1 Differentiate each of the following expressions with respect to x.

(i) $\tan^{-1}\sqrt{x+1}$ [2]

(ii) $\ln\left(\dfrac{x+e^x}{3^x}\right)$ [3]

2 (i) Sketch the graph of $y=\left|b+\dfrac{a}{x+b}\right|$, where $0<a<b$, stating the equations of any asymptotes and the coordinates of the points that crosses the axes. [3]

(ii) Hence solve the inequality $-x\le\left|b+\dfrac{a}{x+b}\right|$. [2]

(iii) Using part (ii), given that $a=1$, $b=2$, solve the inequality $e^x\le\left|b+\dfrac{a}{b-e^x}\right|$. [2]

3 (a) A curve with equation $\dfrac{(x-2)^2}{4}+y^2=1$ undergoes, in succession, the following transformations:

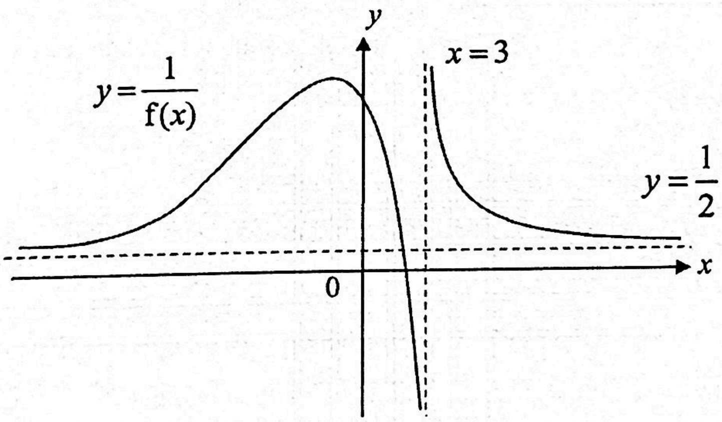
A: A stretch with scale factor $\dfrac{1}{2}$ parallel to the y-axis.

B: A translation of 5 units in the positive x-direction.

C: A reflection about y-axis.

Find the equation of the resulting curve. [3]

(b) The diagram shows a sketch of the curve $y=\dfrac{1}{f(x)}$. The curve cuts the axes at $(2,\ 0)$ and $(0,\ 3)$. It has a stationary point at $(-1,4)$ and equations of asymptotes $x=3$ and $y=\dfrac{1}{2}$.



Sketch the following curves and state the equations of the asymptotes, the coordinates of the turning points and of points where the curve crosses the axes.

(i) $y=f(x)$, [3]

(ii) $y=-\dfrac{1}{f(x)}+4$. [3]

4 **Do not use a calculator in answering this question.**

A curve C has equation $y^2 + 3xy = 13 + x^2$.

(i) Find the coordinates of the stationary points of C. [4]

(ii) For the stationary point with $x > 0$, determine whether it is a maximum or minimum point. [3]

5 **(a)** Functions f and g are defined by

$$\mathrm{f}(x) = \mathrm{e}^{x+3} - a \text{ for } x \in \quad , x \leq 4,$$
$$\mathrm{g}(x) = 2x + 5 \text{ for } x \in \quad .$$

where a is a constant.

(i) Find $\mathrm{f}^{-1}(x)$ and state its domain, leaving your answer in terms of a. [3]

(ii) It is given that the composite function $\mathrm{g}^{-1}\mathrm{f}$ exists and $\mathrm{g}^{-1}\mathrm{f}(-3) = 15$. Without finding $\mathrm{g}^{-1}\mathrm{f}$, find the value of a. [2]

(b) A curve has equation $y = \mathrm{h}(x)$, where

$$\mathrm{h}(x) = \begin{cases} 2^x & \text{for } 0 < x \leqslant 2, \\ 6 - \frac{1}{2}x^2 & \text{for } 2 < x \leqslant 3, \end{cases}$$

and that $\mathrm{h}(x) = \mathrm{h}(x-3)$ for all real values of x.

(i) Find $\mathrm{h}(9)$. [1]

(ii) Sketch the graph of $y = \mathrm{h}(x)$ for $-1 \leq x \leq 4$. [4]

6 Referred to the origin O, the points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively, where vector $\mathbf{p}$ and vector $\mathbf{q}$ are not parallel. Point A lies between P and Q, such that $PA:AQ = \lambda:(1-\lambda)$, where $0 < \lambda < 1$.

(i) Express $\overrightarrow{OA}$ in terms of $\mathbf{p}$, $\mathbf{q}$ and λ. [1]

(ii) The point B lies between O and Q, such that $OB:OQ = 1:3$. Given that $\overrightarrow{AB}$ is parallel to $(\mathbf{q} - 4\mathbf{p})$, find the value of λ. [4]

(iii) It is now given that $\mathbf{p}$ is a unit vector which is perpendicular to $\mathbf{q}$ and $|\mathbf{q}| = 2$, find the area of triangle OAQ, in terms of λ. [4]

7 The curve C has parametric equations

$$x = 4\cos\theta,\ \ y = 3\sin\theta,\ \ 0 < \theta \leq \pi.$$

(a) Point Q lies on C and is moving along C, such that its x coordinate is increasing at the constant rate of 15 units per second. Determine the exact rate of change of θ at the instant when the coordinates of Q is $\left(2, \frac{3\sqrt{3}}{2}\right)$. [3]

It is given that point P also lies on C and has parameter p where $0 < p < \frac{\pi}{2}$.

