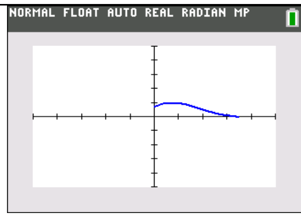
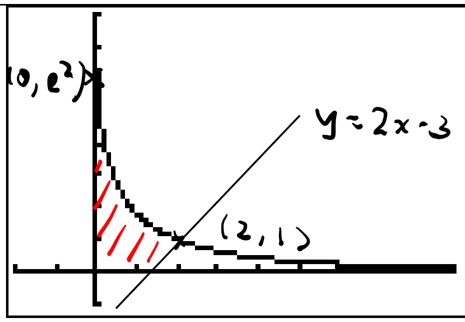
Applications of Integration (Solutions)

Qn	Suggested Solution
1(a)(i)	Area $R = \int_0^{\pi/4} \tan y \, dy + \frac{1}{\frac{\pi}{4}-1}\int_{\pi/4}^{1}(y-1)\,dy$ **Or** Area $R = \int_0^{\pi/4} \tan y \, dy + \frac{1}{2}\left(1-\frac{\pi}{4}\right)(1)$ **Or** Area $R = \int_0^1 \left(\frac{\pi}{4}-1\right)x + 1 \, dx - \int_0^1 \tan^{-1} x \, dx$ **Or** Area $R = \frac{1}{2}\left(1+\frac{\pi}{4}\right)(1) - \int_0^1 \tan^{-1} x \, dx$
	$\int_0^{\pi/4} \tan y \, dy + \frac{1}{2}(1)\left(1-\frac{\pi}{4}\right)$ $= \left[\ln\lvert\sec y\rvert\right]_0^{\pi/4} + \frac{1}{2}\left(1-\frac{\pi}{4}\right)$ $= \ln\sqrt{2} + \frac{1}{2}\left(1-\frac{\pi}{4}\right)$ $= \frac{1}{2}\ln 2 + \frac{4-\pi}{8}$ (shown) **OR** $\frac{1}{2}\left(1+\frac{\pi}{4}\right)(1) - \int_0^1 \tan^{-1} x \, dx$ $= \frac{1}{2}\left(1+\frac{\pi}{4}\right) - \left\{\left[x\tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx\right\}$ $= \frac{1}{2}\left(1+\frac{\pi}{4}\right) - \left\{\frac{\pi}{4} - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1\right\}$ $= \frac{1}{2}\ln 2 + \frac{4-\pi}{8}$
(ii)	Volume generated by R abt y-axis $= \pi\int_0^{\pi/4}(\tan y)^2 dy + \pi\int_{\pi/4}^{1}\left(\frac{y-1}{\frac{\pi}{4}-1}\right)^2 dy$ $= 0.89892 \approx 0.899$ Or Volume generated by R abt y-axis $= \pi\int_0^{\pi/4}(\tan y)^2 dy + \frac{1}{3}\pi(1)^2\left(1-\frac{\pi}{4}\right)$ $= 0.89892 \approx 0.899$ Or Shell method : Volume

	$= 2\pi\int_0^1 x\left(\left(\frac{\pi}{4}-1\right)x+1-\tan^{-1}x\right)\mathrm{d}x$ $= 0.899$
(b)	 Area of required region $= \int_0^{\pi\ln\pi} y\,\mathrm{d}x$ $\qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \ln t + 1 \Rightarrow \mathrm{d}x = (\ln t+1)\,\mathrm{d}t$ $= \int_1^{\pi}(\sin^2 t)(\ln t+1)\,\mathrm{d}t$ $\qquad x=0 : t=1;\ x=\pi\ln\pi : t=\pi$ $= 1.9786$ ≈ 1.98

Qn	Suggested Solution
2(a)	$\int x\mathrm{e}^{-x}\mathrm{d}x = -x\mathrm{e}^{-x} - \int -\mathrm{e}^{-x}\mathrm{d}x$ $= -x\mathrm{e}^{-x} - \mathrm{e}^{-x} + C$ $= -\mathrm{e}^{-x}(x+1) + C$
(b)(i)	$x = \frac{(t+2)^2}{2}, \quad y = \mathrm{e}^{-t}$ $\frac{\mathrm{d}x}{\mathrm{d}t} = t+2, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -\mathrm{e}^{-t}$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$ $= \frac{-1}{\mathrm{e}^{t}(t+2)}$ When $t = 0$, $x = 2, y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}$ So, gradient of normal = 2 Hence, equation of normal at $t = 0$: $y-1 = 2(x-2) \Rightarrow y = 2x-3$

