\mathrm{i}|=|z|$.

Hence, or otherwise, find the exact values of all the complex numbers $z$ that satisfy both (i) and (ii).

*Solution:*

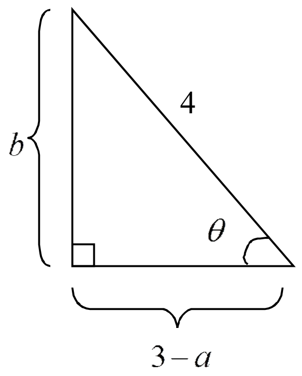
**(i)** $|z-3|=4$ is a circle centred at (3, 0) with radius 4.

**(ii)** $|z-3-3\mathrm{i}|=|z|$ is the perpendicular bisector of the line joining the points (3, 3) and (0, 0).




Why does the perpendicular bisector pass through (3,0)?

Let $A=a+\mathrm{i}b$




$\sin\frac{\pi}{4}=\frac{b}{4}$

$\Rightarrow b=4\left(\frac{\sqrt{2}}{2}\right)$

$\therefore b=2\sqrt{2}$

$\cos\frac{\pi}{4}=\frac{3-a}{4}$

$\Rightarrow 3-a=4\left(\frac{\sqrt{2}}{2}\right)$

$\therefore a=3-2\sqrt{2}$

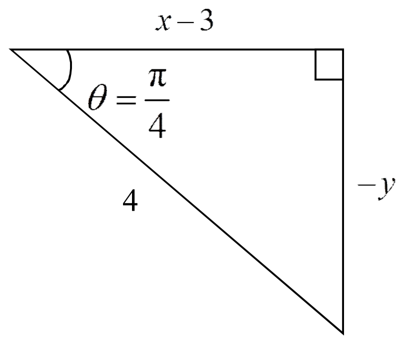
$\therefore A=3-2\sqrt{2}+\mathrm{i}2\sqrt{2}$

Let $B=x+\mathrm{i}y$

$\sin\frac{\pi}{4}=-\frac{y}{4}$

$\Rightarrow y=-4\left(\frac{\sqrt{2}}{2}\right)$

$\therefore y=-2\sqrt{2}$




$\cos\frac{\pi}{4}=\frac{x-3}{4}$

$\Rightarrow x=3+4\left(\frac{\sqrt{2}}{2}\right)$

$\therefore x=3+2\sqrt{2}$

$\therefore B=3+2\sqrt{2}-\mathrm{i}2\sqrt{2}$

Alternatively using GC,

**Step 1**:Find Cartesian equation of circle : $(x-3)^2+y^2=4^2 \;\Rightarrow\; y^2=16-(x-3)^2$

Key in $\mathrm{Y}_1=\sqrt{16-(x-3)^2}$ and $\mathrm{Y}_2=-\sqrt{16-(x-3)^2}$ or $-\mathrm{Y}_1$

**Step 2**: Find Cartesian equation of the perpendicular bisector:

Gradient of line joining (3, 3) and (0, 0) = $\frac{3-0}{3-0}=\frac{3}{3}=1$

Thus gradient of the perpendicular bisector is $-\frac{1}{1}=-1$ and the perpendicular bisector passes through the midpoint of (3, 3) and (0, 0) i.e. $\left(\frac{3+0}{2}, \frac{3+0}{2}\right)$.

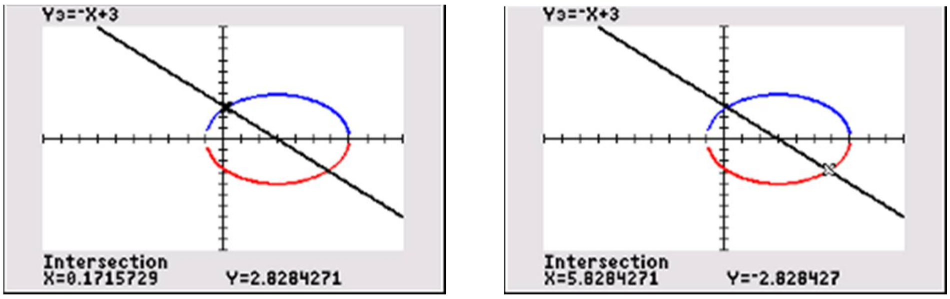
Equation of the perpendicular bisector:

$$y-\frac{3}{2}=(-1)\left(x-\frac{3}{2}\right)$$

$$\Rightarrow \quad y=-x+3$$

Key in $\mathrm{Y}_3=-x+3$

**Step 3**: Press 'Graph'. Enter '2$^{\text{nd}}$', 'trace' and scroll down to press 'intersection' to find the intersection(s) of the circle and the perpendicular bisector.




**Note:** You may use the GC to help you in your calculations involving complex numbers, unless it is stated that the problem given needs to be solved without the use of a calculator, or if an exact solution is required.