(b) Show that the equation of the tangent to C at the point P, is given by

$$y = \left(-\frac{3}{4}\cot p\right)x + 3\operatorname{cosec} p.$$ [3]

(c) The tangent to the curve at P meets the x-axis at the point A and the y-axis at the point B. Find the area of the triangle OAB. [2]

(d) Let M be the mid-point of AB. Find a cartesian equation of the curve traced by M as p varies. [2]

8 **(a)** **(i)** The sum of the first n terms of a sequence $u_1, u_2, u_3, \ldots$ is given by $S_n = e^2n^2 - 3n$. Show that $u_n = e^2(2n-1) - 3$. [2]

(ii) Hence show that the sequence is an arithmetic progression. [2]

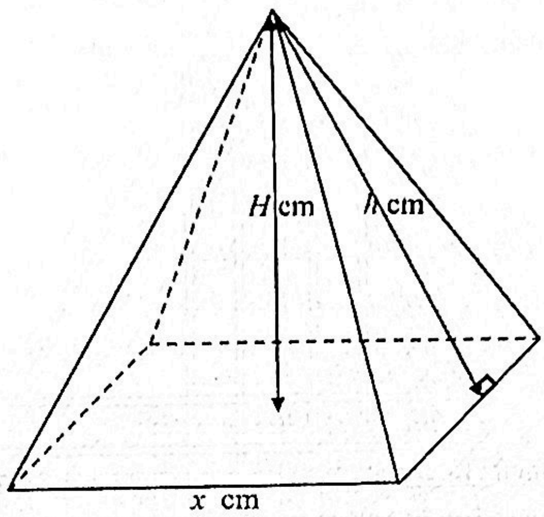
(iii) State the first 3 odd-numbered terms, u_1, u_3 and u_5. Hence find the least value of n such that the sum of the first n odd-numbered terms is more than 500. [3]

(b) **(i)** Find the range of values of x for which the following geometric series converges,

$$e - e^{\frac{1}{x}+1} + e^{\frac{2}{x}+1} - e^{\frac{3}{x}+1} + \ldots\ .$$ [2]

(ii) Given that $x = -1$, find the sum of all the terms of this geometric series after, and including, the 8^{th} term. [3]

9 [It is given that the volume of a pyramid is $\frac{1}{3}$ × base area × height.]



A container in the shape of a right pyramid with vertical height H cm consists of a square base of side length x cm and four isosceles triangles, each with base x cm and perpendicular height h cm (see diagram). The pyramid has a fixed surface area A cm^2.

(a) Show that $h = \dfrac{A - x^2}{2x}$. [1]

(b) Hence show that $36V^2 = A^2x^2 - 2Ax^4$, where V is the volume of the pyramid. [3]

(c) Find, in terms of A, the value of x for which V is maximum, and show that this maximum value of V is $\sqrt{\dfrac{A^3}{288}}$. You do not need to show that this value is a maximum. [3]

10 A sequence $a_1, a_2, a_3, \ldots$ is such that $a_n > 0$, for $n \in \mathbb{Z}^+, n \geq 1$.

(a) Find $\displaystyle\sum_{n=1}^{N} \ln\left(\frac{a_{n+1}}{a_n}\right)$, and express your answer as a single logarithm. [2]

It is also known that this sequence is defined by $\dfrac{a_{n+1}}{a_n} = \dfrac{1}{4}$.

(b) Given that $\displaystyle\sum_{n=1}^{N} \left(a_n - a_{n+1}\right)^2 = \frac{k}{16} \sum_{n=1}^{N} \left(a_n\right)^2$, find the value of k. [2]

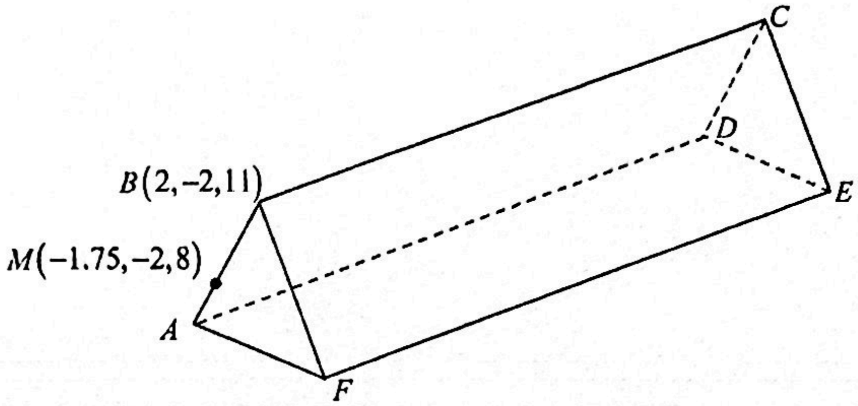
Hence

(i) show that

$$\sum_{n=1}^{N}\left(a_n - a_{n+1}\right)^2 = \frac{3}{5}a_1^2\left(1 - \frac{1}{16^N}\right), \text{ for } N \in \mathbb{Z}^+, N \geq 1,$$ [3]

