National Junior College
2016 – 2017 H2 Further Mathematics
Topic F6: Numerical Methods (Assignment Solutions)

Suggested Solutions

1 Let $\mathrm{f}(x)=\dfrac{2}{x}$ and $h=\dfrac{2-1}{n}=\dfrac{1}{n}$,

$$\mathrm{f}(1)=2,$$

$$\mathrm{f}\left(1+\frac{1}{n}\right)=\frac{2n}{n+1},$$

$$\mathrm{f}\left(1+\frac{2}{n}\right)=\frac{2n}{n+2},$$

$$\vdots$$

$$\mathrm{f}\left(1+\frac{r}{n}\right)=\frac{2n}{n+r},$$

$$\vdots$$

$$\mathrm{f}\left(1+\frac{n-1}{n}\right)=\frac{2n}{n+(n-1)},$$

$$\mathrm{f}(2)=1.$$

Using Trapezium Rule,

$$\int_1^2 \frac{2}{x}\,\mathrm{d}x$$

$$\approx \frac{1}{2n}\left[2+1+2\left(\frac{2n}{n+1}+\frac{2n}{n+2}+\cdots+\frac{2n}{n+(n-1)}\right)\right]$$

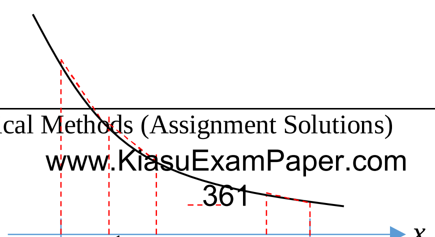
$$=\frac{1}{2n}\left[3+2\left(\frac{2n}{n+1}+\frac{2n}{n+2}+\cdots+\frac{2n}{n+(n-1)}\right)\right]$$

$$=\frac{3}{2n}+\frac{1}{n}\left(\frac{2n}{n+1}+\frac{2n}{n+2}+\cdots+\frac{2n}{n+(n-1)}\right)$$

$$=\frac{3}{2n}+2\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+(n-1)}\right)$$

$$=\frac{3}{2n}+2\sum_{r=1}^{n-1}\frac{1}{n+r}\quad(\text{shown})$$

Since the curve $y=\dfrac{2}{x}$ from $x=1$ to $x=2$ concave upwards, applying the trapezium rule will lead to an overestimation of the actual area as the sum of the areas of the n trapezia given by $\dfrac{3}{2n}+2\displaystyle\sum_{r=1}^{n-1}\frac{1}{n+r}$, is more than the actual area, $\displaystyle\int_1^2 \frac{2}{x}\,\mathrm{d}x$.



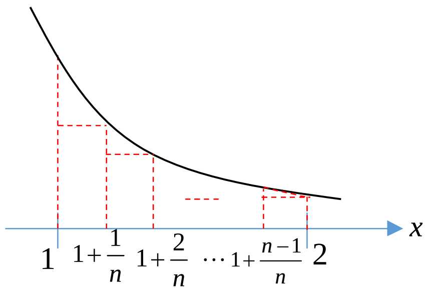
$$\int_1^2 \frac{2}{x}\,\mathrm{d}x = \left[2\ln x\right]_1^2$$
$$= 2\ln 2 - 2\ln 1$$
$$= 2\ln 2$$

Since the sum of the areas of the n trapezia, $\frac{3}{2n} + 2\sum_{r=1}^{n-1}\frac{1}{n+r}$, is more than the actual area,

