RAFFLES INSTITUTION

2014 YEAR 6 PRELIMINARY EXAMINATION

MATHEMATICS PAPER 1 **9740/1**

Higher 2 **16 September 2014**

Total Marks: 100 **3 hours**

Additional materials: Answer Paper
Graph Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

RAFFLES INSTITUTION

Mathematics Department

1 A function f is given by $\mathrm{f}(x)=\dfrac{x^2-ax+b}{x}$ for $x\in\mathbf{R}$, $x\neq 0$ where a and b are positive real constants.

Find $\mathrm{f}'(x)$ and hence sketch the graph of $y=\mathrm{f}'(x)$, stating the equations of any asymptotes and the coordinates of the points where the curve crosses the axes. **[4]**

2 A sequence $u_0, u_1, u_2, \ldots$ is such that $u_0=20$ and

$$u_{n+1}=ru_n\left(1-\frac{u_n}{100}\right), \quad \text{for all } n\geq 0.$$

(i) Given that $r=\dfrac{2}{3}$, find the least value of n such that $u_n<1$. **[2]**

(ii) Find the value of r such that $u_n=20$ for all values of n. **[2]**

(iii) It is given that $r=\dfrac{4}{3}$ and $u_n\to l$ as $n\to\infty$. Showing your working, find the exact value of l. **[2]**

3 Let $\mathrm{f}(x)=\dfrac{1-x}{(1+x^2)^2}$.

(i) Find the binomial expansion of $\mathrm{f}(x)$ in increasing powers of x, up to and including the term in x^4. **[3]**

(ii) Find the coefficient of x^{2r+1} in the expansion for $\mathrm{f}(x)$. **[3]**

4 Prove by the method of mathematical induction that

$$\sum_{r=1}^{n}\sin(2rx)\sin x=\sin(nx)\sin(n+1)x \quad \text{for all positive integers } n.$$ **[4]**

Hence find the exact value of $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\sin 3x\sin 4x}{\sin x}\,\mathrm{d}x$. **[3]**

5 The curve with equation $y = -\dfrac{1}{x}$ undergoes a translation of 1 unit in the positive x-direction, followed by a stretch with factor $\dfrac{1}{2}$ parallel to the x-axis and then a translation of 3 units in the positive y-direction.

Given that the equation of the new curve is $y = \mathrm{f}(x)$. Sketch the curve, stating its asymptotes and the coordinates of the points of intersection with the axes. **[4]**

By sketching the graphs of $y = |\mathrm{f}(x)|$ and $y = \mathrm{f}(|x|)$ on separate diagrams, solve the inequality

$$|\mathrm{f}(x)| > \mathrm{f}(|x|).$$ **[4]**

6 Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. It is given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$ and $|3\mathbf{a} - \mathbf{b}| = 10$.

(i) Give the geometrical interpretation of $\left|\mathbf{a} \cdot \dfrac{\mathbf{b}}{|\mathbf{b}|}\right|$. **[1]**

(ii) Show that $\mathbf{a} \cdot \mathbf{b} = 1$. **[2]**

(iii) Hence find the shortest distance from A to the line OB, and the area of the triangle OAB. **[2]**

(iv) Given that $\mathbf{a}$, $2\mathbf{a}+3\mathbf{b}$ and $\mu\mathbf{a} + 2\mathbf{b}$, where μ is a constant, are position vectors of collinear points, find μ. **[4]**

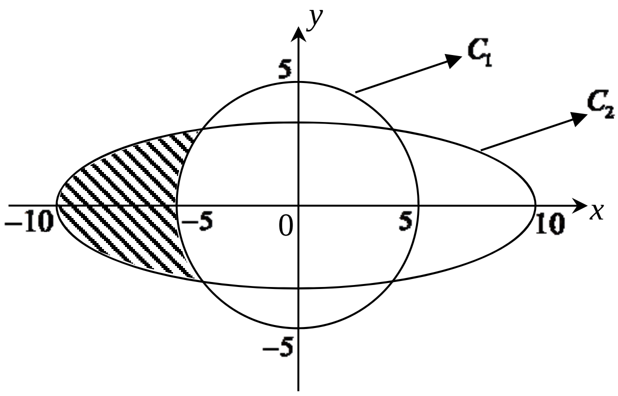
7 Curves C_1 and C_2 are given by the equations

$$C_1: \quad x^2 + y^2 = 25$$
$$C_2: \quad ax^2 + by^2 = 100$$

where a and b are positive real constants such that $a < b$.