(ii) find $\displaystyle\sum_{n=2}^{\infty}\left(a_n - a_{n+1}\right)^2$, leaving your answers in terms of a_1. [3]

11



The diagram shows a roof structure modelled by a prism $ABCDEF$ with parallelogram sides. The triangular ends ABF and DCE are similar and parallel to the xz plane. The origin O is a point on the horizontal ground which is parallel to the xy plane. The coordinates of point B and M are $(2,-2,11)$ and $(-1.75,-2,8)$ respectively. It is given that plane $ABCD$ has equation $48x+5y-60z=-574$ and plane $BCEF$ has equation $12x-y+12z=158$.

(i) Find the acute angle between plane $ABCD$ and plane $BCEF$, giving your answer correct to 2 decimal places. [2]

(ii) Without the use of the calculator, show that the equation of line BC is $x=2,\ \dfrac{y+2}{12}=z-11$. [2]

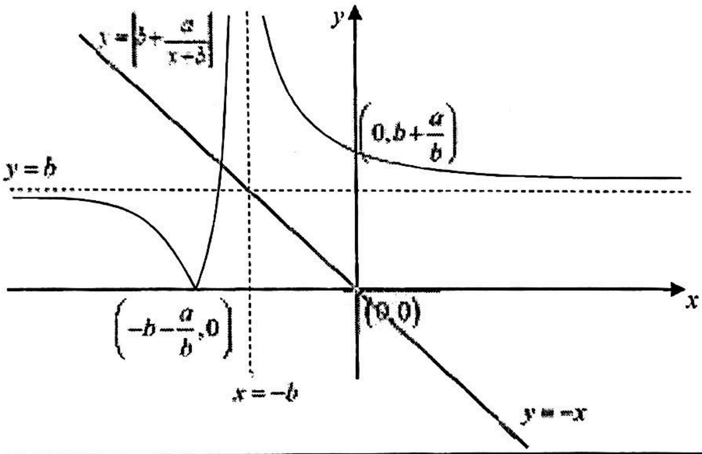
(iii) It is now given that the vertical height of point C from the horizontal ground is 11.25 metres. State the coordinates of point C. [2]

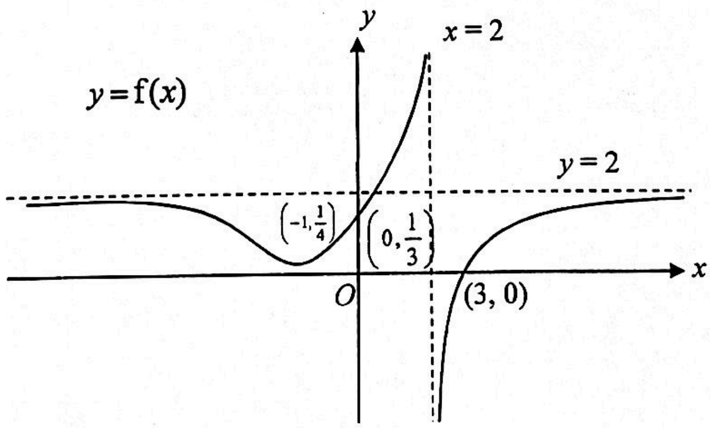
(iv) Find the acute angle between the line BC and the horizontal ground, giving your answer correct to 2 decimal places. [2]

(v) Given that point N lies on BF such that MN is parallel to the x-axis, find the length of MN. [3]

(vi) To strengthen the roof structure and minimise material cost, a steel beam MW is constructed such that point W lies on plane $BCEF$ and is closest to point M. Find the length of the steel beam MW. [3]

ACJC 2023 H2 Math JC1 Promo Solutions

Qn	Solutions
1 (i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\sqrt{x+1}\right)$ $=\frac{1}{1+\left(\sqrt{x+1}\right)^2}\left(\frac{1}{2\sqrt{x+1}}\right)$ $=\frac{1}{2(x+2)\left(\sqrt{x+1}\right)}$
(ii)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(x+e^x\right)-x\ln 3\right)$ $=\frac{1+e^x}{x+e^x}-\ln 3$
2 (i)	
2 (ii)	To find intersection points, $-x=-\left(b+\frac{a}{x+b}\right)$ $x-b=\frac{a}{x+b} \Rightarrow (x-b)(x+b)=a$ $x^2-b^2=a$ $x^2=a+b^2$ Since $x<0, x=-\sqrt{a+b^2}$ Solve