$$\frac{3}{2n} + 2\sum_{r=1}^{n-1}\frac{1}{n+r} > 2\ln 2\,.$$

By considering n rectangles,

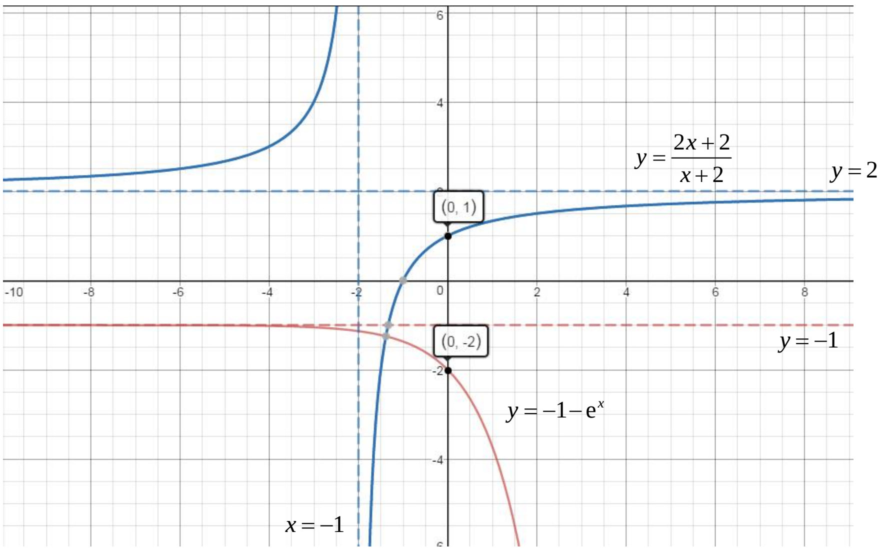
$$\text{total area of } n \text{ rectangles} = \left(\frac{1}{n}\right)\left(\frac{2n}{n+1}\right) + \left(\frac{1}{n}\right)\left(\frac{2n}{n+2}\right) + \left(\frac{1}{n}\right)\left(\frac{2n}{n+3}\right) + \cdots + \left(\frac{1}{n}\right)\left(\frac{2n}{n+n}\right)$$
$$= \left(\frac{2}{n+1}\right) + \left(\frac{2}{n+2}\right) + \left(\frac{2}{n+3}\right) + \cdots + \left(\frac{2}{n+n}\right)$$
$$= 2\sum_{r=1}^{n}\frac{1}{n+r}$$



Since the sum of the areas of the n rectangles, $2\sum_{r=1}^{n}\frac{1}{n+r}$, is less than the actual area,

$$\frac{3}{2n} + 2\sum_{r=1}^{n-1}\frac{1}{n+r} > 2\ln 2 > 2\sum_{r=1}^{n}\frac{1}{n+r}\,.$$

2



From the graph of $y=\frac{2x+2}{x+2}$ and $y=-1-e^x$, the graphs intersect exactly once.

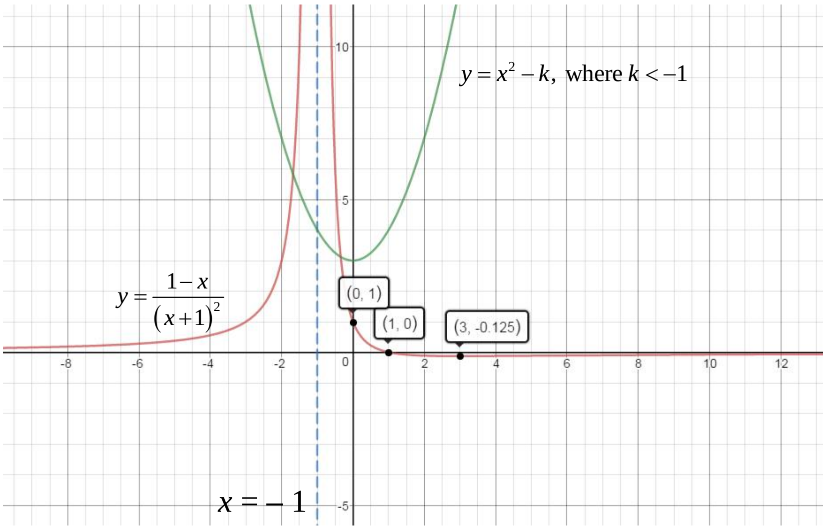
Hence, the equation $\frac{2x+2}{x+2}=-1-e^x$ has exactly one real root.

