H2 MATHEMATICS

TOPIC ***GRAPHING TECHNIQUES*** ***2022/JC1***

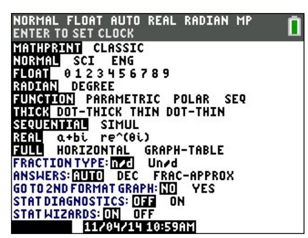
Content Outline

- Use of a graphing calculator to graph a given function.
- Sketching graphs with important characteristics such as symmetry, intersections with axes, turning points and asymptotes of the following:
  - $y=\dfrac{ax+b}{cx+d}$
  - $y=\dfrac{ax^2+bx+c}{dx+e}$
- Determine the equations of asymptotes, axes of symmetry, and restrictions on the possible values of x and/or y.

1 Introduction

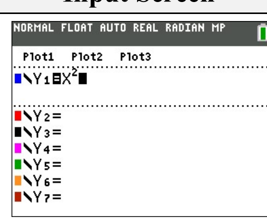
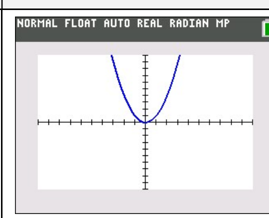
Have you seen graphs (bar, pie, histogram, line) in newspapers, television, magazines and books etc.? The purpose of the graphs is to show numerical facts in visual form so that they can be understood quickly, easily and clearly. Thus graphs are visual representations of data collected. Data can also be presented in the form of a table; however a graphical presentation is easier to understand.

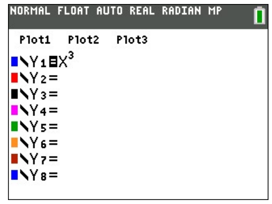
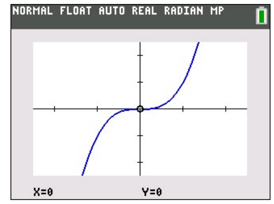
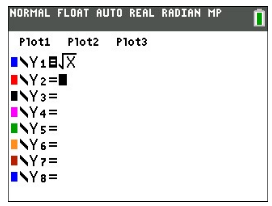
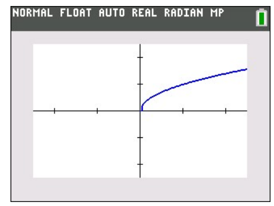
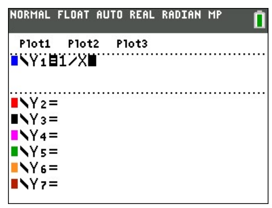
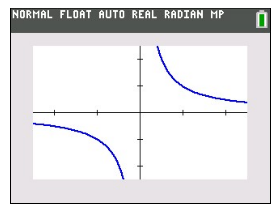
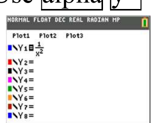
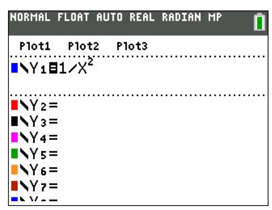
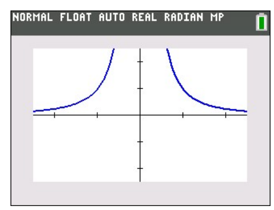
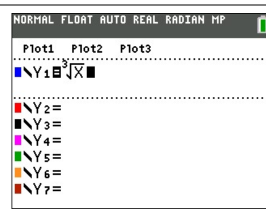
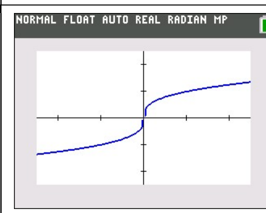
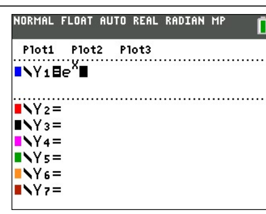
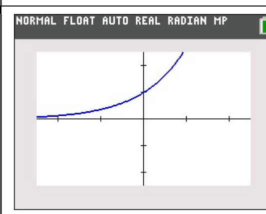
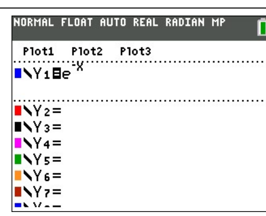
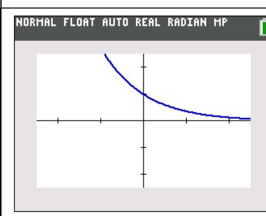
In the H2 Mathematics syllabus, graphing calculator (G.C.) will aid in the sketching of graphs. However, students should be aware that there are limitations inherent in G.C.

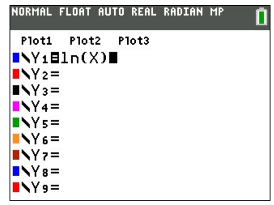
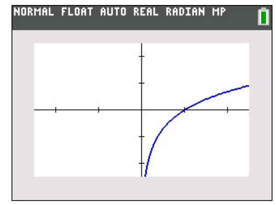
G.C. Basics: Before sketching a graph, ensure that the settings are as shown. Settings can be accessed by pressing $\boxed{\text{mode}}$. **FUNCTION** PARAMETRIC POLAR SEQ . (Use the arrow keys to navigate and $\boxed{\text{enter}}$ key to highlight **FUNCTION**.)	

📖 **Self-Reading**

G.C. can be used to sketch the graphs of the following basic curves:

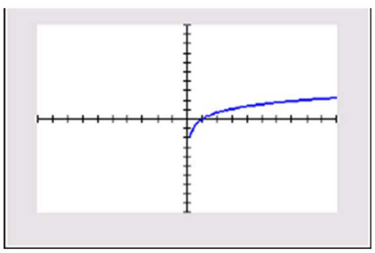
Equation	Press	Input Screen	Result
$y=x^2$	$\boxed{\text{y=}}$ $\boxed{\text{X,T,}\theta\text{,n}}$ $\boxed{x^2}$ $\boxed{\text{graph}}$ displays the plotted graph (of the specified equation) brings us to the input screen for specifying equations		

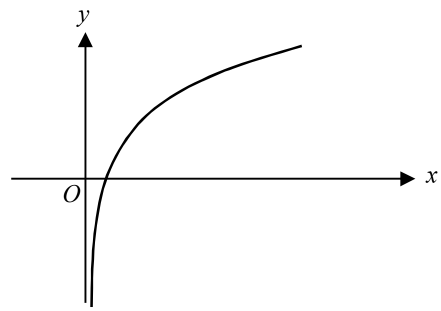
Equation	Press	Input Screen	Result
$y=x^3$	$\boxed{\text{y=}}\ \boxed{\text{X,T,}\theta\text{,n}}\ \boxed{\wedge}\ \boxed{3}\ \boxed{\text{graph}}$ To zoom in at the point $(0,0)$, press : $\boxed{\text{zoom}}\ \boxed{2}\ \boxed{\text{enter}}$		
$y=\sqrt{x}$	$\boxed{\text{y=}}\ \underbrace{\boxed{\text{2nd}}\ \boxed{x^2}}_{\sqrt{\ }}\ \boxed{\text{X,T,}\theta\text{,n}}\ \boxed{\text{graph}}$		
$y=\dfrac{1}{x}$	$\boxed{\text{y=}}\ \boxed{1}\ \boxed{\div}\ \boxed{\text{X,T,}\theta\text{,n}}\ \boxed{\text{graph}}$ **Alternative:** Use $\boxed{\text{alpha}}\ \boxed{\text{y=}}$  		
$y=\dfrac{1}{x^2}$	$\boxed{\text{y=}}\ \boxed{1}\ \boxed{\div}\ \boxed{\text{X,T,}\theta\text{,n}}\ \boxed{x^2}\ \boxed{\text{graph}}$ **Alternative:** Use $\boxed{\text{alpha}}\ \boxed{\text{y=}}$  		
$y=\sqrt[3]{x}$	$\boxed{\text{y=}}\ \underbrace{\boxed{\text{math}}\ \boxed{4}}_{\sqrt[3]{\ }}\ \boxed{\text{X,T,}\theta\text{,n}}\ \boxed{\text{graph}}$		
$y=\mathrm{e}^x$	$\boxed{\text{y=}}\ \underbrace{\boxed{\text{2nd}}\ \boxed{\ln}}_{\mathrm{e}}\ \boxed{\text{X,T,}\theta\text{,n}}\ \boxed{\text{graph}}$		
$y=\mathrm{e}^{-x}$	$\boxed{\text{y=}}\ \underbrace{\boxed{\text{2nd}}\ \boxed{\ln}}_{\mathrm{e}}\ \boxed{(-)}\ \boxed{\text{X,T,}\theta\text{,n}}\ \boxed{\text{graph}}$		

Equation	Press	Input Screen	Result
$y = \ln x$	$\boxed{\text{y=}}$ $\boxed{\text{ln}}$ $\boxed{\text{X,T,}\theta\text{,n}}$ $\boxed{\text{graph}}$		

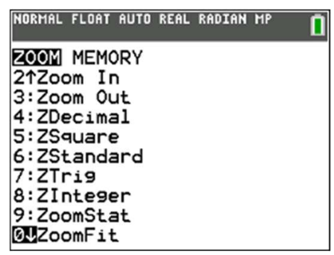
Remark:

Due to the limitation of G.C., there might be a difference between what we see on the $\boxed{\text{zoom}}$ **6: ZStandard** screen and the actual graph that we draw. Take for example $y = \ln x$:





We should be familiar with the other $\boxed{\text{zoom}}$ functions keys available in the G.C. so that we can use them to obtain an optimal view of the graph. Nevertheless, we should take note that G.C. is simply a tool that is used to determine the general shape of a curve.



2 Features of Graphs

In this section, we will investigate the following features of graphs:

- Axial intercepts
- Asymptotes
- Turning points
- Lines of Symmetry
- Possible range of values of x and y

2.1 Axial Intercepts

An axial intercept is a point at which a curve intersects with either axis (x -axis / y -axis).

A point of intersection between the curve and the x -axis is an $\boldsymbol{x}$ **-intercept**, while a point of intersection between the curve and the y -axis is a $\boldsymbol{y}$ **-intercept**.

Example 1:

Find the coordinates of the axial intercepts of the curve $y = x^2 + 2x - 15$.