<table>
<tr><td>(ii)</td><td>


Required Area

$$= \int_0^2 y \, dx - \frac{1}{2}\left(\frac{1}{2}\right)(1)$$

$$= \int_{-2}^{0} y\left(\frac{dx}{dt}\right) dt - \frac{1}{4}$$

$$= \int_{-2}^{0} e^{-t}(t+2)\, dt - \frac{1}{4}$$

$$= \int_{-2}^{0} te^{-t} dt + 2\int_{-2}^{0} e^{-t} dt - \frac{1}{4}$$

$$= \left[-e^{-t}(t+1)\right]_{-2}^{0} - 2\left[e^{-t}\right]_{-2}^{0} - \frac{1}{4} \quad \text{using result in (a)}$$

$$= \left[-1 + e^{2}(-1)\right] - 2\left[1 - e^{2}\right] - \frac{1}{4}$$

$$= \left(e^{2} - \frac{13}{4}\right) \text{ units}^2$$
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<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td>3(a)</td><td>

$$\int_1^4 \frac{|x-2|}{x} dx = \int_1^2 \frac{-(x-2)}{x} dx + \int_2^4 \frac{(x-2)}{x} dx$$

If $x \geq 2$, then $|x-2| = (x-2)$

If $x > 2$, then $|x-2| = -(x-2)$

$$= -\int_1^2 \left(1 - \frac{2}{x}\right) dx + \int_2^4 \left(1 - \frac{2}{x}\right) dx$$

$$= -\left[x - 2\ln x\right]_1^2 + \left[x - 2\ln x\right]_2^4$$

$$= -\left(2 - 2\ln 2 - 1\right) + [4 - 2\ln 4 - (2 - 2\ln 2)]$$

$$= 1$$
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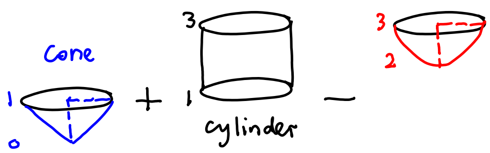
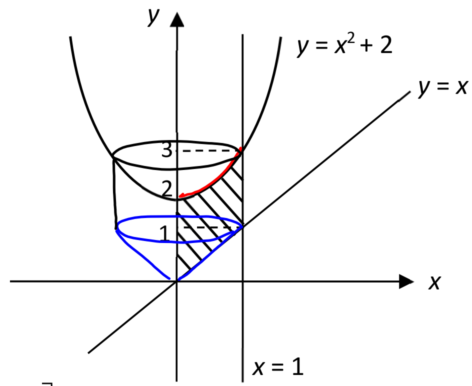
(b)





Volume of solid generated

$$= \frac{1}{3}\pi(1)^2(1) + \left[\pi(1)^2(2) - \pi\int_2^3 (y-2)\,\mathrm{d}y\right]$$

$$= \frac{7}{3}\pi - \left[\frac{y^2}{2} - 2y\right]_2^3$$

$$= \frac{7}{3}\pi - \frac{1}{2}\pi$$

$$= \frac{11}{6}\pi$$

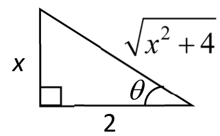
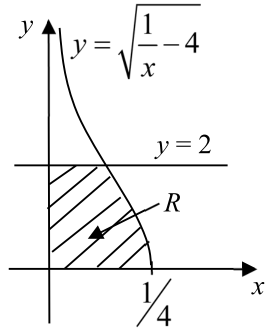
Alternative : Shell method

$$\text{Volume } = 2\pi\int_0^1 x\left(x^2+2-x\right)\,\mathrm{d}x$$

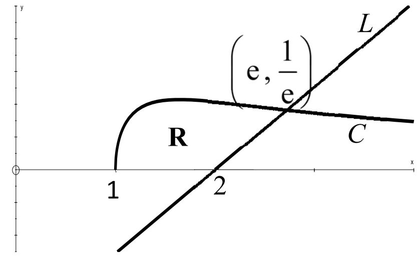
$$= 2\pi\int_0^1 \left(x^3+2x-x^2\right)\,\mathrm{d}x$$

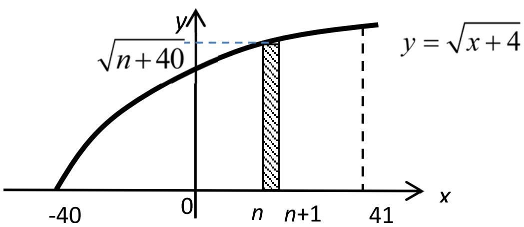
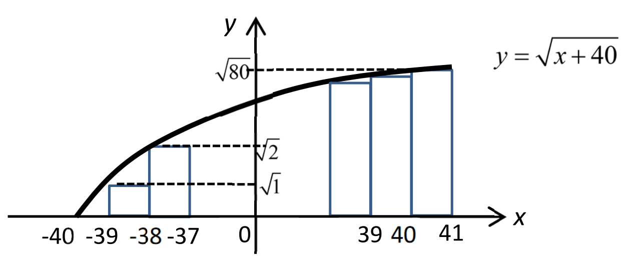
$$= 2\pi\left[\frac{x^4}{4} + x^2 - \frac{x^3}{3}\right]_0^1$$