	$-x \le \left\lvert b+\dfrac{a}{x+b}\right\rvert$ $-\sqrt{a+b^2} \le x < -b$ or $x > -b$ Alternative presentation $x \ge -\sqrt{a+b^2}$, $x \ne -b$
2 iii	Given that $a=1,\ b=2$, $e^x \le \left\lvert b+\dfrac{a}{-e^x+b}\right\rvert$ Replace x by $-e^x$, $-\sqrt{5} \le -e^x < -2$ or $-e^x > -2$ $2 < e^x \le \sqrt{5}$ or $e^x < 2$ $\ln 2 < x \le \ln\sqrt{5}$ or $x < \ln 2$ Alternative presentation $x \le \ln\sqrt{5}$, $x \ne \ln 2$
3 (a)	From $\dfrac{(x-2)^2}{4}+y^2=1$ **A**: A stretch with scale factor $\dfrac{1}{2}$ parallel to the y-axis. $\dfrac{(x-2)^2}{4}+(2y)^2=1 \Rightarrow \dfrac{(x-2)^2}{4}+4y^2=1$ **B** : A translation of 5 units in the positive of x-direction. $\dfrac{(x-5-2)^2}{4}+4y^2=1 \Rightarrow \dfrac{(x-7)^2}{4}+4y^2=1$ **C**: A reflection about y-axis. $\dfrac{(-x-7)^2}{4}+4y^2=1 \Rightarrow \dfrac{(x+7)^2}{4}+4y^2=1$
3b (i)	

5a (i)

Let $y = \mathrm{f}(x)$

$$y = \mathrm{e}^{x+3} - a$$

$$\mathrm{e}^{x+3} = y + a$$

$$x = \ln(y+a) - 3$$

$$\mathrm{f}^{-1}(x) = \ln(x+a) - 3$$

$$\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_{\mathrm{f}} = (-a,\ \mathrm{e}^{7} - a]$$

5a (ii)

$$\mathrm{g}^{-1}\mathrm{f}(-3) = 15$$

$$\mathrm{f}(-3) = \mathrm{g}(15)$$

$$\mathrm{e}^{-3+3} - a = 2(15) + 5$$

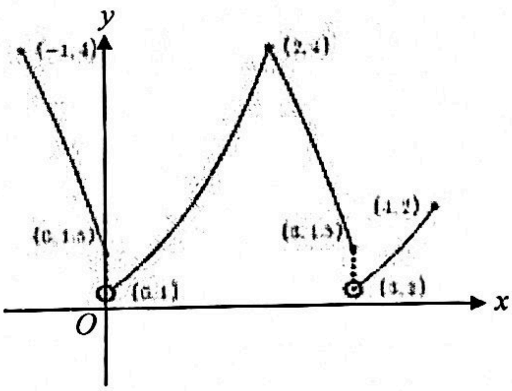
$$1 - a = 35$$

$$a = -34$$

5b (i)

$$\begin{aligned} \mathrm{h}(9) &= \mathrm{h}(6+3) = \mathrm{h}(6) \\ &= \mathrm{h}(3+3) = \mathrm{h}(3) \\ &= 6 - \frac{1}{2}(3)^2 = \frac{3}{2} \end{aligned}$$

5b (ii)



6i	Using Ratio Theorem, $\overrightarrow{OA}=\dfrac{\lambda\mathbf{q}+(1-\lambda)\mathbf{p}}{\lambda+(1-\lambda)}=\lambda\mathbf{q}+(1-\lambda)\mathbf{p}$
6ii	$\overrightarrow{OB}=\frac{1}{3}\mathbf{q}$ $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$ $=\frac{1}{3}\mathbf{q}-[\lambda\mathbf{q}+(1-\lambda)\mathbf{p}]=\frac{1}{3}\mathbf{q}-\lambda\mathbf{q}-(1-\lambda)\mathbf{p}$ $=\left(\frac{1}{3}-\lambda\right)\mathbf{q}+(\lambda-1)\mathbf{p}$ Since $\overrightarrow{AB}$ is parallel to $(\mathbf{q}-4\mathbf{p})$, $\therefore \overrightarrow{AB}=m(\mathbf{q}-4\mathbf{p})$, where m is a scalar. $\left(\frac{1}{3}-\lambda\right)\mathbf{q}+(\lambda-1)\mathbf{p}=m(\mathbf{q}-4\mathbf{p})$ $\left(\frac{1}{3}-\lambda\right)\mathbf{q}+(\lambda-1)\mathbf{p}=m\mathbf{q}-4m\mathbf{p}$ Since vector $\mathbf{p}$ and vector $\mathbf{q}$ are not parallel, by comparison, $\frac{1}{3}-\lambda=m$ ----------(1) $\lambda-1=-4m$ ----------(2) Solving (1) & (2): $\lambda=\frac{1}{9}$.