Solution:

Method ①: Algebraic/Analytical

To find x-intercepts:

Substitute $y = 0$,

$$\begin{aligned} x^2 + 2x - 15 &= 0 \\ (x+5)(x-3) &= 0 \\ x &= -5 \text{ or } 3 \end{aligned}$$

To find y-intercept:

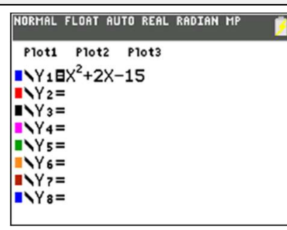
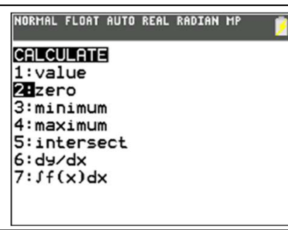
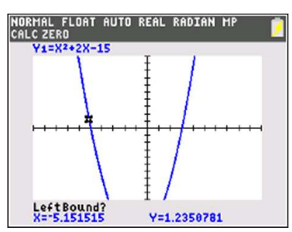
Substitute $x = 0$,

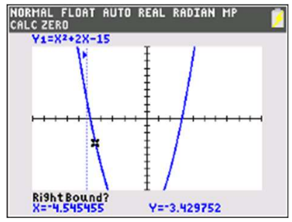
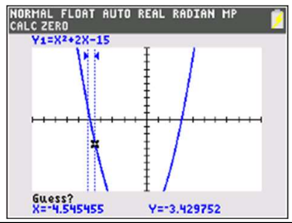
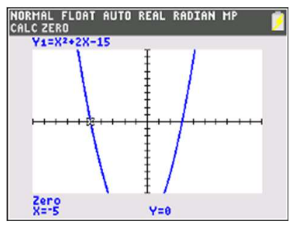
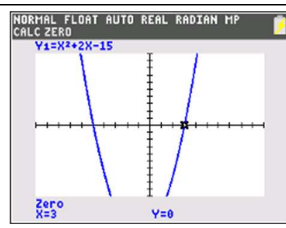
$$\begin{aligned} y &= 0^2 + 2(0) - 15 \\ &= -15 \end{aligned}$$

$\therefore$ axial intercepts are $(-5,0)$, $(3,0)$ and $(0,-15)$.

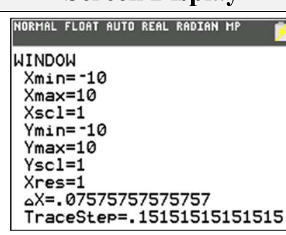
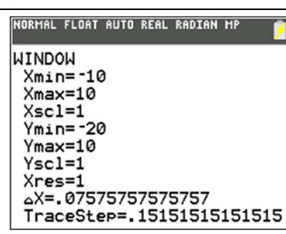
Method ②: Using G.C.

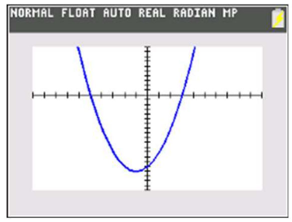
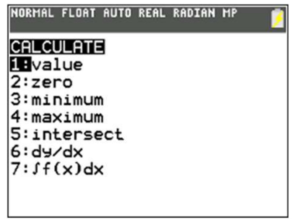
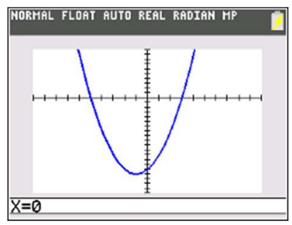
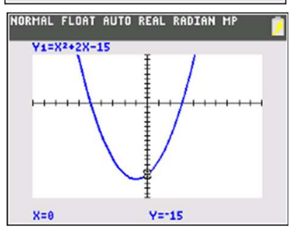
To find the x-intercepts:

	Press	Screen Display	Remarks
1	Press $\boxed{\text{y=}}$. Key $x^2 + 2x - 15$ into **Y₁**.		
2	Press $\boxed{\text{2nd}}$ $\boxed{\text{trace}}$ to access the **CALCULATE** menu. Select **2: zero** to find the x-intercepts.		
3	Press arrow keys to move the cursor to the left of first root followed by $\boxed{\text{enter}}$. OR You can just enter a number that is to the left of the first root.		

	Press	Screen Display	Remarks
4	Press arrow keys to move the cursor to the right of first root followed by enter. OR You can just press a number that is to the right of the first root.		The left and right boundary should only contain one root, else G.C. may find the other root instead.
5	For guess?, you can move the cursor close to the first root, then press enter.	 	G.C. uses a numerical method to find the root. In some situations, the answer given may be -4.9999999 and you will need to round off accordingly after verifying your answer. You can verify your answer by substituting $x=-5$ into $y=x^2+2x-15$ and check that $y=0$.
6	Repeat steps 2 – 5 to find the other root.		

Before proceeding to find the y-intercept, notice that the G.C. did not show the curve in its entirety as the minimum turning point is not shown. We will need to change the window settings.

	Press	Screen Display	Remarks
1	Press window to access the window settings menu.		The default window settings are: $\text{Xmin}=-10$, $\text{Xmax}=10$, $\text{Ymin}=-10$, $\text{Ymax}=10$.
2	For this question, we can set Ymin to be -20.		Changing the window setting is a trial and error process, i.e. there are no "correct" values to set.

	Press	Screen Display	Remarks
3	Press $\boxed{\text{graph}}$. The minimum point can now be displayed.		
4	Press $\boxed{\text{2nd}}$ $\boxed{\text{trace}}$ to access the **CALCULATE** menu. Select **1:value** to find the value of y-coordinate given the x-coordinate.		
5	Press $\boxed{0}$ then $\boxed{\text{enter}}$.		

Remark:

- If question states explicitly to find **exact** coordinates of the axial intercepts, then **method ① algebraic/analytical** must be used.
- For **method ②: using G.C., do not** press the arrow key to **trace** on the curve and simply read off the values as it is not accurate.

Self-Practice 1:

Find the exact coordinates of the axial intercepts for the following curves:

(a) $y=\dfrac{1}{3-x}$ **(b)** $y=\dfrac{2x+1}{x-2}$ **(c)** $y=\dfrac{x^2}{x-1}$ **(d)** $y=\dfrac{x^2+2x-3}{x+2}$.

Answers:

(a) y-intercept : $\left(0,\ \dfrac{1}{3}\right)$; no x-intercept **(b)** y-intercept: $\left(0,\ -\dfrac{1}{2}\right)$; x-intercept: $\left(-\dfrac{1}{2},\ 0\right)$

(c) y-intercept / x-intercept : $(0,\ 0)$ **(d)** y-intercept: $\left(0,-\dfrac{3}{2}\right)$; x-intercept: $(-3,0)$, $(1,0)$

2.2 Asymptotes

2.2.1 Vertical Asymptotes

If there is a constant k such that along a curve, as $x \to k$, $y \to -\infty$ or $y \to \infty$, then the vertical line $x = k$ is a **vertical asymptote** of the curve.

In many curves, the value(s) of x which makes the function y undefined indicates the presence of a vertical asymptote at that value(s). In such curves, it will never intersect with its vertical asymptote(s).

2.2.2 Horizontal Asymptotes

If there is a constant k such that along a curve, as $x \to -\infty$ or $x \to \infty$, $y \to k$, then the horizontal line $y = k$ is a **horizontal asymptote** of the curve. The curve **might intersect** with its horizontal asymptote(s) (Refer to **Example 8c and 8d**).

Example 2:

Given the curve $y = 1 + \dfrac{1}{x-3}$, find the equations of the asymptotes and sketch the curve.

Solution:

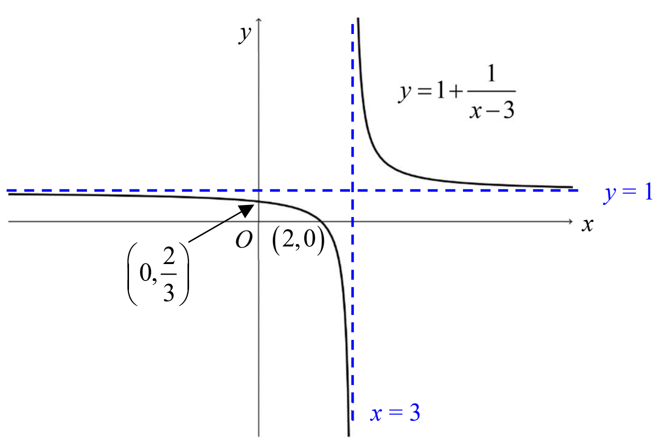
When $x =$ ___3___ , y is ___undefined $\left(1+\frac{1}{0}\right)$___.

Explanation: As $x \to 3^+$, $3^+ - 3 = 0^+$, i.e. a very small positive value. $\therefore \dfrac{1}{0^+} \to +\infty$.

As $x \to 3^-$, $3^- - 3 = 0^-$, i.e. a very small negative value. $\therefore \dfrac{1}{0^-} \to -\infty$.

Thus as $x \to 3$, $y \to \pm\infty$. Hence $x = 3$ is a **vertical asymptote**.

As $x \to \pm\infty$, $\dfrac{1}{x-3} \to 0$, and $y \to 1$, hence $y = 1$ is a **horizontal asymptote**.



Remark:
Asymptotes should be determined and drawn using **dotted lines** before sketching the curve.

Self-Practice 2:
Find the equation(s) of the vertical asymptote(s) for the following curves:

(a) $y=\dfrac{1}{3-x}$ **(b)** $y=\dfrac{2x+1}{x-2}$ **(c)** $y=\dfrac{x^2}{x-1}$ **(d)** $y=\dfrac{x^2+2x-3}{x+2}$.

Answers:
(a) $x=3$ **(b)** $x=2$ **(c)** $x=1$ **(d)** $x=-2$

2.2.3 Oblique Asymptotes

An **oblique asymptote** is a slanted line of the form $y=mx+c$, $m\neq 0$ that the graph of a function approaches as $x\to-\infty$ or $x\to\infty$.

Example 3:

Given the curve $y=\dfrac{x^2-3x}{x+1}$, find the equations of the asymptotes and sketch the curve.

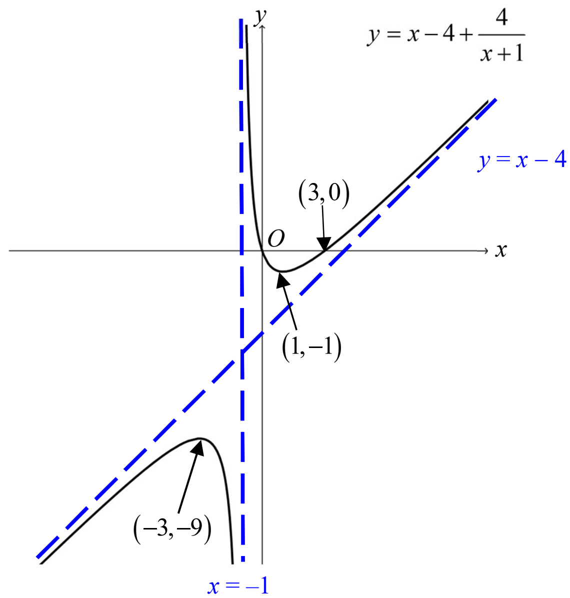
Solution:

Notice that when $x=-1$, y is undefined. $\therefore$ $x=-1$ is a **vertical asymptote**.