$$= 2\pi\left[\frac{1}{4} + 1 - \frac{1}{3}\right] = \frac{11}{6}\pi$$

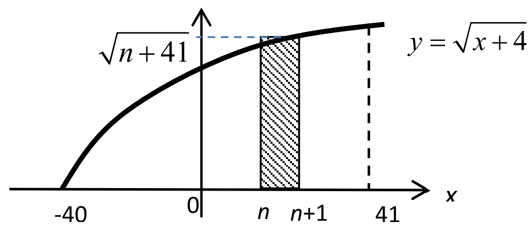
Qn	Suggested Solution
4(i)	$\int \frac{1}{(4+x^2)^2}\,dx$ $= \int \frac{1}{(4+4\tan^2\theta)^2}\,2\sec^2\theta\ d\theta$ $= \int \frac{2\sec^2\theta}{16(1+\tan^2\theta)^2}\,d\theta$ $= \frac{1}{8}\int \frac{1}{\sec^2\theta}\,d\theta$ $= \frac{1}{8}\int \cos^2\theta\ d\theta$ $= \frac{1}{16}\int (\cos 2\theta + 1)\,d\theta$ $= \frac{1}{16}\left(\frac{\sin 2\theta}{2} + \theta\right) + C$ $= \frac{1}{16}(\sin\theta\cos\theta + \theta) + C$ $= \frac{1}{16}\left(\frac{2x}{x^2+4} + \tan^{-1}\frac{x}{2}\right) + C$ Let $x = 2\tan\theta$ $\frac{dx}{d\theta} = 2\sec^2\theta \Rightarrow dx = 2\sec^2\theta d\theta$ $\tan\theta = \frac{x}{2}$  $\sin\theta = \frac{x}{\sqrt{x^2+4}}$ $\cos\theta = \frac{2}{\sqrt{x^2+4}}$
(ii)	Required Volume $= \pi\int_0^2 \frac{1}{(y^2+4)^2}\,dy$ $= \frac{\pi}{16}\left[\frac{2y}{y^2+4} + \tan^{-1}\left(\frac{y}{2}\right)\right]_0^2$ use (a) $= \frac{\pi}{16}\left(\frac{1}{2} + \tan^{-1}(1)\right)$ $= \frac{\pi}{16}\left(\frac{1}{2} + \frac{\pi}{4}\right)$ $= \frac{\pi}{64}(2+\pi)$ units^3 

Qn	Suggested Solution
5(a)	$\int \frac{6+2x}{\sqrt{1-4x-x^2}}\,dx = \int \frac{2-(-4-2x)}{\sqrt{1-4x-x^2}}\,dx$ $= \int \frac{2}{\sqrt{1-4x-x^2}}\,dx - \int \frac{(-4-2x)}{\sqrt{1-4x-x^2}}\,dx$ $= \int \frac{2}{\sqrt{5-(x+2)^2}}\,dx - \int \frac{-4-2x}{\sqrt{1-4x-x^2}}\,dx$ $= 2\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) - 2\sqrt{1-4x-x^2} + c$

(b)	Volume $= \pi\int_1^{e}\left(\frac{\sqrt{\ln x}}{x}\right)^2 dx - \frac{\pi}{3}\left(\frac{1}{e}\right)^2(e-2)$ $= \pi\int_1^{e}\frac{\ln x}{x^2}dx - \frac{\pi(e-2)}{3e^2}$ $= \pi\left[(\ln x)\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right)\frac{1}{x}dx\right]_1^{e} - \frac{\pi(e-2)}{3e^2}$ $= \pi\left[\left(-\frac{\ln x}{x}\right) - \frac{1}{x}\right]_1^{e} - \frac{\pi(e-2)}{3e^2}$ $= \pi\left[1-\frac{2}{e}\right] - \frac{\pi(e-2)}{3e^2}$ $= \pi - \frac{2\pi}{e} - \frac{\pi}{3e} + \frac{2\pi}{3e^2} = \pi\left(1 - \frac{7}{3e} + \frac{2}{3e^2}\right)$ 