6iii	**Area of** $\Delta OAQ = \frac{1}{2}\left\|\overline{OA} \times \overline{OQ}\right\|$ $= \frac{1}{2}\|\mathbf{a} \times \mathbf{q}\|$ $= \frac{1}{2}\left\|\left[\lambda\mathbf{q} + (1-\lambda)\mathbf{p}\right] \times \mathbf{q}\right\|$ $= \frac{1}{2}\left\|\lambda\mathbf{q} \times \mathbf{q} \;+\; (1-\lambda)\mathbf{p} \times \mathbf{q}\right\|$ $= \frac{1}{2}\left\|\mathbf{0} \;+\; (1-\lambda)\|\mathbf{p}\|\|\mathbf{q}\|\sin 90°\right\|$ $= \frac{1}{2}\left\|(1-\lambda)(1)(2)(1)\right\|$ $= \|1-\lambda\|$ $= 1-\lambda.$ **(since** $0 < \lambda < 1$**)**
7 (a)	$x = 4\cos\theta$ $\frac{dx}{d\theta} = -4\sin\theta$ $\frac{d\theta}{dt} = \frac{d\theta}{dx}.\frac{dx}{dt} = -\frac{1}{4\sin\theta}.(15)$ At $\left(2, \frac{3\sqrt{3}}{2}\right)$, $\theta = \frac{\pi}{3}$ $\frac{d\theta}{dt} = -\frac{1}{4\sin\frac{\pi}{3}}.(15) = -\frac{5\sqrt{3}}{2} = -\frac{15}{2\sqrt{3}}$
7 (b)	$\frac{dy}{dx} = -\frac{3\cos\theta}{4\sin\theta} = -\frac{3}{4}\cot\theta$ At point $P(4\cos p, 3\sin p)$, $\theta = p$, $y - 3\sin p = -\frac{3}{4}\cot p(x - 4\cos p)$ $y = \left(-\frac{3}{4}\cot p\right)x + 3\sin p + 3\frac{\cos^2 p}{\sin p}$ $y = \left(-\frac{3}{4}\cot p\right)x + 3\left(\frac{\sin^2 p + \cos^2 p}{\sin p}\right)$ $y = \left(-\frac{3}{4}\cot p\right)x + 3\operatorname{cosec} p$

7 (c)	$A\left(\dfrac{4}{\cos p},0\right)$ $B\left(0,\dfrac{3}{\sin p}\right)$ Area of triangle $OAB=\dfrac{1}{2}\left(\dfrac{4}{\cos p}\right)\left(\dfrac{3}{\sin p}\right)=\dfrac{12}{\sin 2p}$ $=\dfrac{6}{\sin p\cos p}$
7 (d)	mid-point, $M\left(\dfrac{2}{\cos p},\dfrac{3}{2\sin p}\right)$ parametric equation: $x=\dfrac{2}{\cos p},\ y=\dfrac{3}{2\sin p}$ Since $\sin^2 p+\cos^2 p=1$, cartesian equation: $\left(\dfrac{3}{2y}\right)^2+\left(\dfrac{2}{x}\right)^2=1$
8 (a) (i)	$S_n=e^2n^2-3n$ $u_n=S_n-S_{n-1}$ $=e^2n^2-3n-\left[e^2(n-1)^2-3(n-1)\right]$ $=e^2n^2-3n-\left[e^2\left(n^2-2n+1\right)-3n+3\right]$ $=e^2n^2-3n-e^2n^2+2e^2n-e^2+3n-3$ $=e^2(2n-1)-3$ (shown)
8 (a) (ii)	$u_n-u_{n-1}=e^2(2n-1)-3-\left[e^2(2n-3)-3\right]$ $=2e^2$ is a constant Hence the sequence is an arithmetic progression. (shown)
8 (a) iii	$u_1=e^2-3$ $u_3=5e^2-3$ $u_5=9e^2-3$ Common difference $=4e^2$ $\dfrac{n}{2}\left[2\left(e^2-3\right)+(n-1)\left(4e^2\right)\right]>500$ Using GC table, When $n=6$, $\dfrac{n}{2}\left[2\left(e^2-3\right)+(n-1)\left(4e^2\right)\right]=469.68$

	When $n=7, \frac{n}{2}\left[2\left(e^2-3\right)+(n-1)\left(4e^2\right)\right]=651.4$ Least $n=7$
8 (b) (i)	Ratio $=r=\frac{-e^{\frac{1}{x}+1}}{e}=-e^{\frac{1}{x}}$ For geometric series to be convergent, $\lvert r\rvert<1 \Rightarrow \left\lvert -e^{\frac{1}{x}}\right\rvert<1 \Rightarrow -1<e^{\frac{1}{x}}<1$ $e^{\frac{1}{x}}<1$ $\frac{1}{x}<\ln 1$ $\frac{1}{x}<0$ $x<0$
8 (b) (ii)	Given that $x=-1, r=-e^{\frac{1}{-1}}=-\frac{1}{e}$ **<u>Method 1</u>** $S_\infty - S_7=\frac{a}{1-r}-\frac{a\left(1-r^7\right)}{1-r}$ $=\frac{ar^7}{1-r}$ $=\frac{e\left(-\frac{1}{e}\right)^7}{1-\left(-\frac{1}{e}\right)}$ $=-0.0018121$ $=-0.00181$ (to 3 s.f.)
	<u>Method 2</u> $u_8=ar^7=e\left(-\frac{1}{e}\right)^7=-\frac{1}{e^6}$

	Required sum $= \dfrac{u_8}{1-r}$ $= \dfrac{-\frac{1}{e^6}}{1-\left(-\frac{1}{e}\right)}$ $= -0.00181$ (to 3 s.f.)
9 **(a)**	$A = x^2 + \left(\frac{1}{2}xh\right)4$ $h = \dfrac{A-x^2}{2x}$