By long division, we have $y=\dfrac{x^2-3x}{x+1}=$ $x-4+\dfrac{4}{x+1}$.

$$\begin{array}{r} x-4 \\ x+1\overline{)\,x^2-3x} \\ -\left(x^2+x\right) \\ \hline -4x \\ -\left(-4x-4\right) \\ \hline 4 \end{array}$$

As $x\to\infty$, $\dfrac{1}{x+1}\to 0$, and $y\to x-4$. $\therefore$ $y=x-4$ is an **oblique asymptote**.



Remarks:

- A graph of an improper rational function has a horizontal asymptote if the degree of the numerator is same as the degree of the denominator, e.g. $y=\dfrac{3x-7}{x+1}$ and $y=\dfrac{x^2-1}{5x^2+4}$.
- A graph of an improper rational function has an oblique asymptote if the degree of the numerator is one more than the degree of the denominator, e.g. $y=\dfrac{x^2-3x}{x+1}$, $y=\dfrac{x^3-3x}{x^2-1}$.
- Long division is an essential technique in re-expressing a rational function in order to determine its horizontal/oblique asymptote.

Self-Practice 3:

By carrying out long division to simplify the rational function (if applicable), find the equation(s) of the horizontal asymptote(s) or oblique asymptote(s) for the following curves:

(a) $y=\dfrac{1}{3-x}$ **(b)** $y=\dfrac{2x+1}{x-2}$ **(c)** $y=\dfrac{x^2}{x-1}$ **(d)** $y=\dfrac{x^2+2x-3}{x+2}$.

Answers:

(a) $y=\dfrac{1}{3-x}$; Horizontal asymptote: $y=0$

(b) $y=2+\dfrac{5}{x-2}$; Horizontal asymptote: $y=2$

(c) $y=x+1+\dfrac{1}{x-1}$; Oblique asymptote: $y=x+1$

(d) $y=x-\dfrac{3}{x+2}$; Oblique asymptote: $y=x$

2.3 Turning Points

The turning points of the curves can be found using differentiation or G.C..

Example 4:

Find the coordinates of the turning points of the curve $y=\dfrac{x^2-3x}{x+1}$, indicating the nature of each turning point.

Solution:

Method ①: Algebraic/Analytical (🕮 Self-Reading)

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(x+1)(2x-3)-(x^2-3x)(1)}{(x+1)^2} \quad \text{(quotient rule)}$$

$$=\frac{x^2+2x-3}{(x+1)^2}$$

At turning points,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\frac{x^2+2x-3}{(x+1)^2}=0$$

$$x^2+2x-3=0$$

$$(x+3)(x-1)=0$$

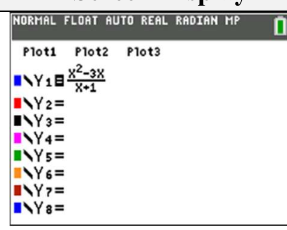
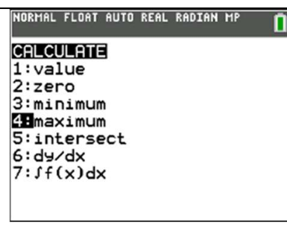
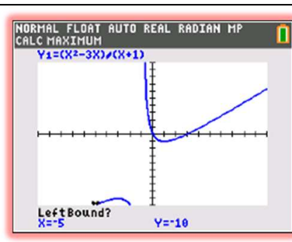
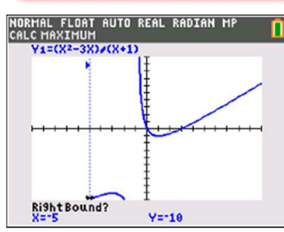
$$x=-3 \text{ or } 1$$

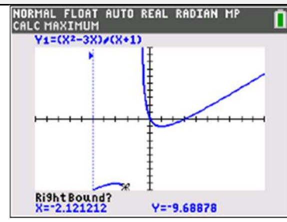
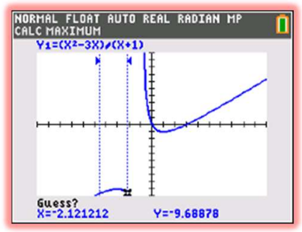
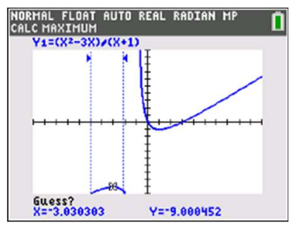
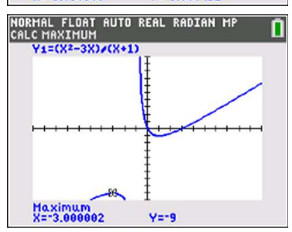
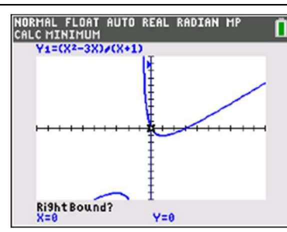
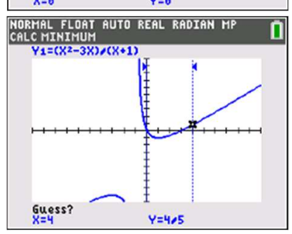
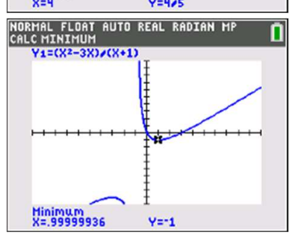
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{(x+1)^2(2x+2)-(x^2+2x-3)(2)(x+1)}{(x+1)^4} \quad \text{(quotient rule)}$$

When $x=-3$, $y=-9$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1<0$. $\therefore$ $(-3,-9)$ is a maximum point.

When $x=1$, $y=-1$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=1>0$. $\therefore$ $(1,-1)$ is a minimum point.

<u>Method ②:</u> Using G.C.

	Press	Screen Display	Remarks
1	Key in the equation into **Y**$_1$.		
2	Press $\boxed{\text{2nd}}$ $\boxed{\text{trace}}$ to access the **CALCULATE** menu. Select **4:maximum** to find the x-intercepts.		
3	Press the arrow keys to move the cursor to the left of maximum point, then press $\boxed{\text{enter}}$. OR You can just press a number that is left of the maximum point.	 	This screenshot will appear after pressing $\boxed{\text{enter}}$ for left bound.

	Press	Screen Display	Remarks
4	Press the arrow keys to move the cursor to the right of maximum point, then press $\boxed{\text{enter}}$. OR You can just press a number that is right of the maximum point.	 	Like finding roots, the G.C. uses a numerical method to find the maximum point, you will need to round off accordingly and verify your answer. This screenshot appears before pressing $\boxed{\text{enter}}$ for right bound.
5	For the guess, you can move the cursor close to the maximum point, then press $\boxed{\text{ENTER}}$.	 	
6	Repeat the process to find the minimum point.	  	

$\therefore$ $(-3,-9)$ is a maximum point and $(1,-1)$ is a minimum point.

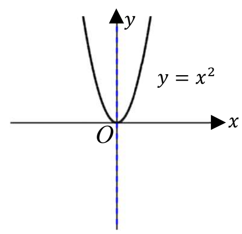
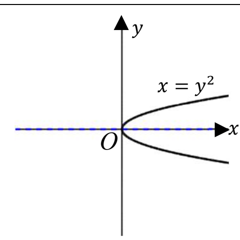
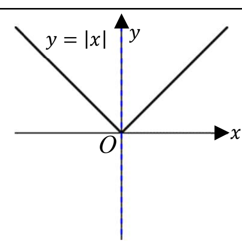
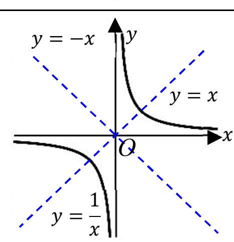
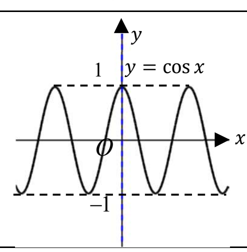
Remark:

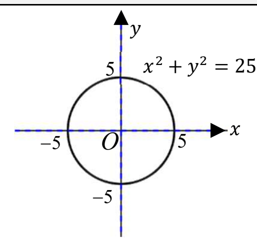
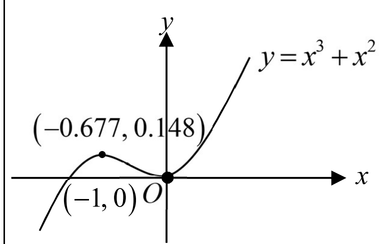
The use of calculus to determine the **exact** coordinates of the turning points will be further discussed in **Module IV Calculus: Applications of Differentiation**.

2.4 Lines of Symmetry

Recall that **line of symmetry** is the imaginary line that passes through the centre of the shape or object and divides it into identical halves.

Observe the line of symmetry of the following graphs.

Equation	Curve	Line of symmetries
$y = x^2$		$x = 0$
$x = y^2$		$y = 0$
$y = \|x\|$		$x = 0$
$y = \frac{1}{x}$		$y = x$ and $y = -x$
$y = \cos x$		$x = 0$

<table>
<tr><th>Equation</th><th>Curve</th><th>Line of symmetries</th></tr>
<tr><td>$x^2 + y^2 = 25$</td><td>
</td><td>Any straight line that passes though the origin,
e.g. $x = 0$ and $y = 0$</td></tr>
<tr><td>$y = x^3 + x^2$</td><td>
</td><td>None</td></tr>
</table>

Remark:

	Line/Axis of Symmetry
If the equation consists of **even powers of *x*** only.	$x = 0$
If the equation consists of **even powers of *y*** only.	$y = 0$

Explanation:

Since $x^2 = (-x)^2$, when we substitute an x value, x_1 or $-x_1$ into the equation, we will get the same y value, y_1, i.e. $x = 0$ will be the line of symmetry.

Similarly, if an equation contains only even power of y, $y = 0$ will be the line of symmetry.

2.5 Possible Ranges of *x* and *y*

Often, there may be restrictions to the possible values of x and y, depending on the equation of a curve. These restrictions may not always be stated clearly, but we can determine these restrictions by observing the equations and/or graphs.

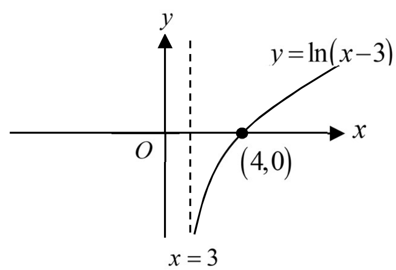
Example 5:

Find the range of values of x and y for the following curves:

(a) $y=\ln(x-3)$ **(b)** $y=-\sqrt{3-x}$ **(c)** $y=\dfrac{x^2+2}{x+1}$

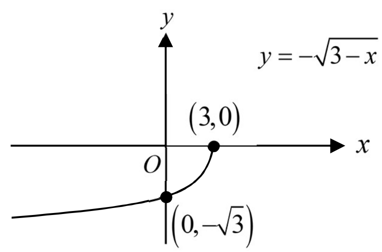
Solution:

(a) For $y=\ln(x-3)$ to be defined, $x-3>0 \Rightarrow x>3$.



From the graph, $y\in\mathbb{R}$.