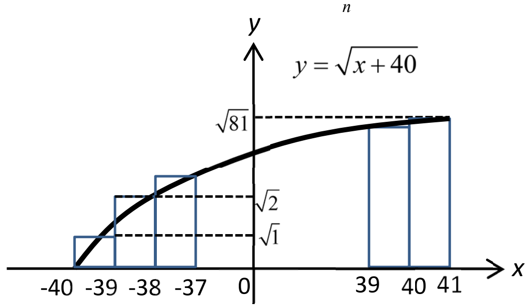
Qn	Suggested Solution
6	 area of shaded rectangle < Area under the curve from $x = n$ to $x = n+1$ hence $1 \text{ x } \sqrt{n+40} < \int_n^{n+1}\sqrt{x+40}\,dx$  Drawing similar triangles as shown above, Sum of area of rectangles < actual area under curve $\sqrt{0}\times1 + \sqrt{1}\times1 + \sqrt{2}\times1 + ... + \sqrt{80}\times1 < \int_{-40}^{41}\sqrt{x+40}\,dx$ Hence $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{80} < \int_{-40}^{41}\sqrt{x+40}\,dx$ --- (1)

Draw rectangle as shown below:



area of shaded rectangle $>$ Area under the curve from $x = n$ to $x = n+1$

hence 1 x $\sqrt{n+41} > \int_{n}^{n+1} \sqrt{x+40}\ \mathrm{d}x$



By drawing rectangles as shown above,
Sum of area of rectangles > actual area under curve

$$\sqrt{1}\times 1+\sqrt{2}\times 1+....+\sqrt{81}\times 1 > \int_{-40}^{41} \sqrt{x+40}\ \mathrm{d}x$$

Hence $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{81} > \int_{-40}^{41} \sqrt{x+40}\ \mathrm{d}x$ --- (2)

$$\int_{-40}^{41} \sqrt{x+40}\ \mathrm{d}x = \left[\frac{2}{3}(x+40)^{\frac{3}{2}}\right]_{-40}^{41} \quad \left[\frac{2}{3}(x+40)^{\frac{3}{2}}\right]_{-40}^{41} = 486 \text{ (or use GC)}$$

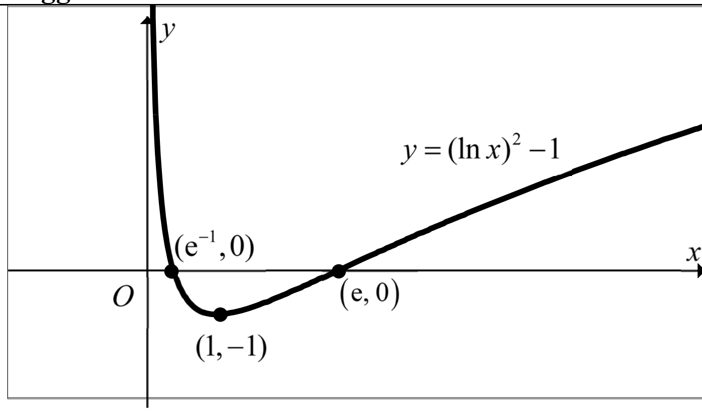
From (1) and (2),

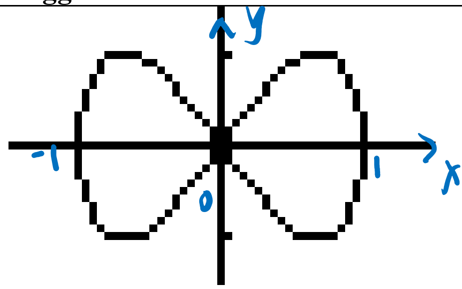
$$\int_{-40}^{41} \sqrt{x+40}\ \mathrm{d}x - \sqrt{81} < \sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{80} < \int_{-40}^{41} \sqrt{x+40}\ \mathrm{d}x$$

$$477 < \sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{80} < 486$$

$$9(53) < \sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{80} < 9(54)$$

$a = 53$

Qn	Suggested Solution
7(i)	 To obtain x-intercepts, let $y=0$ $(\ln x)^2=1$ $\ln x=\pm 1$ $x=e^1$ or e^{-1} To obtain the turning point, find $\frac{dy}{dx}=2\ln x$. Let $\frac{dy}{dx}=0 \Rightarrow 2\ln x=0 \Rightarrow x=1$ Thus coordinates of turning point is $(1,-1)$.
(ii)	Area of region R $=\int_{e^{-1}}^{e} -\left((\ln x)^2-1\right)\,dx$ $=-\left[x(\ln x)^2\right]_{e^{-1}}^{e}+\int_{e^{-1}}^{e} x\frac{2\ln x}{x}\,dx+\left[x\right]_{e^{-1}}^{e}$ $=-\left(e-e^{-1}\right)+2\left(\left[x\ln x\right]_{e^{-1}}^{e}-\int_{e^{-1}}^{e} 1\,dx\right)+\left(e-e^{-1}\right)$ $=-\left(e-e^{-1}\right)+2\left(\left(e+e^{-1}\right)-\left(e-e^{-1}\right)\right)+\left(e-e^{-1}\right)$ $=4e^{-1}$
(iii)	Make x the subject: $y=(\ln x)^2-1$ $\ln x=\pm\sqrt{y+1}$ $x=e^{\pm\sqrt{y+1}}$ Thus the volume obtained $=\pi\int_{-1}^{0}\left(e^{\sqrt{y+1}}\right)^2-\left(e^{-\sqrt{y+1}}\right)^2\,dy=12.2$ (to 3 s.f.) Alternative : Shell method Volume $=2\pi\int_{e^{-1}}^{e} x[1-(\ln x)^2]\,dx$ $=12.2$