9 (b)	$H^2 = h^2 - \left(\frac{x}{2}\right)^2$ Volume, V $= \frac{1}{3}x^2 H$ $= \frac{1}{3}x^2 \sqrt{h^2 - \left(\frac{x}{2}\right)^2}$ $= \frac{1}{3}x^2 \sqrt{\left(\frac{A - x^2}{2x}\right)^2 - \left(\frac{x}{2}\right)^2}$ $= \frac{1}{3}x^2 \sqrt{\left(\frac{A - x^2}{2x}\right)^2 - \left(\frac{x}{2}\right)^2}$ $= \frac{1}{3}x^2 \sqrt{\frac{A^2 - 2Ax^2 + x^4 - x^4}{4x^2}}$ $V = \frac{1}{3}x^2 \sqrt{\frac{A^2 - 2Ax^2}{4x^2}}$ $V^2 = \frac{1}{9}x^4 \left(\frac{A^2 - 2Ax^2}{4x^2}\right)$ $36V^2 = A^2x^2 - 2Ax^4$

9 (c)	$72V\dfrac{dV}{dx} = A^2(2x) - 2A(4x^3)$ Since V is maximum, $\dfrac{dV}{dx} = 0$ $A^2(2x) - 2A(4x^3) = 0$ $A = 4x^2$ $x = \sqrt{\dfrac{A}{4}}$ or $x = -\sqrt{\dfrac{A}{4}}$ (rej) $36V^2 = A^2x^2 - 2Ax^4$ $V = \sqrt{\dfrac{A^2x^2 - 2Ax^4}{36}}$ $= \sqrt{\dfrac{A^2\left(\dfrac{A}{4}\right) - 2A\left(\dfrac{A^2}{16}\right)}{36}}$ $= \sqrt{\dfrac{A^3}{288}}$

10 (a)	$$\sum_{n=1}^{N}\left[\ln\left(\frac{a_{n+1}}{a_n}\right)\right]=\sum_{n=1}^{N}\left[\ln\left(a_{n+1}\right)-\ln\left(a_n\right)\right]$$ $$=\begin{bmatrix} \ln a_2 - \ln a_1 \\ +\ln a_3 - \ln a_2 \\ +\ln a_4 - \ln a_3 \\ \vdots \\ +\ln a_N - \ln a_{N-1} \\ +\ln a_{N+1} - \ln a_N \end{bmatrix}$$ $$=\ln\left(\frac{a_{N+1}}{a_1}\right)$$
	Alternatively, $\sum_{n=1}^{N}\left[\ln\left(\frac{a_{n+1}}{a_n}\right)\right]=\ln\left(\frac{a_2}{a_1}\right)+\ln\left(\frac{a_3}{a_2}\right)+\ln\left(\frac{a_4}{a_3}\right)+\ldots+\ln\left(\frac{a_N}{a_{N-1}}\right)+\ln\left(\frac{a_{N+1}}{a_N}\right)$ $$=\ln\left(\frac{a_2}{a_1}\times\frac{a_3}{a_2}\times\frac{a_4}{a_3}\times\ldots\times\frac{a_N}{a_{N-1}}\times\frac{a_{N+1}}{a_N}\right)$$ $$=\ln\left(\frac{a_{N+1}}{a_1}\right)$$
10 b	$$\sum_{n=1}^{N}\left(a_n-a_{n+1}\right)^2$$ $$=\sum_{n=1}^{N}\left(a_n-\frac{1}{4}a_n\right)^2$$ $$=\sum_{n=1}^{N}\left(\frac{3}{4}a_n\right)^2$$ $$=\frac{9}{16}\sum_{n=1}^{N}\left(a_n\right)^2$$ $k=9$

3b (ii)	$x = 3$ $y = \frac{7}{2}$
4 (i)	$y^2 + 3xy = 13 + x^2$ ------(1) $2y\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y - 2x = 0$ $\frac{dy}{dx} = \frac{2x - 3y}{3x + 2y}$ when $\frac{dy}{dx} = 0$, $y = \frac{2x}{3}$ ----(2) sub into (1), we have, $\left(\frac{2x}{3}\right)^2 + 3x\left(\frac{2x}{3}\right) = 13 + x^2$ $13x^2 = 117$ $x = 3$ or $x = -3$ sub into (2), $y = 2$ or $y = -2$ Stationary points: (3,2), $(-3,-2)$.
4 (ii)	$\frac{dy}{dx} = \frac{2x - 3y}{3x + 2y}$ $\frac{d^2y}{dx^2} = \frac{\left(2 - 3\frac{dy}{dx}\right)(3x + 2y) - \left(3 + 2\frac{dy}{dx}\right)(2x - 3y)}{(3x + 2y)^2}$ sub (3,2) and $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = \frac{26}{169} = \frac{2}{13} > 0$ (3,2) is a minimum point.