(b) For $y=-\sqrt{3-x}$ to be defined, $3-x\geq 0 \Rightarrow x\leq 3$.



From the graph, $y\leq 0$.

(c) For $y=\dfrac{x^2+2}{x+1}$ to be defined, $x\neq -1$.

$\therefore\ x<-1$ or $x>-1$.

Method ①: Algebraic/Analytical (For exact answers)

To find the range of values of y, we first make x the subject.

$$y=\frac{x^2+2}{x+1}$$

$$xy+y=x^2+2$$

$$x^2+(-y)x+(2-y)=0$$

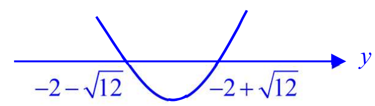
$$x=\frac{-(-y)\pm\sqrt{(-y)^2-4(1)(2-y)}}{2(1)}$$

$$x=\frac{y\pm\sqrt{y^2+4y-8}}{2}$$

For x to be defined, $y^2+4y-8\geq 0$.

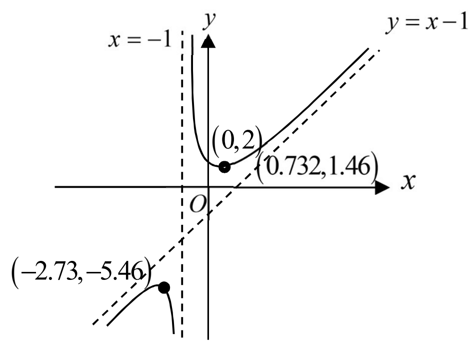
Consider

$$y^2+4y-8=0$$
$$(y+2)^2-12=0$$
$$y=-2\pm\sqrt{12}$$



$\therefore\ y\leq -2-\sqrt{12}$ or $y\geq -2+\sqrt{12}$.

<u>Method ②:</u> Graphical (For non- exact answers)



From the graph, $y\leq -5.46$ or $y\geq 1.46$.

🗎 **<u>Remark:</u>**

If we can identify the possible values of x and y, the values of Xmin, Xmax, Ymin and Ymax in the G.C. can be changed so that the entire graph can be seen. The ability to adjust settings in the $\boxed{\text{window}}$ function is an important skill to be acquired in H2 Mathematics.

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3 Graphs of Common Functions

In this section, we will learn more about the graphs of these common functions:

- Polynomial
- Exponential
- Logarithmic
- Trigonometric
- Inverse Trigonometric
- Rational
- Modulus

Basic **Exponential and Logarithmic functions** have been covered in **Section 5 of Revision (From 'O' Levels)** set. Similarly, Basic **Trigonometric and Inverse Trigonometric functions h**ave been covered in **Section 7 of Revision (From 'O' Levels)** set. Therefore, in this section we will focus on Polynomial, Exponential, Rational and Modulus Functions.

3.1 Polynomial Functions

A polynomial is a function of the form

$$\mathrm{f}(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a non-negative integer and $a_0, a_1, a_2, \cdots, a_n$ are constants (called the **coefficients** of the polynomial). If the leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the **degree** of the polynomial.

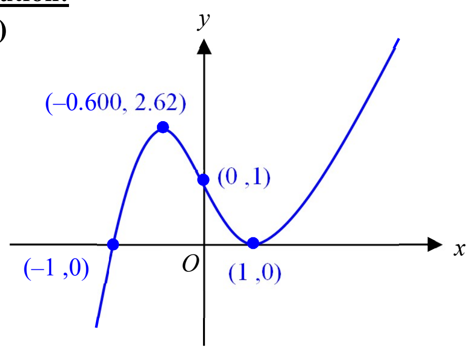
A linear function such as $\mathrm{f}(x) = 2x+1$ is a polynomial of degree 1 while a quadratic function such as $\mathrm{f}(x) = x^2 + x + 1$ is a polynomial of degree 2.

Example 6:
Sketch the following curves, indicating clearly the coordinates of any intersections with the axes and turning points:

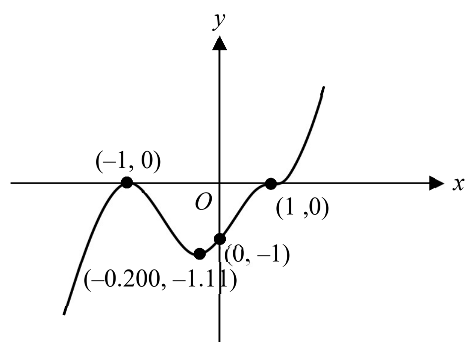
(a) $y = (x+1)(x-1)^4$ **(b)** $y = (x+1)^2(x-1)^3$

Solution:

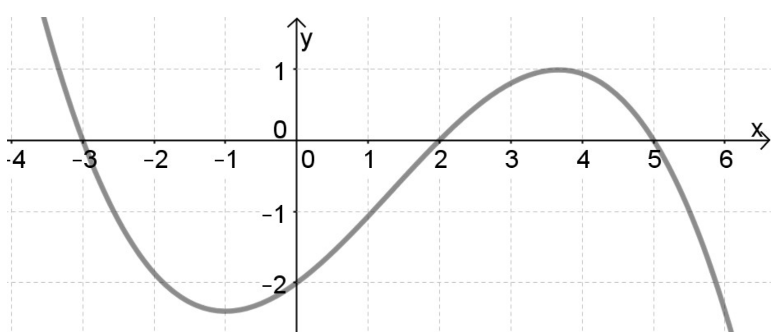
(a)



(b)



Example 7: A cubic polynomial curve, C, is shown below, find a possible equation for C.



Solution:

Since the graph crosses the x-axis three times at -3, 2, and 5, the function has factors $[x-(-3)]$, $(x-2)$ and $(x-5)$. Thus we can let $f(x)=a(x+3)(x-2)(x-5)$, where a is a constant to be determined.

Also the graph also crosses the y-axis at -2,

$$f(0)=-2$$
$$a(0+3)(0-2)(0-5)=-2$$
$$a=-\frac{1}{15}$$

Hence, one possible equation for C is $f(x)=-\frac{1}{15}(x+3)(x-2)(x-5)$.

3.2 Exponential Functions

Since we were taught how to sketch basic Exponential functions in secondary school, we will now learn how to sketch graphs involving combinations of common functions.

Example 8:

Sketch the graph of the following curves, indicating clearly the coordinates of any intersections with the axes, turning points and the equations of any asymptotes:

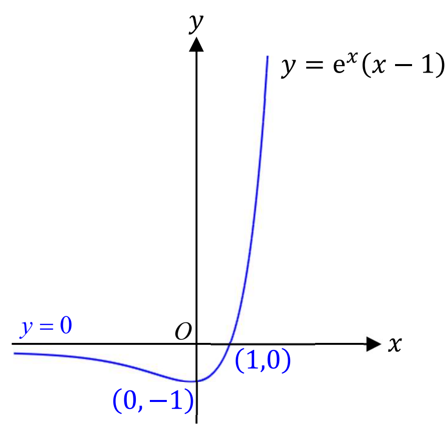
(a) $y=e^x(x-1)$ **(b)** $y=\dfrac{e^x-1}{e^{2x}+1}$

Solution:

(a) **Asymptotes:**

As $x\to-\infty$, $e^x\to 0$ and $y\to 0$.

$\therefore$ $y=0$ is a horizontal asymptote.



Intercepts:

When $x=0$, $y=-1$ $\Rightarrow$ $(0,-1)$

When $y=0$, $x=1$ $\Rightarrow$ $(1,0)$

Stationary Points

$$\frac{dy}{dx}=e^x(1)+(x-1)e^x$$
$$=xe^x$$

Let $\dfrac{dy}{dx}=0$

$$xe^x=0$$
$$x=0$$

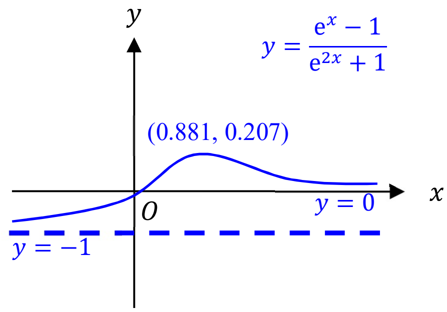
$\therefore$ stationary point is $(0,-1)$.

(b) **Asymptotes:**

As $x \to -\infty$, $e^x \to 0$ and $e^{2x} \to 0$,

$y \to \frac{-1}{1} = -1$.

$\therefore$ $y = -1$ is a horizontal asymptote.



Notice that as $x \to \infty$, $e^x \to \infty$ and $e^{2x} \to \infty$

$y \to \frac{\infty}{\infty}$ which is not well defined. We will need to do some manipulation first.

$$y = \frac{e^x - 1}{e^{2x} + 1}$$

$$= \frac{\frac{e^x - 1}{e^{2x}}}{\frac{e^{2x} + 1}{e^{2x}}}$$

Divide throughout by e^{2x}

$$= \frac{\frac{1}{e^x} - \frac{1}{e^{2x}}}{1 + \frac{1}{e^{2x}}}$$

$$= \frac{e^{-x} - e^{-2x}}{1 + e^{-2x}}$$

Now, as $x \to \infty$, $e^{-x} \to 0$ and $e^{-2x} \to 0$,

$y \to \frac{0}{1} = 0$.

$\therefore$ $y = 0$ is a horizontal asymptote.

Intercepts:

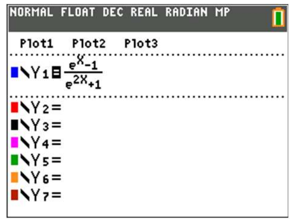
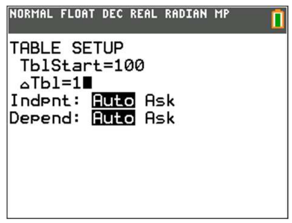
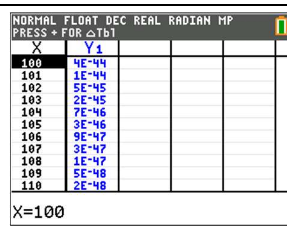
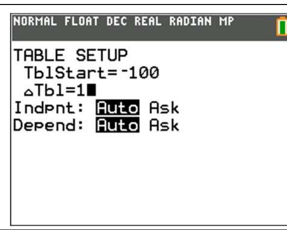
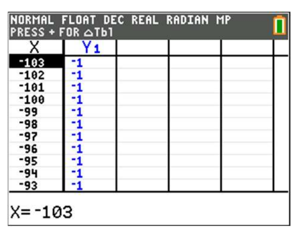
When $x = 0$, $y = 0$

When $y = 0$, $x = 0$ $\Rightarrow$ $(0,0)$

Stationary Points

Use G.C.