Qn	Suggested Solution
8(i)	
(ii)	Since the figure is symmetrical, Area $= 4\int_0^1 y \ \mathrm{d}x \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = -\sin t \Rightarrow \mathrm{d}x = -\sin t\mathrm{d}t$ $= 4\int_{\frac{\pi}{2}}^0 (\sin 2t)(-\sin t) \ \mathrm{d}t \qquad x = 0, t = \frac{\pi}{2}; x = 1, t = 0$ $= 4\int_0^{\frac{\pi}{2}} \sin t \sin 2t \ \mathrm{d}t$ $= 8\int_0^{\frac{\pi}{2}} \sin^2 t \cos t \ \mathrm{d}t$ $= 8\left[\frac{\sin^3 t}{3}\right]_0^{\frac{\pi}{2}}$ $= \frac{8}{3}$
(iii)	Volume $= 2\pi\int_0^1 y^2 \ \mathrm{d}x$ $= 2\pi\int_{\frac{\pi}{2}}^0 \sin^2 2t \ (-\sin t) \ \mathrm{d}t$ refer to previous part on area $= 8\pi\int_0^{\frac{\pi}{2}} \sin^3 t \cos^2 t \ \mathrm{d}t$ $= 8\pi\int_0^{\frac{\pi}{2}} \sin t(1-\cos^2 t)\cos^2 t \ \mathrm{d}t$ $= 8\pi\left[-\frac{\cos^3 t}{3} + \frac{\cos^5 t}{5}\right]_0^{\frac{\pi}{2}}$ $= 8\pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{16}{15}\pi$

Qn	Suggested Solution
9	$s=\int_0^3\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ $=\int_0^3\sqrt{1+\left(\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{\frac{1}{2}}\right)^2}\,\mathrm{d}x$ $=\int_0^3\sqrt{1+\frac{1}{4x}-\frac{1}{2}+\frac{1}{4}x}\,\mathrm{d}x$ $=\int_0^3\sqrt{\frac{1}{4x}+\frac{1}{2}+\frac{1}{4}x}\,\mathrm{d}x$ $=\int_0^3\sqrt{\left(\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{\frac{1}{2}}\right)^2}\,\mathrm{d}x$ $=\int_0^3\left(\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{\frac{1}{2}}\right)\mathrm{d}x$ $=\left[x^{\frac{1}{2}}+\frac{1}{3}x^{\frac{3}{2}}\right]_0^3$ $=\sqrt{3}+\frac{1}{3}\left(3\sqrt{3}\right)=2\sqrt{3}$ (shown)
	$S=2\pi\int_0^3 y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ $=2\pi\int_0^3\left(x^{\frac{1}{2}}-\frac{1}{3}x^{\frac{3}{2}}+\lambda\right)\left(\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{\frac{1}{2}}\right)\mathrm{d}x$ $=\pi\int_0^3\left(1+x-\frac{1}{3}x-\frac{1}{3}x^2+\lambda x^{-\frac{1}{2}}+\lambda x^{\frac{1}{2}}\right)\mathrm{d}x$ $=\pi\left[2\lambda x^{\frac{1}{2}}+x+\frac{2}{3}\lambda x^{\frac{3}{2}}+\frac{1}{3}x^2-\frac{1}{9}x^3\right]_0^3$ $=\pi\left[2\lambda\sqrt{3}+3+2\lambda\sqrt{3}+3-3\right]$ $=\pi\left[4\lambda\sqrt{3}+3\right]$