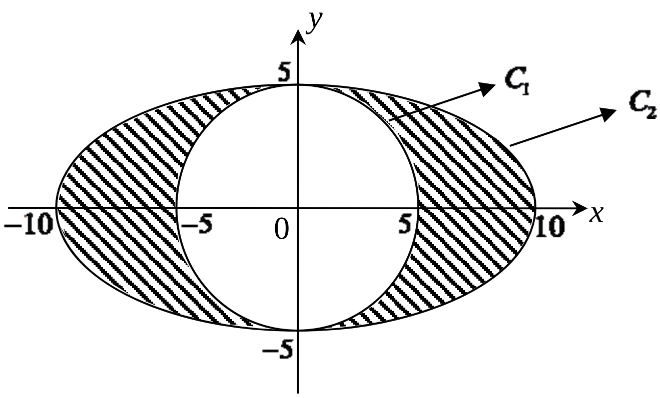
(a) Find the condition(s) on the values of a and b such that C_1 and C_2 intersect at exactly 4 points. **[2]**

(b) The diagram shows the curves C_1 and C_2 when $a = 1$ and $b = 9$.



Find the area of the shaded region. **[3]**

(c) The diagram shows the curves C_1 and C_2 when $a = 1$ and $b = 4$.



Find the exact volume when the shaded region is rotated through π radians about the y-axis. **[4]**

8 **(a)** Without using a calculator, solve the equation

$$z^4 - 8\left(-\sqrt{3} + \mathrm{i}\right) = 0,$$

giving the roots in the form $r\mathrm{e}^{\mathrm{i}\alpha}$, where $r > 0$ and $-\pi < \alpha \le \pi$. **[4]**

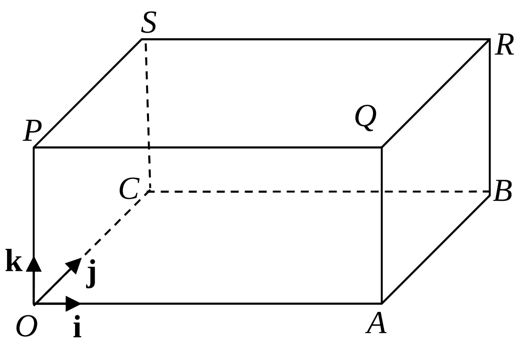
Show the roots on an Argand diagram. **[2]**

(b) One root of the equation $z^2 + az^* + b = 0$, where a and b are real, is w.

Show that w^* is also a root of this equation. **[2]**

Solve the equation $z^2 + 6z^* + 9 = 0$, giving your answers in the form $x + \mathrm{i}y$. **[4]**

9



The diagram shows a cuboid with horizontal rectangular base *OABC*, where $OA = 5$ units, $OC = 3$ units and $OP = 2$ units. The edges *OP*, *AQ*, *BR* and *CS* are vertical, and *PQRS* is the top of the cuboid. Using *O* as the origin, unit vectors **i**, **j** and **k** are taken along *OA*, *OC* and *OP* respectively.

The plane π_1 contains the points *A*, *C* and *R*, and the plane π_2 contains the points *Q*, *S* and *B*.

(i) Find, in scalar product form, an equation of π_1. **[3]**

(ii) Find the acute angle between π_1 and the horizontal base. **[2]**

(iii) Hence state the acute angle between π_1 and π_2. **[1]**

The point *X* lies on *PS* such that $\overrightarrow{SX} = \alpha\overrightarrow{SP}$, where $\alpha > 0$ is a constant, and *Y* is the point with coordinates $(5, 2, 1)$.

(iv) Find the equations of the lines *XY* and *OR* in vector form. **[3]**

(v) Given that the lines *XY* and *OR* intersect at a point *W*, find α and the ratio $OW : OR$. **[4]**

 [Turn over

10 A curve C is defined by the parametric equations $x = \tan\theta,\ y = \sec\theta$ for $0 < \theta < \dfrac{\pi}{2}$.

(i) Find the cartesian equation of C. Sketch C, giving the equation(s) of any asymptote(s). **[3]**

The tangent and normal at $P(\tan\theta, \sec\theta)$ meets the x-axis at Q and R respectively.

(ii) Show that the area, A of the circle passing through P, Q and R can be expressed as

$$A = \pi\left(\tan\theta + \frac{1}{2\tan\theta}\right)^2.$$ **[7]**

(iii) Show that $\left(t + \dfrac{1}{2t}\right)^2 - 2 \geq 0$ for all $t \in \mathbb{R}, t \neq 0$. **[2]**

Deduce the minimum value of A . **[1]**

11 The variables x and y are related by the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -y. \qquad \text{-----------------} \ (1)$$

It is also given that $y = \dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

(i) Write down the value of $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x = 0$. **[1]**

(ii) By further differentiation, obtain the values of $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$ and $\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4}$ when $x = 0$.

Hence write down the first five terms of the Maclaurin series for y. **[2]**

(iii) Show that the substitution $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{u}$ reduces (1) to $\dfrac{\mathrm{d}u}{\mathrm{d}y} = -2y$.

Find u in terms of y, given that $u = 1$ when $y = 1$. **[4]**

(iv) Using your answer from **(iii)**, find y in terms of x, giving your answer in the form $y = P\sin(x + Q)$, where P and Q are constants to be determined. **[4]**

(v) Use the standard series in MF15 to find the Maclaurin's expansion for your answer in **(iv)** up to and including the term in x^4. **[2]**

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