Using G.C. to find Horizontal Asymptotes:

	Press	Screen Display	Remarks
1	Key in the equation into **Y**$_1$.		
2	Press $\boxed{\text{2nd}}$ $\boxed{\text{window}}$ to access **tblset** menu. Choose a large value for **TblStart**, say 100.		ΔTbl is the increment. By default, the increment is 1.
3	Press $\boxed{\text{2nd}}$ $\boxed{\text{graph}}$ to access **table** menu. Scroll up to observe values for y-values as x increases.		Notice that as x-value increases, y-value approaches 0. $\therefore\ y=0$ is a horizontal asymptote.
4	Press $\boxed{\text{2nd}}$ $\boxed{\text{window}}$ to access **tblset** menu. Choose a large value for **TblStart**, say -100.		
5	Press $\boxed{\text{2nd}}$ $\boxed{\text{graph}}$ to access **table** menu. Scroll up to observe values for y-values as x decreases.		Notice that as x-value decreases, y-value shows -1. $\therefore\ y=-1$ is a horizontal asymptote.

3.3 Rational Functions

A **rational function** f is the ratio of two polynomials, i.e. $\mathrm{f}(x)=\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ where P and Q are polynomials. There are two types of rational functions:

- If the degree of $\mathrm{P}(x) <$ the degree of $\mathrm{Q}(x)$, it is known as a **proper** rational function
- If the degree of $\mathrm{P}(x) \geq$ the degree of $\mathrm{Q}(x)$. It is known as an **improper** rational function.

For instance, $\dfrac{3}{x+1}$ is proper while $\dfrac{3x}{x+1}$ and $\dfrac{3x^2}{x-1}$ are improper rational functions.

Remark:

Before sketching a rational function, we should ensure that any improper rational function is converted to a proper one via long division. This will enable us to determine the vertical asymptotes and horizontal/oblique asymptotes. The shape can then be obtained with the use of G.C.

Example 9:

Sketch the graph of the following curves, indicating clearly the coordinates of any intersections with the axes, turning points and the equations of any asymptotes:

(a) $y=\dfrac{2x-3}{x-4}$ **(b)** $y=\dfrac{x^2-2}{x^2-1}$ **(c)** $y=\dfrac{2x^2-x+1}{x-2}$

(d) $y=\dfrac{x^2-x-12}{x-2}$ **(e)** $y=\dfrac{(x+5)^2}{x^2-5x}$.

Solution:

(a) $y=\dfrac{2x-3}{x-4}=2+\dfrac{5}{x-4}$

Asymptotes:

$y=2$

$x=4$

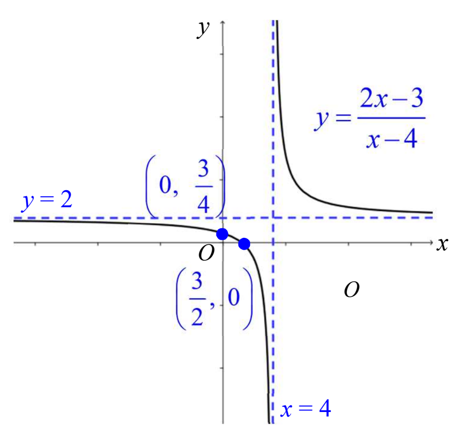
Intercepts:

When $x=0$, $y=\dfrac{3}{4}$ $\Rightarrow$ $\left(0,\dfrac{3}{4}\right)$

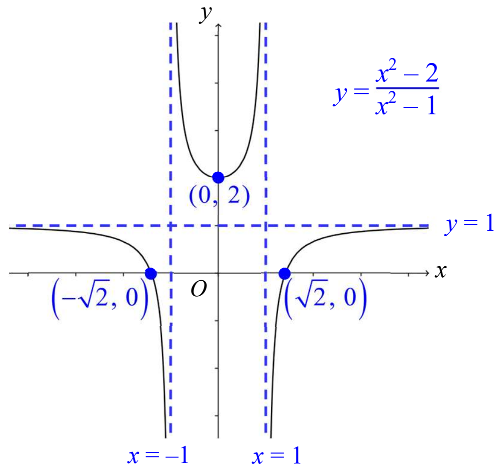
When $y=0$, $x=\dfrac{3}{2}$ $\Rightarrow$ $\left(\dfrac{3}{2},0\right)$

Stationary Points:

None



(b) $y=\dfrac{x^2-2}{x^2-1}=1-\dfrac{1}{x^2-1}=1-\dfrac{1}{(x-1)(x+1)}$

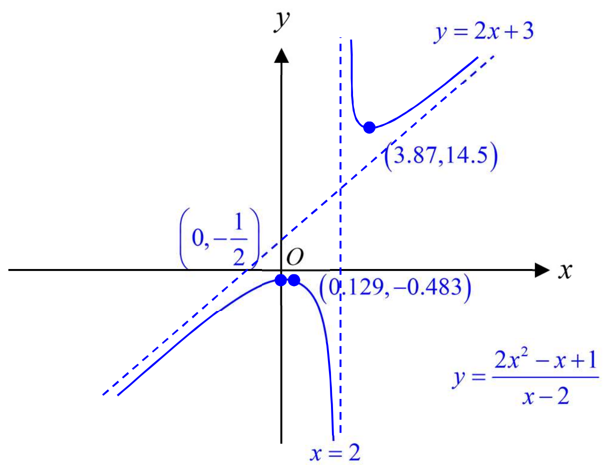


Asymptotes:
$y=1$
$x=-1$
$x=1$

Intercepts:
When $x=0$, $y=2$ $\Rightarrow$ $(0,-1)$
When $y=0$, $x^2-2=0$
$x=\pm\sqrt{2}$
$\Rightarrow$ $(-\sqrt{2},0)$ and $(\sqrt{2},0)$

Stationary Points:
Use G.C.

(c) $y=\dfrac{2x^2-x+1}{x-2}=2x+3+\dfrac{7}{x-2}$



Asymptotes:
$y=2x+3$
$x=2$

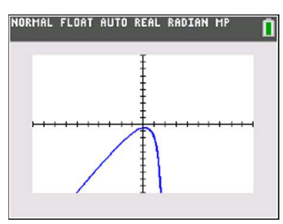
Intercepts:
When $x=0$, $y=-\dfrac{1}{2}$ $\Rightarrow$ $\left(0,-\dfrac{1}{2}\right)$

When $y=0$, $2x^2-x+1=0$ has no real solutions.

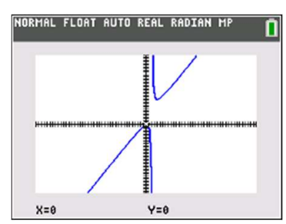
Stationary Points:
Use G.C.

Remarks:

G.C. in $\boxed{\text{zoom}}$ **6: ZStandard** does not display the entire curve. We need to either **change the window settings** or **zoom out.**

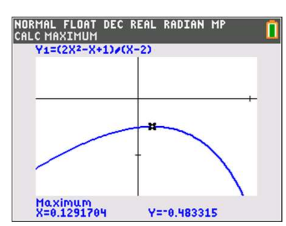


6: ZStandard



After changing window settings

From the G.C., it may seem like the maximum point is lying on the y-axis. However, it is not. We can check by pressing $\boxed{\text{zoom}}$ **1: ZBox** to enlarge a chosen section of the curve.

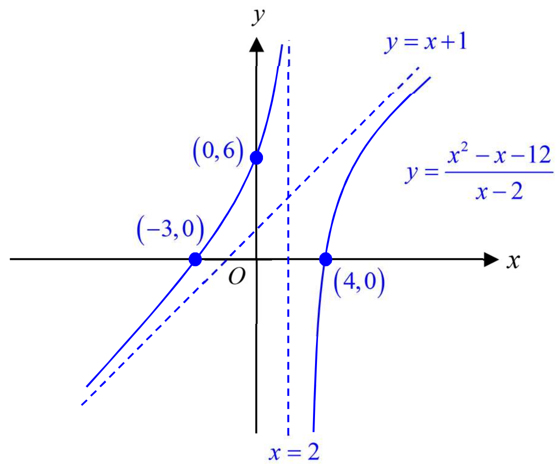


(d) $y=\dfrac{x^2-x-12}{x-2}=x+1-\dfrac{10}{x-2}$

Asymptotes:
$y=x+1$
$x=2$

Intercepts:
When $x=0$, $y=6 \Rightarrow (0,6)$
When $y=0$, $x^2-x-12=0$
$x=-3$ or 4
$\Rightarrow (-3,0)$ and $(4,0)$

Stationary Points:
None

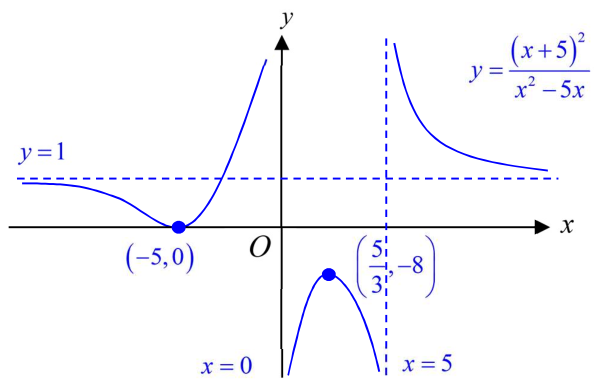


(e) $y=\dfrac{(x+5)^2}{x^2-5x}=1+\dfrac{15x+25}{x(x-5)}$

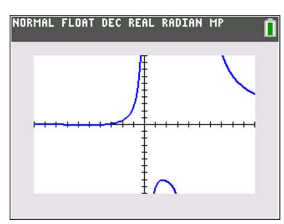
Asymptotes:
$y=1$
$x=0$
$x=5$

Intercepts:
When $x=0$, y is not defined.
When $y=0$, $x=-5 \Rightarrow (-5,0)$

Stationary Points:
Use G.C.



Remark:
G.C. in $\boxed{\text{zoom}}$ **6: ZStandard** does not display the entire curve clearly for **Example 9(e)**. In addition, from the G.C., it seems that as $x\to-\infty$, $y\to 0$. However, this is not true. As such, it is a good practice to determine the asymptotes and intercepts of the graph analytically/algebraically before using the G.C. to obtain the shape.



Self-Practice 4:

Sketch the graph of the following curves, indicating clearly axial intercept(s), asymptote(s) and stationary point(s), if any:

(i) $y=\dfrac{1}{3-x}$ **(ii)** $y=\dfrac{2x+1}{x-2}$ **(iii)** $y=\dfrac{x^2}{x-1}$ **(iv)** $y=\dfrac{x^2+2x-3}{x+2}$

Answers:

(iii) Max.: $(0,\ 0)$; Min.: $(2,\ 4)$

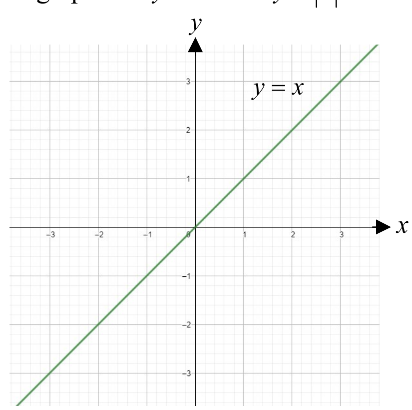
3.4 Modulus (Absolute) Functions

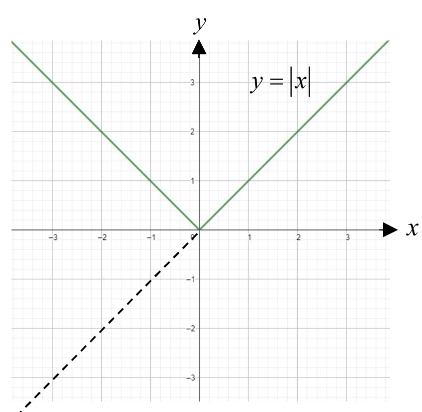
If $y=x$, y is negative when x is negative. But if $y=|x|$, y takes the numerical value and not the sign of x, i.e. y is always non-negative.