Qn	Suggested Solution
10	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(1+\sqrt{1+x^2}\right)-\ln x\right]$ $=\frac{\frac{x}{\sqrt{1+x^2}}}{1+\sqrt{1+x^2}}-\frac{1}{x}$ $=\frac{x^2-\left[\sqrt{1+x^2}(1+\sqrt{1+x^2})\right]}{x\sqrt{1+x^2}(1+\sqrt{1+x^2})}$ $=\frac{x^2-\left[\sqrt{1+x^2}+1+x^2\right]}{x\sqrt{1+x^2}(1+\sqrt{1+x^2})}=-\frac{1}{x\sqrt{1+x^2}}, x>0$ (shown)
	Length of arc Length of arc $=\int_{\frac{1}{\sqrt{3}}}^{1}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ $=\int_{\frac{1}{\sqrt{3}}}^{1}\sqrt{1+\left(-\frac{1}{x}\right)^2}\,\mathrm{d}x$ $=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{\sqrt{1+x^2}}{x}\mathrm{d}x$ since $x>0$ $=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{1+x^2}{x\sqrt{1+x^2}}\,\mathrm{d}x$ $=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{1}{x\sqrt{1+x^2}}\,\mathrm{d}x+\int_{\frac{1}{\sqrt{3}}}^{1}\frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x$ $=\left[-\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)\right]_{\frac{1}{\sqrt{3}}}^{1}+\left[\sqrt{1+x^2}\right]_{\frac{1}{\sqrt{3}}}^{1}$ $=\left[-\ln(1+\sqrt{2})+(\ln\left(\frac{1+\sqrt{1+1/3}}{1/\sqrt{3}}\right)\right]+\left[\sqrt{2}-\sqrt{1+\frac{1}{3}}\right]$ $=\ln(\frac{2+\sqrt{3}}{1+\sqrt{2}})+\sqrt{2}-\frac{2}{3}\sqrt{3}$

Qn	Suggested Solution
11(i)	$x=\tan t-t,\ y=\ln(\sec t)$ for $0\le t<\frac{\pi}{2}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\sec^2 t-1=\tan^2 t,\ \frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\sec t\tan t}{\sec t}=\tan t$ Arc length of OP $=\int_0^p \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$ $=\int_0^p \sqrt{\tan^4 t+\tan^2 t}\ \mathrm{d}t$ $=\int_0^p \tan t\sec t\ \mathrm{d}t$ $=\left[\sec t\right]_0^p$ $=\sec p-1$ (shown)
(ii)	If $OP=1, \sec p-1=1 \therefore p=\frac{\pi}{3}$. Area of curved surface $=2\pi\int_0^{\frac{\pi}{3}} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$ $=2\pi\int_0^{\frac{\pi}{3}} (\ln\sec t)\tan t\sec t\ \mathrm{d}t$ $=2\pi\left\{\left[(\sec t)\ln\sec t\right]_0^{\frac{\pi}{3}}-\int_0^{\frac{\pi}{3}}\sec t\tan t\ \mathrm{d}t\right\}$ $=2\pi\left\{2\ln 2-\left[\sec t\right]_0^{\frac{\pi}{3}}\right\}$ $=2\pi\left\{2\ln 2-2+1\right\}=2\pi\left\{2\ln 2-1\right\}$

Qn	Suggested Solution
12(i)	$x = ae^{-t}\cos t, y = ae^{-t}\sin t$ where $t \ge 0$ $\frac{dx}{dt} = ae^{-t}\left[-\cos t - \sin t\right], \frac{dy}{dt} = ae^{-t}\left[-\sin t + \cos t\right]$ $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $= \frac{-\sin t + \cos t}{-\cos t - \sin t}$ $= \frac{\tan t - 1}{1 + \tan t}$ $= \frac{\tan t - \tan\frac{\pi}{4}}{1 + \tan t \tan\frac{\pi}{4}}$ $= \tan(t - \frac{\pi}{4})$ (shown)
(ii)	Length of arc AB $= \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ $= \int_0^{\pi} \sqrt{\left(ae^{-t}\left[-\cos t - \sin t\right]\right)^2 + \left(ae^{-t}\left[-\sin t + \cos t\right]\right)^2}\, dt$ $= a\int_0^{\pi} e^{-t}\sqrt{\cos^2 t + 2\sin t\cos t + \sin^2 t + \cos^2 t - 2\sin t\cos t + \sin^2 t}\, dt$ $= \sqrt{2}a\int_0^{\pi} e^{-t} dt$ $= \sqrt{2}a\left[-e^{-t}\right]_0^{\pi}$ $= \sqrt{2}a\left(1 - e^{-\pi}\right)$
(iii)	**Su**rface area $= 2\pi\int_0^{\pi} y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ $= 2\sqrt{2}\pi a^2\int_0^{\pi} e^{-2t}\sin t\ dt$ $\int_0^{\pi} e^{-2t}\sin t\ dt = \left[-e^{-2t}\cos t\right]_0^{\pi} - \int_0^{\pi} 2e^{-2t}\cos t\ dt$ $= \left(e^{-2\pi} + 1\right) - 2\left\{\left[e^{-2t}\sin t\right]_0^{\pi} - \int_0^{\pi} -2e^{-2t}\sin t\ dt\right\}$ $= \left(e^{-2\pi} + 1\right) - 0 - 4\int_0^{\pi} e^{-2t}\sin t\ dt$ $\therefore \int_0^{\pi} e^{-2t}\sin t\ dt = \frac{1}{5}\left(e^{-2\pi} + 1\right)$ Therefore surface area $= \frac{2\sqrt{2}}{5}\pi a^2\left(e^{-2\pi} + 1\right)$