Let $f(x)=\frac{2x+2}{x+2}+1+e^x$, applying linear interpolation once,

$$\begin{aligned}\alpha_1&=\frac{(-1.9)\left|f(-1)\right|+(-1)\left|f(-1.9)\right|}{\left|f(-1)\right|+\left|f(-1.9)\right|}\\&=-1.0676\\&\approx-1.1\ (\text{to 1 decimal place})\end{aligned}$$

Since $y=f(x)$ has a vertical asymptote at $x=-2$, $f'(x)$ changes rapidly in the interval $[-1.9,-1]$, α_1 is further away from α. Therefore, α_1 is not a good approximation to α.

3



(i) From the graph, for $\frac{1-x}{(x+1)^2} = x^2 + (-k)$ to have 2 negative roots,

$-k > 1 \Rightarrow \therefore k < -1.$

(ii) For $1 < k < 9\frac{1}{8}$, the graph of $y = x^2 + (-k)$ lies between $y = x^2 - 1$ and

$y = x^2 - 9\frac{1}{8}$, it will intersect $y = \frac{1-x}{(x+1)^2}$ at a point where $x = \alpha$ and $\alpha \in (1,3)$.

If $k = 3$, $\frac{1-x}{(x+1)^2} = x^2 - 3 \Rightarrow \frac{1-x}{(x+1)^2} - x^2 + 3 = 0$.

Let $\mathrm{f}(x) = \frac{1-x}{(x+1)^2} - x^2 + 3$,

$\mathrm{f}'(x) = \frac{x-3}{(x+1)^3} - 2x$.

By Newton-Raphson Method, $x_0 = 2$,

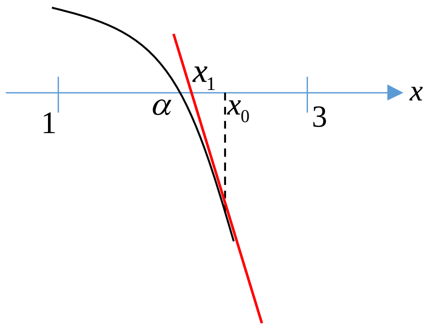
$x_1 = 2 - \frac{\mathrm{f}(2)}{\mathrm{f}'(2)} = 1.72477 \approx 1.725$ (to 3 decimal places).

$$f''(x)=\frac{(x+1)^3-3(x-3)(x+1)^2}{(x+1)^6}-2$$
$$=\frac{-2x+10}{(x+1)^4}-2$$
$$=-2\left[\frac{x-5}{(x+1)^4}+1\right]$$

For $x\in(1,3)$, $f'(x)<0$ and $f''(x)<0$.

The curve strictly decreases and concave downwards for $x\in(1,3)$. The tangent line at $x_0=2$ cuts the x-axis at x_1 which is on the right of α.

Therefore, α is smaller than x_1.



4

Let $f(x)=x^4-2, f'(x)=4x^3$.

By Newton-Raphson method,

$$\alpha_2=\alpha_1-\frac{f(\alpha_1)}{f'(\alpha_1)}$$
$$=\alpha_1-\frac{(\alpha_1^{\,4}-2)}{4\alpha_1^{\,3}}$$
$$=\alpha_1-\frac{\alpha_1}{4}+\frac{1}{2\alpha_1^{\,3}}$$
$$=\frac{3\alpha_1}{4}+\frac{1}{2\alpha_1^{\,3}}\quad\text{(shown)}$$

Taking $\alpha_1=1$,

$$\alpha_2=\frac{3\alpha_1}{4}+\frac{1}{2\alpha_1^{\,3}}=\frac{5}{4}$$
$\alpha_3=1.1935$

$\alpha_4=1.18923$

$\alpha_5=1.18921$

$\therefore x\approx 1.189$ (to 3 decimal places)