Taking values of x from -3 to $+3$ we have

x	-3	-2	-1	0	1	2	3
$y=\lvert x\rvert$	3	2	1	0	1	2	3

Thus the graphs of $y=x$ and $y=|x|$ are

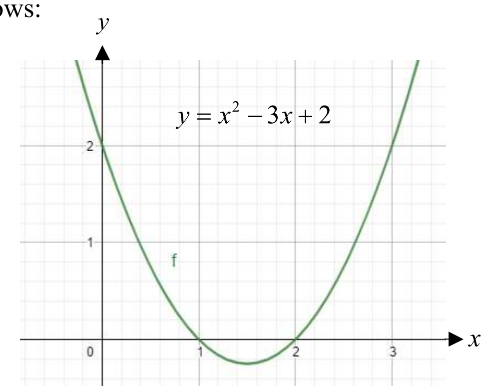


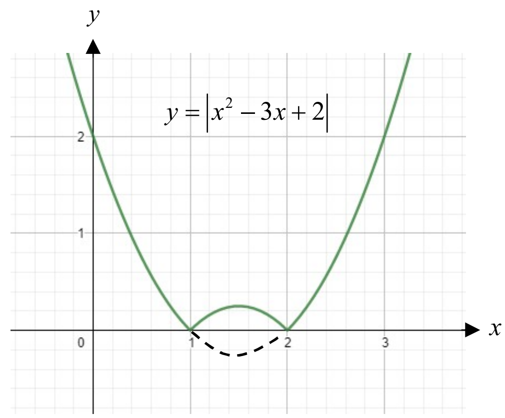


It can be seen that, for negative values of x, $y=|x|$ is the reflection of $y=x$ in the x-axis.

In general, the graph C_1 with equation $y=|\mathrm{f}(x)|$ can be obtained from the graph C_2 of $y=\mathrm{f}(x)$ by reflecting in the x-axis those points on C_2 for which $\mathrm{f}(x)<0$, the remaining sections of C_2 being retained without change.

For instance, the graph of $y=\left|x^2-3x+2\right|$ can be obtained from the graph of $y=x^2-3x+2$ as follows:





When a section of a graph $y=\mathrm{f}(x)$ is reflected in the x-axis, the y coordinate of every point on that section of the graph changes sign.

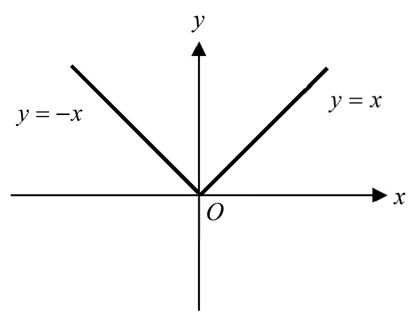
Thus the equation of the reflected section becomes

$$y=-\mathrm{f}(x)$$

e.g. when $y=|x|$

the Cartesian equations are

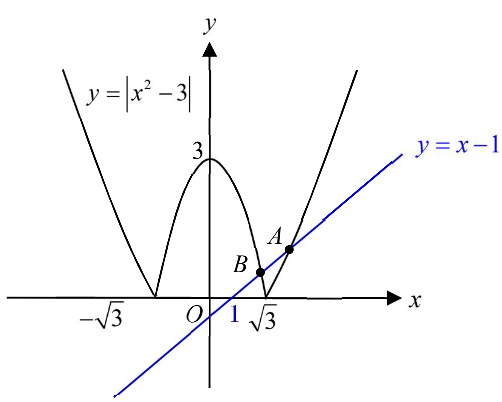
$$\begin{cases} y=x & \text{for } x\ge 0 \\ y=-x & \text{for } x<0 \end{cases}$$



Example 10:

Sketch the graphs of $y=x-1$ and $y=\left|x^2-3\right|$ on the same axes and determine the coordinates of the point of intersections of these two graphs.

Solution:



The Cartesian equations of the various sections of the line and curve show that

at A $\quad y=x-1$ cuts $\quad y=x^2-3$

at B $\quad y=x-1$ and $\quad y=-\left(x^2-3\right)$

Thus the coordinates of A are given by the equation

$$x-1=x^2-3$$
$$x^2-x-2=0$$
$$(x+1)(x-2)=0$$
$$x=-1 \quad \text{or} \quad x=2$$

But it is clear from the diagram that $x \neq -1$.

So, at A, $\quad x=2 \quad$ and $\quad y=x-1 \Rightarrow y=1$

$\therefore A(2, 1)$

Similarly the coordinates of B are given by

$$x-1=-(x^2-3)$$
$$x^2+x-4=0$$
$$x=\frac{-1\pm\sqrt{1-4(1)(-4)}}{2}$$
$$=\frac{-1\pm\sqrt{17}}{2}$$

Again from the diagram we see that $x \neq \frac{-1-\sqrt{17}}{2}$.

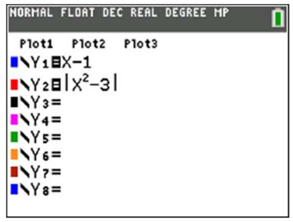
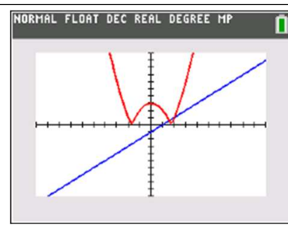
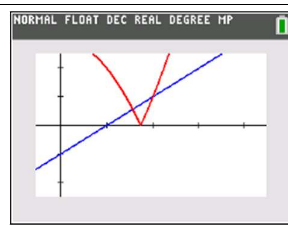
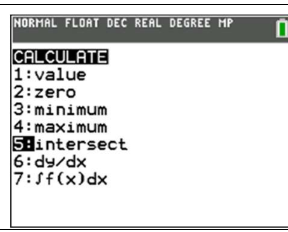
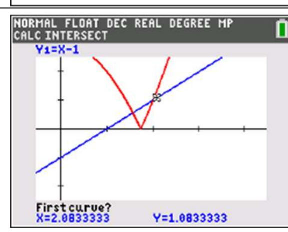
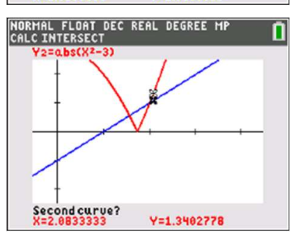
So, at B, $\quad x=\frac{-1+\sqrt{17}}{2} \quad$ and $\quad y=\frac{-1+\sqrt{17}}{2}-1 \Rightarrow y=\frac{-3+\sqrt{17}}{2}$

$\therefore B\left(\frac{-1+\sqrt{17}}{2}, \frac{-3+\sqrt{17}}{2}\right)$

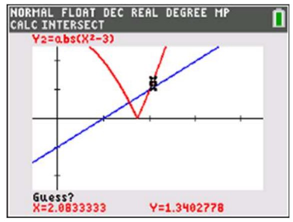
🗎 **Remark:**

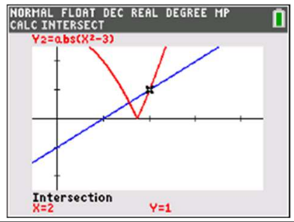
If intersection occurs between two lines or curves whose equations involve a modulus, care must be taken to use the correct pair of Cartesian equations when locating the point(s) of intersection.

Using G.C. to find Coordinates of Points of Intersections of Two Graphs:

	Press	Screen Display	Remarks
1	Key in the equation $y = x - 1$ in **Y$_1$** and $y = \|x^2 - 3\|$ in **Y$_2$** .		Modulus can be obtained from math, followed by ▶ and select **1:abs(**. Alternatively, it can also be obtained from alpha window and select **1:abs(**.
2	Press zoom and select **6:ZStandard**.		**6:ZStandard** allows us to view the graph in the standard window setting, i.e. $-10 \le x \le 10$ and $-10 \le y \le 10$.
3	Press zoom and select **2:Zoom In**. Move the cursor nearer to the points of intersections and press enter. .		**2:Zoom In** allows us to zoom in on a particular portion of the curve.
4	Press 2nd followed by trace and select **5:intersect**.		**5:intersect** allows us to find the point of intersection of the graphs.
5	**To find *A*:** The cursor should indicate the first curve **Y$_1$**. Move the cursor closer to point *A* and press enter. The cursor should now indicate the second curve **Y$_2$**. Press enter.	 	

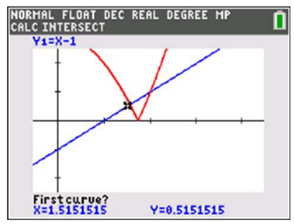
<table>
<tr><th></th><th>Press</th><th>Screen Display</th><th>Remarks</th></tr>
<tr><td></td><td>The G.C. will now ask you to guess the x-coordinate of A. Simply press $\boxed{\text{enter}}$.

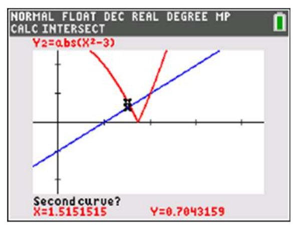
The point intersection will be displayed at the bottom of the G.C.</td><td>

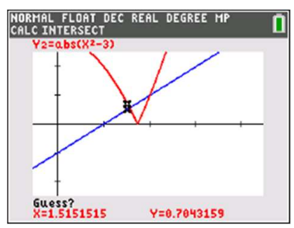

</td><td></td></tr>
<tr><td>6</td><td>To find B:
The cursor should indicate the first curve $\mathbf{Y_1}$. Move the cursor closer to point B and press $\boxed{\text{enter}}$.

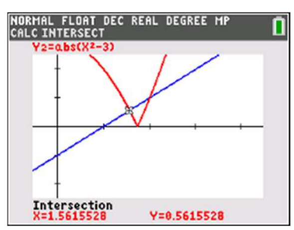
The cursor should now indicate the second curve $\mathbf{Y_2}$. Press $\boxed{\text{enter}}$.

The G.C. will now ask you to guess the x-coordinate of B. Simply press $\boxed{\text{enter}}$.

The point intersection will be displayed at the bottom of the G.C.</td><td>






</td><td></td></tr>
</table>

Remark:

If question requires exact values then algebraic/analytical method must be used, as G.C. will provide a non-exact value.