Qn	Suggested Solution
13	$x=a(\cos t-1), y=a(\sin t-t),$ where $0\le \mathrm{t}\le \frac{\pi}{2}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=-a\sin t, \frac{\mathrm{d}y}{\mathrm{d}t}=a(\cos t-1)$ $\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2=a^2\sin^2 t+a^2\left(\cos^2 t-2\cos t+1\right)$ $=a^2\left(2-2\cos t\right)$ $=2a^2\left(2\sin^2\left(\frac{t}{2}\right)\right)$ $=4a^2\sin^2\left(\frac{t}{2}\right)$ $A=2\pi\int_0^{\frac{\pi}{3}} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$ $=2\pi\int_0^{\frac{\pi}{3}} a(\sin t-t)\left(2a\sin\frac{t}{2}\right)\mathrm{d}t$ $=4\pi a^2\int_0^{\frac{\pi}{3}}(\sin t-t)\sin\left(\frac{t}{2}\right)\mathrm{d}t$ (shown) $=4\pi a^2\left\{\int_0^{\frac{\pi}{3}} 2\sin^2\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)-\int_0^{\frac{\pi}{3}} t\sin\left(\frac{t}{2}\right)\mathrm{d}t\right\}$ $=4\pi a^2\left\{\left[\frac{4}{3}\sin^3\left(\frac{t}{2}\right)\right]_0^{\frac{\pi}{3}}-\left(\left[-2t\cos\left(\frac{t}{2}\right)\right]_0^{\frac{\pi}{3}}-\int_0^{\frac{\pi}{3}}-2\cos\left(\frac{t}{2}\right)\mathrm{d}t\right)\right\}$ $=4\pi a^2\left\{\frac{1}{6}-\left(\left[-2\left(\frac{\pi}{3}\right)\left(\frac{\sqrt{3}}{2}\right)\right]+\left[4\sin\left(\frac{t}{2}\right)\right]_0^{\frac{\pi}{3}}\right)\right\}$ $=4\pi a^2\left\{\frac{1}{6}-\left(-\frac{\sqrt{3}}{3}\pi+2\right)\right\}$ $=\frac{2}{3}\pi a^2\left(2\sqrt{3}\pi-11\right)$

Qn	Suggested Solution
14 ai	$I_n = \int_0^{\pi/2} \sin^{2n-1} x \sin x \, dx$ $= \left[\sin^{2n-1} x(-\cos x)\right]_0^{\pi/2}$ $-\int_0^{\pi/2} (-\cos x)(2n-1)\sin^{2n-2} x \cos x \, dx$ $= 0 + (2n-1)\int_0^{\pi/2} \sin^{2n-2} x \cos^2 x \, dx$ $= (2n-1)\int_0^{\pi/2} \sin^{2n-2} x \, (1-\sin^2 x) \, dx$ $= (2n-1)\int_0^{\pi/2} \sin^{2n-2} x - \sin^{2n} x) \, dx$ $\therefore I_n = (2n-1)\left[I_{n-1} - I_n\right]$ $I_n = \frac{2n-1}{2n} I_{n-1}$
aii	$I_0 = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2}$ $I_5 = \frac{9}{10} I_4 = \frac{9}{10}\frac{7}{8} I_3$ $= \frac{9\times7\times5\times3\times1}{10\times8\times6\times4\times2} I_0$ $= \frac{63}{512}\pi$
b	$V = \int_1^2 2\pi x[(x-1)-(x-1)^2] \, dx$ $= 2\pi\int_1^2 x^2 - x - x(x^2-2x+1) \, dx$ $= 2\pi\int_1^2 x^2 - x - x^3 + 2x^2 - x \, dx$ $= \frac{\pi}{2}$