10 b (i)	$\sum_{n=1}^{N}(a_n-a_{n+1})^2=\frac{9}{16}\sum_{n=1}^{N}(a_n)^2$ $=\frac{9}{16}\left[a_1^2+a_2^2+a_3^2+\ldots+a_N^2\right]$ $=\frac{9}{16}\left[a_1^2+\left(a_1\left(\frac{1}{4}\right)\right)^2+\left(a_1\left(\frac{1}{4}\right)^2\right)^2+\ldots+\left(a_1\left(\frac{1}{4}\right)^{N-1}\right)^2\right]$ $=\frac{9}{16}(a_1^2)\left[1+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^4+\ldots+\left(\frac{1}{4}\right)^{2N-2}\right]$ $=\frac{9}{16}(a_1^2)\frac{(1)\left(1-\left[\left(\frac{1}{4}\right)^2\right]^N\right)}{1-\left(\frac{1}{4}\right)^2}$ $=\frac{9}{15}(a_1^2)\left[1-\left(\frac{1}{4}\right)^{2N}\right]$ $=\frac{3}{5}a_1^2\left[1-\frac{1}{16^N}\right]$
10 b (ii)	As $N\to\infty$, $\frac{1}{16^N}\to 0$, $\quad\therefore \sum_{n=1}^{N}(a_n-a_{n+1})^2\to\frac{3}{5}(a_1^2)$ $\therefore \sum_{n=1}^{\infty}(a_n-a_{n+1})^2=\frac{3}{5}a_1^2$ $\sum_{n=2}^{\infty}(a_n-a_{n+1})^2=\sum_{n=1}^{\infty}(a_n-a_{n+1})^2-\sum_{n=1}^{1}(a_n-a_{n+1})^2$ $=\sum_{n=1}^{\infty}(a_n-a_{n+1})^2-\left[a_1-a_2\right]^2$ $=\frac{3}{5}a_1^2-\left[a_1-\frac{1}{4}a_1\right]^2$ $=\frac{3}{5}a_1^2-\left[\frac{3}{4}a_1\right]^2$ $=\frac{3}{5}a_1^2-\frac{9}{16}a_1^2$ $=\frac{3}{80}a_1^2$
11	Let θ be the acute angle between plane $ABCD$ and plane $BCEF$.

	Plane $ABCD$: $\mathbf{r}\cdot\begin{pmatrix}48\\5\\-60\end{pmatrix}=-574$. Plane $BCEF$: $\mathbf{r}\cdot\begin{pmatrix}12\\-1\\12\end{pmatrix}=158$. $\cos\theta=\dfrac{\left\|\begin{pmatrix}48\\5\\-60\end{pmatrix}\cdot\begin{pmatrix}12\\-1\\12\end{pmatrix}\right\|}{\sqrt{48^2+5^2+(-60)^2}\sqrt{12^2+(-1)^2+12^2}}$ $\cos\theta=\dfrac{\|-149\|}{\sqrt{5929}\sqrt{289}}$ $\cos\theta=\dfrac{149}{(77)(17)}$ $\theta=\cos^{-1}\left(\dfrac{149}{1309}\right)$ $\theta=83.5^{\circ}$ (1d.p)
11 **ii**	Finding direction vector of line BC: $\begin{pmatrix}48\\5\\-60\end{pmatrix}\times\begin{pmatrix}12\\-1\\12\end{pmatrix}=\begin{pmatrix}60-60\\-(576+720)\\-48-60\end{pmatrix}=\begin{pmatrix}0\\-1296\\-108\end{pmatrix}=-108\begin{pmatrix}0\\12\\1\end{pmatrix}$ Line BC: $\mathbf{r}=\begin{pmatrix}2\\-2\\11\end{pmatrix}+\lambda\begin{pmatrix}0\\12\\1\end{pmatrix}$, where $\lambda\in\mathbb{R}$. Let $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$,

	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 12 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2+12\lambda \\ 11+\lambda \end{pmatrix} \Rightarrow \begin{matrix} x=2 \\ \lambda = \frac{y+2}{12} \\ \lambda = z-11 \end{matrix}$ Hence line *BC* is $x = 2,\ \frac{y+2}{12} = z-11$.
11 iii	Line *BC*: $x = 2,\ \frac{y+2}{12} = z-11$ When $z = 11.25$, $\frac{y+2}{12} = 11.25-11$ $y = 1$ Hence $C(2,\ 1,\ 11.25)$.
11 iv	Let α be the acute angle between horizontal ground and line *BC*. <u>Method 1:</u> Normal vector of horizontal ground is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$$\sin\alpha = \frac{\begin{pmatrix}0\\12\\1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{0^2+12^2+1^2}\sqrt{1^2}}$$

$$\sin\alpha = \frac{1}{\sqrt{145}}$$

$$\alpha = \sin^{-1}\left(\frac{1}{\sqrt{145}}\right)$$

$$\alpha = 4.76^\circ \text{ (2d.p)}$$

Method 2:

$$\cos\left(90^\circ - \alpha\right) = \frac{\begin{pmatrix}0\\12\\1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{145}\sqrt{1^2}}$$

$$90^\circ - \alpha = \cos^{-1}\frac{1}{\sqrt{145}}$$

$$\alpha = 90^\circ - \cos^{-1}\frac{1}{\sqrt{145}} = 4.76^\circ \text{ (2d.p)}$$