4 Further Examples

Example 11: **[2019/CJC/Promo/2]**

The curve C has an equation $y=\dfrac{ax^2+bx+a}{x^2-b}$ where a and b are constants.

(i) Given that two of the asymptotes of C are $x=2$ and $y=3$, determine the values of a and b.

(ii) Sketch the curve C, giving the coordinates of the point(s) where the curve crosses the axes, stationary points and the equations of all asymptotes.

Solution:

(i)
$$y=\frac{ax^2+bx+a}{x^2-b}$$
$$=\frac{ax^2+bx+a}{\left(x-\sqrt{b}\right)\left(x+\sqrt{b}\right)}$$

$$\begin{array}{r} a \\ x^2-4\,\big)\,\overline{ax^2+4x+a} \\ -\left(ax^2 \qquad -4a\right) \\ \hline 4x+5a \end{array}$$

So, $x=\sqrt{b}$ and $x=-\sqrt{b}$ are asymptotes.

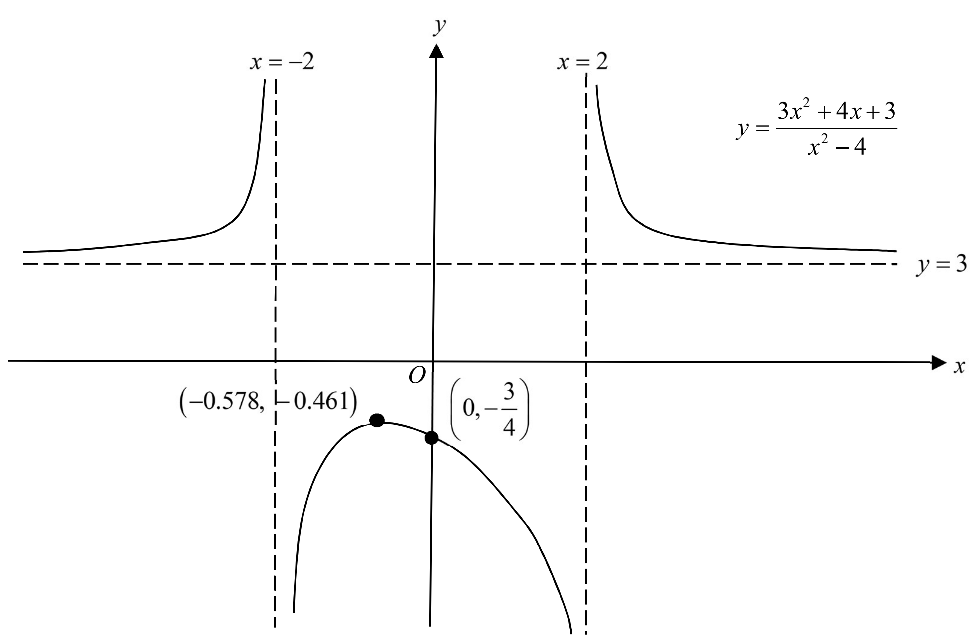
Hence, $\sqrt{b}=2 \quad \Rightarrow \quad b=4$.

$$y=\frac{ax^2+bx+a}{x^2-b}=a+\frac{4x+5a}{x^2-4}$$

So, $y=a$ is an asymptote.

Hence, $a=3$.

(ii)



Example 12: The curve C has equation $y=\dfrac{x^2-2x+2}{x+1}$.

(i) Determine the asymptotes of C.

(ii) Sketch C, showing clearly any axial intercepts, turning points and asymptotes.

(iii) Deduce the number of real roots of the equation $x^4-2x^3+2x^2-x-1=0$.

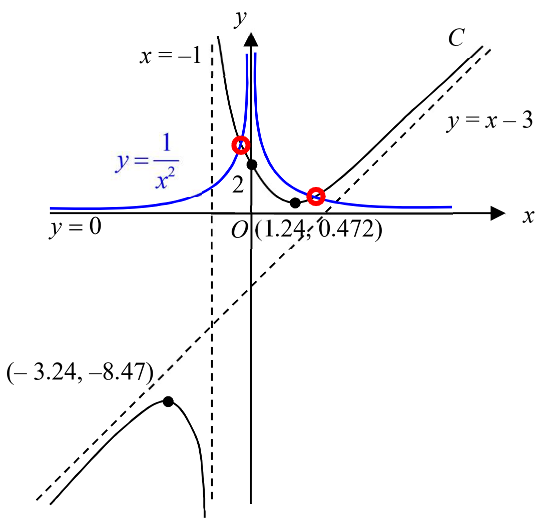
Solution:

(i) $y=\dfrac{x^2-2x+2}{x+1}=x-3+\dfrac{5}{x+1}$

$\therefore$ asymptotes are $y=x-3$ and $x=-1$.

$$\begin{array}{r} x-3 \\ x+1\overline{)\,x^2-2x+2} \\ -\left(x^2+x\right) \\ \hline -3x+2 \\ -\left(-3x-3\right) \\ \hline 5 \end{array}$$

(ii)



(iii)

$$x^4-2x^3+2x^2-x-1=0$$
$$x^4-2x^3+2x^2=x+1$$
$$x^2(x^2-2x+2)=x+1$$
$$\frac{x^2-2x+2}{x+1}=\frac{1}{x^2}$$
$$y=\frac{1}{x^2}$$

Sketch $y=\dfrac{1}{x^2}$ on the same axes in (ii).

Since there are 2 points of intersection, there are 2 real roots.

Example 13: **[2018/RI/Promo/9]**

The curve C has equation $y=\dfrac{x^2+3x-1}{x-2}$, $x\in\mathbb{R}$, $x\neq 2$.

(i) Using an algebraic method, find the range of values that y can take.

(ii) Sketch C, stating the equations of the asymptotes, axial intercepts and the coordinates of the turning points, if any.

(iii) State the range of values that m can take such that the equation $\dfrac{x^2+3x-1}{x-2}=m(x-2)+7$ has 2 distinct solutions, justifying your answer.

Solution:

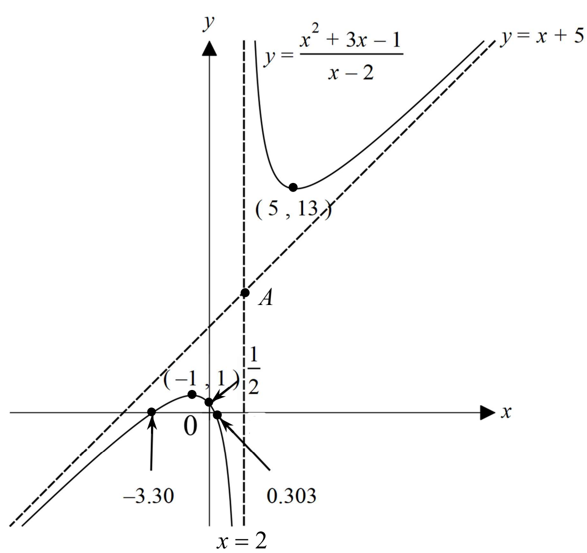
(i) $$y=\frac{x^2+3x-1}{x-2}$$

$$xy-2y=x^2+3x-1$$
$$x^2+(3-y)x+(2y-1)=0$$

If y is a value the expression can take, the final quadratic equation will have a real solution for x. Therefore,

$$(3-y)^2-4(1)(2y-1)\geq 0$$
$$y^2+9-6y-8y+4\geq 0$$
$$y^2-14y+13\geq 0$$
$$(y-1)(y-13)\geq 0$$
$$y\leq 1 \quad\text{or}\quad y\geq 13$$

(ii) $y=\dfrac{x^2+3x-1}{x-2}=x+5+\dfrac{9}{x-2}$



(iii) For $\dfrac{x^2+3x-1}{x-2}=m(x-2)+7$ to have 2 distinct solutions, C must intersect the line l $y=m(x-2)+7$ at 2 different points. Note that l passes through the point $A(2,7)$ with gradient m. Since A is also the intersection of the 2 asymptotes of C (refer to sketch in **(ii)**), l must be steeper than the oblique asymptote $y=x+5$ and its gradient must remain positive. $\therefore$ $m>1$.

MASTERY LEARNING OBJECTIVES	Graphing Techniques

At the end of this chapter, I should be able to:

	At the end of lecture…	At the end of tutorial…
Features of Graphs		
• Find axial intercepts • Find asymptotes • Find turning points • Find lines of symmetry • Find possible range of values of x and y		
Using a G.C. to		
• obtain the graph of a given function (apply essential G.C. functions such as Zoom In, ZBox, window settings etc.).		
• obtain the axial intercept(s) of a graph, if any.		
• obtain the coordinates of stationary point(s) on a curve, if any.		
Use an analytical method to		
• determine the asymptote(s) of a graph, if any.		
• obtain the axial intercept(s) of a graph, if anfy.		
• obtain the coordinates of stationary point(s) on a curve, if any.		

H2 MATHEMATICS TUTORIAL
TOPIC ***GRAPHING TECHNIQUES*** ***2022/JC1***

DISCUSSION

1 Sketch the graph represented by each of the following equations, clearly indicating the equations of asymptote(s), axial intercept(s), and the coordinates of the turning point(s), if any.

(a) $y = 2x^3 - 5x^2 + x - 2$

(b) $y = 4x^2 e^x$

(c) $y = \dfrac{\ln x}{2x}$

(d) $y = \dfrac{x^2 + 2x - 8}{x^2 + 2x - 3}$

(e) $y = \left|2x - x^2\right|$

(f) $y = \dfrac{ax+1}{x-b}$, where constants a , b are given by

(i) $-1 < a < 0$ and $0 < b < 1$;

(ii) $a < -1$ and $b > 1$.

What could be said about the graph if $ab + 1 = 0$?

2 **[2013/YJC/I/12]**
The curve C has equation

$$y = \frac{x^2}{x-2}.$$

(i) Find the equation(s) of the asymptote(s) of C .

(ii) Sketch the curve C , labelling the equation(s) of its asymptote(s) and coordinates of any axial intercepts and turning points.

(iii) Hence find the range of values of k for which the equation $x^2 = k\left(x^2 - 4\right)$ has no real roots.

3 **[2009/IJC/II/1]**
The curve C has equation

$$y = \frac{x^2 + 9x + 16}{x+2}.$$

(i) Find the equations of the asymptotes of C .

(ii) Find, leaving your answers in exact form, the range of values of k for which the equation $k = \dfrac{x^2 + 9x + 16}{x+2}$ has no real roots.

(iii) Sketch C . Show, on your diagram, the equations of the asymptotes, the coordinates of the stationary points, and the coordinates of the points of intersection of C with the x- and y-axes.

4 **[2014/NJC/MYE/7(part)]**

The curve C_1 is given by the equation

$$y=\frac{ax^2+bx}{x+c}.$$

(i) The curve C_1 has a vertical asymptote at $x=2$. The linear oblique asymptote of C_1 passes through the points with coordinates $(0, 10)$ and $(-2, 0)$. Determine the values of the unknown constants a, b and c.