Qn	Suggested Solution
15i	$\frac{\mathrm{d}x}{\mathrm{d}\theta}=3\sin^2\theta\cos\theta$ $\frac{\mathrm{d}y}{\mathrm{d}\theta}=-3\cos^2\theta\sin\theta$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-3\cos^2\theta\sin\theta}{3\sin^2\theta\cos\theta}=-\cot\theta$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\cdot\frac{\mathrm{d}\theta}{\mathrm{d}x}$ $=\operatorname{cosec}^2\theta\frac{1}{3\sin^2\theta\cos\theta}=\frac{1}{3}\operatorname{cosec}^4\theta\sec\theta$
(ii)	Arc length = $\int_0^{\pi/2}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$ $=\int_0^{\pi/2}\sqrt{9\sin^4\theta\cos^2\theta+9\cos^4\theta\sin^2\theta}\,\mathrm{d}\theta$ $=\int_0^{\pi/2}3\sin\theta\cos\theta\sqrt{\sin^2\theta+\cos^2\theta}\,\mathrm{d}\theta$ $=\frac{3}{2}\int_0^{\pi/2}\sin 2\theta\,\mathrm{d}\theta$ $=\frac{3}{2}$
(iii)	Area = $\int_0^{\pi/2}2\pi\lvert y\rvert\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$ $=\int_0^{\pi/2}2\pi(2-\cos^3\theta)(3\sin\theta\cos\theta)\,\mathrm{d}\theta$ $=6\pi\int_0^{\pi/2}2\sin\theta\cos\theta-\sin\theta\cos^4\theta\,\mathrm{d}\theta$ $=6\pi\int_0^{\pi/2}\sin 2\theta-\sin\theta\cos^4\theta\,\mathrm{d}\theta$ $=6\pi\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2}+6\pi\left[\frac{\cos^5\theta}{5}\right]_0^{\pi/2}$ $=\frac{24\pi}{5}$

Qn	Suggested Solution
16	

(i) $$D=\int_0^{\frac{\pi}{2}}\sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2+r^2}\ \mathrm{d}\theta$$

$$=\int_0^{\frac{\pi}{2}}\sqrt{\left(\frac{0.3\sin\theta}{(1+0.3\cos\theta)^2}\right)^2+\left(\frac{1}{1+0.3\cos\theta}\right)^2}\ \mathrm{d}\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{1}{1+0.3\cos\theta}\sqrt{\left(\frac{0.3\sin\theta}{1+0.3\cos\theta}\right)^2+1}\ \mathrm{d}\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{1}{(1+0.3\cos\theta)^2}\sqrt{0.09\sin^2\theta+(1+0.3\cos\theta)^2}\ \mathrm{d}\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{1}{(1+0.3\cos\theta)^2}\sqrt{1.09+0.6\cos\theta}\ \mathrm{d}\theta\ \text{(shown)}$$

(ii) For 5 ordinates, the width of a trapezium = $\frac{\pi}{2}\div 4=\frac{\pi}{8}$

Using trapezium rule,

$$D=\frac{\pi}{16}\left(\mathrm{f}(0)+2\mathrm{f}(\tfrac{\pi}{8})+2\mathrm{f}(\tfrac{2\pi}{8})+2\mathrm{f}(\tfrac{3\pi}{8})+\mathrm{f}(\tfrac{4\pi}{8})\right)$$

$$=\frac{\pi}{16}\left(0.76923+2(0.78614)+2(0.83753)+2(0.92432)+(1.04403)\right)$$

= 1.35662

= 1.3566 (to 4 dp.)

(iii) For 5 ordinates, the width of one strip = $\frac{\pi}{2}\div 4=\frac{\pi}{8}$

Using Simpson's rule,

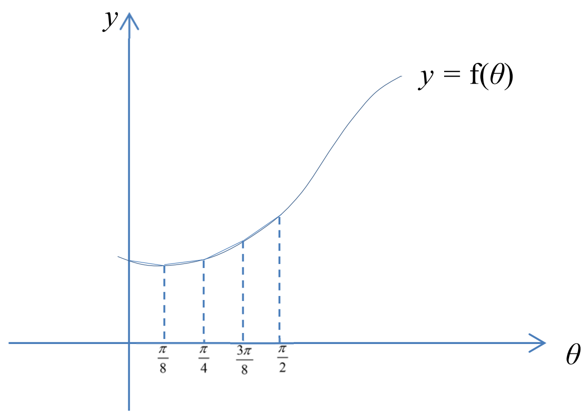
$$D=\frac{\pi}{24}\left(\mathrm{f}(0)+4\mathrm{f}(\tfrac{\pi}{8})+2\mathrm{f}(\tfrac{2\pi}{8})+4\mathrm{f}(\tfrac{3\pi}{8})+\mathrm{f}(\tfrac{4\pi}{8})\right)$$

$$=\frac{\pi}{24}\left(0.76923+4(0.78614)+2(0.83753)+4(0.92432)+(1.04403)\right)$$

= 1.35221

= 1.3522 (to 4 dp.)

	(iv)



Since the slope of the trapezium strips lie above the curve, the approximation using trapezium rule in part (ii) is an overestimation.

The approximation with Simpson's rule using quadratic curve which concaves upwards is similar to the graph of y = f(θ), therefore the area between the quadratic curves and the original curve y = f(θ) is smaller. Therefore the approximation using Simpson's rule is better.

(v) Area swept $= \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \mathrm{d}\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{1}{1+0.3\cos\theta}\right)^2 \mathrm{d}\theta$

$= 0.564416$

$0.56441 \approx k \times \frac{1.35221}{0.2} \Rightarrow k \approx 0.08348$ (4 dp.) (allow range ±0.0005)