$x^4 - 2 = 0 \Rightarrow x = 2^{\frac{1}{4}}$

$\therefore 2^{\frac{1}{4}} \approx 1.189$

Taking $\alpha_1 = 1$,

$x_2 = \left(\dfrac{2}{x_1}\right)^{\frac{1}{3}} = 1.25992,\ \ x_3 = \left(\dfrac{2}{x_2}\right)^{\frac{1}{3}} = 1.16653,$

$x_4 = \left(\dfrac{2}{x_3}\right)^{\frac{1}{3}} = 1.19686,\ \ x_5 = \left(\dfrac{2}{x_4}\right)^{\frac{1}{3}} = 1.18667,$

$x_6 = \left(\dfrac{2}{x_5}\right)^{\frac{1}{3}} = 1.19006,\ \ x_7 = \left(\dfrac{2}{x_6}\right)^{\frac{1}{3}} = 1.18892,$

$x_7 = \left(\dfrac{2}{x_6}\right)^{\frac{1}{3}} = 1.18930,$

$\therefore 2^{\frac{1}{4}} \approx 1.189$

5

(i) Rate at which the pollutants enter the vessel = $\dfrac{qx}{100}$ litres/min.

Rate at which the pollutants leave the vessel = $\dfrac{py}{100}$ litres/min.

Let U be the volume of pollutant in the vessel at time t, then $\dfrac{\mathrm{d}U}{\mathrm{d}t} = \dfrac{qx - py}{100}$.

Also, $\dfrac{\mathrm{d}V}{\mathrm{d}t} = x + w - y$ and $p = \dfrac{U}{V} \times 100$.

Thus,

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{qx - py}{100}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{pV}{100}\right) = \frac{qx - py}{100}$$

$$\Rightarrow p\frac{\mathrm{d}V}{\mathrm{d}t} + V\frac{\mathrm{d}p}{\mathrm{d}t} = qx - py$$

$$\Rightarrow p(x + w - y) + V\frac{\mathrm{d}p}{\mathrm{d}t} = qx - py$$

$$\Rightarrow \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{qx - py - px - pw + py}{V} \Rightarrow \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{qx - (w + x)p}{V} \text{ (shown)}$$

(ii) When $x = y = w = 1000$, and $q = 40$,

n	t_n	$p_{n,\text{ approx}}$	$\mathrm{d}p/\mathrm{d}t$
0	0.0	10.0000	2.0000
1	1.0	12.0000	1.4545
2	2.0	13.4545	1.0909
3	3.0	14.54545	

$$\frac{\mathrm{d}V}{\mathrm{d}t} = x + w - y = 1000 \Rightarrow V = 1000t + C.$$

Using $V = 10000$ when $t = 0$, we get $C = 10000$.
Thus, $V = 1000t + 10000$.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{40(1000) - (1000+1000)p}{V} = \frac{40000 - 2000p}{1000t + 10000} = \frac{40 - 2p}{t + 10}.$$

Overestimate, because gradient is positive but decreasing.

6

(i) Exact solution is $y(t) = \dfrac{1}{1+t^2}$. So exact/true value of $y(2) = 0.2$.

Using Euler's method to approximate this solution over the interval $0 \le t \le 2$ for step size $\Delta t = 0.2$, we obtain $y_{\text{approx}} = 0.1858$. The approximation is an under-estimate of the true value of $y(2)$, and has an error $e_1 = 0.0142$.

Using Euler's method to approximate this solution over the interval $0 \le t \le 2$ for step size $\Delta t = 0.1$, we obtain $y_{\text{approx}} = 0.1933$. The approximation is an under-estimate of the true value of $y(2)$, and has an error $e_2 = 0.0067$.

Observe that $\dfrac{e_1}{e_2} \approx 2.12$.

(ii) Using improved Euler's method with a step size of $\Delta t = 0.1$, we obtain $y_{\text{approx}} = 0.200695$ which differs from the true value by $e_3 = 6.95 \times 10^{-4}$.

Observe that $\dfrac{e_2}{e_3} \approx 9.5$.

Thus, we can say that the improved Euler's method is about 10 times more accurate than the Euler's method.