(ii) Hence sketch C_1, stating clearly the equation(s) of any asymptote(s) and the coordinates of any stationary point(s).

5 **[2007/HCI/I/13(modified)]**

The curve C has equation

$$y=\frac{x^2+ax+4}{x+b}.$$

It is given that C has a vertical asymptote $x=-1$ and a stationary point at $x=2$.

(i) Determine the values of a and b.

(ii) Find the equation of the other asymptote of C.

(iii) Find, without using a calculator, the set of values that y cannot take.

(iv) Sketch C, showing clearly any axial intercepts, asymptotes and stationary points.

(v) Deduce the number of real roots of the equation $\left(4-x^2\right)\left(x+1\right)^2=\left(x^2-4x+4\right)^2$.

6 **[2009/RJC/I/10]**

The curve C has equation

$$y=\frac{2x^2+ax+3a}{x+b}, \text{ where } a \text{ and } b \text{ are non-zero constants.}$$

It is given that $x=2$ is an asymptote of C.

(i) State the value of b.

(ii) By differentiation, find the range of values of a for which C has no stationary points.

(iii) Given that $a=-3$, sketch C giving the equations of the asymptotes and the coordinates of any points of intersection with the axes.

Answers:

1 **(a)** Axial intercepts: $(0,-2)$, $(2.46,0)$; Min. $(1.56,-5.02)$ Max. $(0.107,-1.95)$

(b) Axial intercept/min. pt.: $(0,0)$; Max. pt.: $(-2.00,2.17)$; Asymptote: $y=0$

(c) Asymptotes: $y=0$, $x=0$; Axial intercept: $(1,0)$**;** Max. $(2.72,0.184)$

(d) Asymptotes: $y=1$, $x=-3$, $x=1$; Axial intercepts: $\left(0,\frac{8}{3}\right),(-4,0),(2,0)$;

Min. pt.: $(-1, 2.25)$

(e) Axial intercepts: $(0,0)$, $(2,0)$; Max. $(1,1)$

(f) **(i)** Asymptotes: $x=b$, $y=a$; Axial intercepts: $\left(0,-\frac{1}{b}\right),\left(-\frac{1}{a},0\right)$

(ii) Asymptotes: $x=b$, $y=a$; Axial intercepts: $\left(0,-\frac{1}{b}\right),\left(-\frac{1}{a},0\right)$

[**Note:** Graphs represented in **(i)** and **(ii)** are different.]

2 **(i)** Asymptotes: $y=x+2$, $x=2$

(ii) Axial intercept/Max. pt.: $(0,0)$; Min. pt.: $(4,8)$ **(iii)** $0<k\le 1$

3 **(i)** Asymptotes: $y=x+7$, $x=-2$ **(ii)** $\left\{k\in\mathbb{R}:5-2\sqrt{2}<k<5+2\sqrt{2}\right\}$

(iii) Min. pt.: $(-0.586,7.83)$, Max. pt.: $(-3.41,2.17)$;

Axial intercepts: $(0,8)$, $(-6.56,0)$, $(-2.44,0)$

4 **(i)** $a=5$, $b=0$, $c=-2$

(ii) Axial intercept/Max. pt.: (0, 0), Min. pt.: (4, 40)

5 **(i)** $a=-4$, $b=1$ **(ii)** $y=x-5$ **(iii)** $\{y\in\mathbb{R}:-12<y<0\}$

(iv) Max. pt.: $(-4,-12)$, Min. pt.: (2, 0); Axial intercepts: $(0,4)$, $(2,0)$ **(v)** 2 real roots

6 **(i)** $b=-2$ **(ii)** $a<-\frac{8}{5}$

(iii) Asymptotes: $y=2x+1$, $x=2$; Axial intercepts: $\left(0,\frac{9}{2}\right)$, $\left(-\frac{3}{2},0\right)$, $(3,0)$

ESSENTIAL PRACTICE

1 **(i)** Determine the coordinates of the turning point(s) in the graph represented by $y=-(x+a)(x+b)^2$, where a, b are constants.

(ii) Sketch the graph represented by $y=-(x+a)(x+b)^2$, clearly indicating the coordinates of the axial intercept(s) and turning point(s),

(a) if $a>0$ and $b<0$;

(b) if $a<b<0$.

2 **[2013/CJC/CA1/2]**

The curve C has equation

$$y=\frac{x^2+3x+1}{2x-1}.$$

(i) Sketch C, indicating clearly any axial intercepts, asymptotes and stationary points. [5]

(ii) Hence, by drawing a sketch of another suitable curve in the diagram of **(i)**, deduce the number of real roots of the equation $2x^4+6x^3+2x^2+2x-1=0$. [3]

3 **[2013/SRJC/MidYear/9(a)(modified)]**

The curve C has equation

$$y=\frac{-9x^2+6x-1}{x-1}.$$

(i) Find, without using a calculator, the range of values of y for which there are no points on the curve. [3]

(ii) Sketch the curve C, showing clearly the coordinates of the turning points, axial intercepts and the equations of the asymptotes. [3]

4 **[2010/NYJC/Promo/10]**

The curve C has equation

$$y=\frac{x^2+a}{x-a}, x\neq a \text{ and } a \text{ is a non-zero constant.}$$

(i) Given that the oblique asymptote of C is $y=x+2$, find the value of a. [1]

(ii) Find the set of values of a if C has two turning points. [3]

(iii) Using the value of a found in part **(i)** and using an algebraic method, show that C cannot exist for $y_1<y<y_2$ where values of y_1 and y_2 are to be determined in exact form. [3]

(iv) Hence sketch the graph of C, showing clearly the equations of the asymptotes and coordinates of the stationary points. [3]

5 **[2010/RI(JC)/Promo/3]**

Sketch the graph of $y=\dfrac{2x^2+1}{x-1}$, showing clearly the coordinates of any turning point(s) and axial intercept(s), and the equation(s) of any asymptote(s). Hence, state the range of values for a such that there are no real solutions to the equation $2x^2+1=a(x-1)\cos(bx+c)$, $x\neq 1$, for all values of b and c. [6]

6 **[2012/SRJC/Promo/8]**

The curve C has equation $y=\mathrm{f}(x)$ where $\mathrm{f}(x)=\dfrac{4x^2+bx+c}{4x+8}$, where b and c are real constants. It is given that C has an oblique asymptote $y=x-5$.

(i) Find b. [2]

(ii) Given that $c>9$, show that the curve C does not cut the x-axis. [1]

(iii) Sketch the graph when $c=13$, indicating clearly any asymptotes, coordinates of axial intercepts and turning points. [3]

7 **[2013/MI/I/6]**

The curve C has equation

$$y=\frac{x^2+2x+3}{x+1}.$$

(i) Find the range of values of y for which C does not lie within, in exact form. [3]

(ii) Sketch C, showing clearly the axial intercept, asymptotes and stationary points. [3]

(iii) By adding an additional sketch for $-\pi\leq x\leq\pi$, solve

$$3(x+1)\cos x-1=x^2+2x+2.$$ [3]

8 **[2015/VJC/MYE/8(part)]**

The curve G has equation

$$y=\frac{2x^2+kx-1}{x+1}, \text{ where } k \text{ is a constant.}$$

(i) Given that G has no stationary point, show that $k>1$. [2]

(ii) Given that $y=2x+1$ is an asymptote of G, show that $k=3$. Hence, sketch G. [4]

9 **[2015/SRJC/MYE/9]**

The curve C has equation

$$y=\frac{-2x^2-2x+a}{x+1} \text{ where } a \text{ is non-zero constant.}$$

Curve C exists when $y \le -2$ or $y \ge b$, where b is a positive constant.

(i) Using an analytical method, show that $a=-2$ and hence find the value of b. [5]

(ii) Sketch the curve C, showing clearly the coordinates of the turning points and the equations of the asymptotes. [3]

(iii) Show that the point $(-3,6)$ lies on the line $y=px+3p+6$, $p \in \mathbb{R}$. Hence using the sketch in **(ii)**, find the range of values of p for which the equation $px+3p+6=\dfrac{-2x^2-2x-2}{x+1}$ has two distinct real roots. [2]

Answers:

1 **(i)** Turning pts.: $\left(-\frac{1}{3}(2a+b),\ \frac{4}{27}(b-a)^3\right)$, $(-b,\ 0)$

(ii) Axial intercepts: $\left(0,-ab^2\right),\left(-a,\ 0\right),\left(-b,0\right)$

[**Note:** The graphs represented in **(a)** and **(b)** are different.]

2 **(i)** Asymptotes: $x=\frac{1}{2}$ & $y=\frac{1}{2}x+\frac{7}{4}$; Axial intercepts: $\left(-2.62,0\right)$, $\left(-0.382,0\right)$, $\left(0,-1\right)$;

Turning pts.: $\left(-1.16,0.342\right)$, $\left(2.16,3.66\right)$

(ii) 2 real roots

3 **(i)** $-24<y<0$

(ii) Asymptotes: $y=-9x-3$ & $x=1$; Axial intercepts: $\left(\frac{1}{3},0\right)$, $\left(0,1\right)$;

Turning pts.: $\left(0.333,0\right)$, $\left(1.67,-24\right)$

4 **(i)** $a=2$ **(ii)** $\{a\in\mathbb{R}:a<-1 \text{ or } a>0\}$

(iii) $y_1=4-2\sqrt{6}$, $y_2=4+2\sqrt{6}$

(iv) Asymptotes: $y=x+2$ & $x=2$; Turning pts.: $\left(2+\sqrt{6},4+2\sqrt{6}\right)$, $\left(2-\sqrt{6},4-2\sqrt{6}\right)$

5 Asymptotes: $y=2x+2$ & $x=1$; Axial intercepts: $\left(0,-1\right)$;

Turning pts.: $\left(-0.225,-0.899\right)$, $\left(2.22,8.90\right)$; $-0.899<a<0.899$

6 **(i)** $b=-12$

(iii) Asymptotes: $y=x-5$ & $x=-2$; Axial intercepts: $\left(0,\frac{13}{8}\right)$;

Turning pts.: $\left(1.64,0.280\right)$, $\left(-5.64,-14.3\right)$

7 **(i)** $-2\sqrt{2}<y<2\sqrt{2}$

(ii) Asymptotes: $y=x+1$ & $x=-1$; Axial intercepts: $\left(0,3\right)$;

Turning pts.: $\left(-2.41,-2.83\right)$, $\left(0.414,2.83\right)$

(iii) $x=0$ or 0.335 (3 s.f.)

8 **(ii)** Asymptotes: $y=2x+1$ & $x=-1$; Axial intercepts: $\left(0,-1\right)$, $\left(-1.78,0\right)$, $\left(0.281,0\right)$

9 **(i)** $b=6$

(ii) Asymptotes: $y=-2x$ & $x=-1$; Axial intercepts: $\left(0,\ -2\right)$;

Turning pts.: $\left(-2,\ 6\right)$, $\left(0,\ -2\right)$

(iii) $p>0$ or $p<-2$