Method 3:

$$\alpha = \tan^{-1}\left(\frac{1}{12}\right)$$

$$= 4.76^\circ \text{ (2 d.p)}$$

11
v

Line *MN*: $\mathbf{r} = \begin{pmatrix} -1.75 \\ -2 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, where $\mu \in \square$. -------- (1)

Plane *BCEF*: $\mathbf{r} \square \begin{pmatrix} 12 \\ -1 \\ 12 \end{pmatrix} = 158$. --------- (2)

Since *N* lies on the line *MN*,

$$\overrightarrow{ON} = \begin{pmatrix} -1.75 \\ -2 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ for } \underline{\text{some}} \ \mu \in \square.$$

Since *N* also lies on the plane *BCEF*, we must have

$$\begin{pmatrix} -1.75 + \mu \\ -2 \\ 8 \end{pmatrix} \square \begin{pmatrix} 12 \\ -1 \\ 12 \end{pmatrix} = 158$$

$$-21 + 12\mu + 2 + 96 = 158$$

$$\mu = \frac{81}{12} = 6.75$$

Sub. $\mu = \frac{81}{12} = 6.75$ into (1):

$$\overrightarrow{ON} = \begin{pmatrix} -1.75 \\ -2 \\ 8 \end{pmatrix} + (6.75) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 8 \end{pmatrix}$$

$$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \begin{pmatrix} 5 \\ -2 \\ 8 \end{pmatrix} - \begin{pmatrix} -1.75 \\ -2 \\ 8 \end{pmatrix} = \begin{pmatrix} 6.75 \\ 0 \\ 0 \end{pmatrix}$$

$\left|\overrightarrow{MN}\right| = 6.75.$

<u>Alternative method:</u>

$$\overrightarrow{MN} = k\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\overrightarrow{ON} - \overrightarrow{OM} = k\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\overrightarrow{ON} - \begin{pmatrix}-1.75\\-2\\8\end{pmatrix} = k\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\overrightarrow{ON} = \begin{pmatrix}-1.75+k\\-2\\8\end{pmatrix} \text{ ----------------(3)}$$

Sub (3) into (2):

$$\begin{pmatrix}-1.75+k\\-2\\8\end{pmatrix} \square \begin{pmatrix}12\\-1\\12\end{pmatrix} = 158$$

$$-21+12k+2+96=158$$

$$k = 6.75$$

$$\left|\overrightarrow{MN}\right| = \left|k\begin{pmatrix}1\\0\\0\end{pmatrix}\right| = \left|6.75\begin{pmatrix}1\\0\\0\end{pmatrix}\right| = 6.75.$$

11
vi

$$\overrightarrow{BM} = \overrightarrow{OM} - \overrightarrow{OB} = \begin{pmatrix}-1.75\\-2\\8\end{pmatrix} - \begin{pmatrix}2\\-2\\11\end{pmatrix} = \begin{pmatrix}-3.75\\0\\-3\end{pmatrix}$$

$$WM = \left|\overrightarrow{BM} \square \frac{\begin{pmatrix}12\\-1\\12\end{pmatrix}}{\left|\begin{pmatrix}12\\-1\\12\end{pmatrix}\right|}\right| = \left|\frac{\begin{pmatrix}-3.75\\0\\-3\end{pmatrix} \square \begin{pmatrix}12\\-1\\12\end{pmatrix}}{\left|\begin{pmatrix}12\\-1\\12\end{pmatrix}\right|}\right|$$

$$= \frac{\left|-45+0-36\right|}{\sqrt{12^2+(-1)^2+12^2}} = \frac{81}{\sqrt{289}} = \frac{81}{17} = 4.76$$

Alternative method

Since W lies on the line MW which must be perpendicular to the plane $BCEF$, we have

Line MW: $\mathbf{r}=\begin{pmatrix}-1.75\\-2\\8\end{pmatrix}+\mu\begin{pmatrix}12\\-1\\12\end{pmatrix}$ for some $\mu\in\square$.

Since W lies on the line MW,

$\therefore\overrightarrow{OW}=\begin{pmatrix}-1.75\\-2\\8\end{pmatrix}+\mu\begin{pmatrix}12\\-1\\12\end{pmatrix}$ for some $\mu\in\square$.

Since W also lies on the plane $BCEF$, we must have

$$\begin{pmatrix}-1.75+12\mu\\-2-\mu\\8+12\mu\end{pmatrix}\square\begin{pmatrix}12\\-1\\12\end{pmatrix}=158$$

$$-21+144\mu+2+\mu+96+144\mu=158$$

$$\mu=\frac{81}{289}$$

Length of steel beam

$=\left|\overrightarrow{MW}\right|$

$=\left|\overrightarrow{OW}-\overrightarrow{OM}\right|$

$$=\left|\mu\begin{pmatrix}12\\-1\\12\end{pmatrix}\right|=\frac{81}{289}\left|\begin{pmatrix}12\\-1\\12\end{pmatrix}\right|=\frac{81}{\sqrt{289}}=4.76$$