1 **(i)** Find the expressions for $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\big(\mathrm{f}(x)\mathrm{g}(x)\big)$ in terms of the functions and their derivatives for $n=1,\ 2.$ [2]

(ii) Hence, solve

(a) $\left(\sin^{-1}x\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{y}{\sqrt{1-x^2}}=\mathrm{e}^x$, given $y=1$ when $x=1$. [3]

(b) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-\left(2\tan x\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}-y=x.$ [4]

2 Let the sequence of numbers $T_1,\ T_2,\ T_3,\ \ldots,\ T_n,\ \ldots,$ for $n\ge 1$ with $T_1=1$ satisfy the recurrence relation

$$T_{a+b}=T_a+T_b+ab$$

for all integers $a,\ b$ greater than or equal to 1.

(i) Show that $T_n=\dfrac{n(n+1)}{2}$ for $n\ge 1$. [3]

(ii) Verify that $T_{ab}=T_aT_b+T_{a-1}T_{b-1}$. [1]

(iii) Show that $T_n\equiv 0$ or $1\ (\mathrm{mod}\,3)$ for all positive integers n. [3]

Three positive integers $x,\ y,\ z$ are collectively denoted as a Pythagorean Triple if $x^2+y^2=z^2$.

(iv) Show that $8T_n+1$ is a perfect square for all positive integers n. [1]

(v) Hence or otherwise, show that there exist Pythagorean Triples with $y=4T_n$ for any positive integer n. [2]

3 Let f be a monotone decreasing function defined on the interval $[N, \infty)$ where N is an integer. The integral test for convergence states that the infinite series $\sum_{n=N}^{\infty} \mathrm{f}(n)$ converges if and only if the integral $\int_{N}^{\infty} \mathrm{f}(n)\,\mathrm{d}n$ is finite.

(i) Show $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges otherwise. [3]

(ii) By considering an appropriate comparison with the series in part **(i)** with an appropriate value of *p*, show that $\sum_{n=1}^{\infty} \frac{n}{4n^4+1}$ converges. [2]

(iii) Show that $\sum_{n=1}^{\infty} \frac{n}{4n^4+1}$ converges by directly applying the integral test for convergence. [2]

(iv) Verify that $\left(2n^2+2n+1\right)\left(2n^2-2n+1\right) = 4n^4+1$.

Hence or otherwise, show that $\sum_{n=1}^{\infty} \frac{n}{4n^4+1} = \frac{1}{4}$. [3]

4 A function f is continuous at the point k if $\lim_{x \to k} \mathrm{f}(x) = \mathrm{f}(k)$. Let $\mathrm{f} : \mathbb{R} \to \mathbb{R}$ be a function satisfying the following conditions:

- $\mathrm{f}\left(x+y\right) = \mathrm{f}\left(x\right) + \mathrm{f}\left(y\right)$, for all $x, y \in \mathbb{R}$,
- $\lim_{h \to 0} \frac{\mathrm{f}\left(h\right)}{h} = 1$.

(i) Find $\mathrm{f}\left(0\right)$. [2]

(ii) Suppose that $\mathrm{f}\left(1\right) = 1$, show that $\mathrm{f}\left(n\right) = n$, for any positive integer *n*. [2]

(iii) Using part **(ii)**, prove that $\mathrm{f}\left(x\right) = x$, for any positive rational number *x*. [3]

(iv) By considering $\lim_{h \to 0} \mathrm{f}\left(h\right)$, show that f is continuous at any point $x \in \mathbb{R}$. [4]

5 There are three types of flowers: roses, sunflowers and tulips.

(i) Twelve flowers are selected to form a bunch. In how many ways can this be done

(a) if there are no other restrictions, [2]

(b) if at least two of each type of flower are selected? [3]

(ii) Twelve flowers are now selected and placed in a row on the table. In how many ways can that be done

(a) if there are no other restrictions, [1]

(b) if at least one of each type of flower is selected? [4]

(iii) Ten sunflowers are distributed among three identical vases such that there is at least one flower in each vase. In how many ways can that be done? [2]

(iv) Instead of just the ten sunflowers, one rose and one tulip are added. The twelve flowers are to be distributed among three identical vases such that there is at least one flower in each vase, but the rose and the tulip must be in different vases. In how many ways can that be done? [2]

6 **(a)** Let $M_1 = 2+1,\ M_2 = 2\cdot 3+1,\ M_3 = 2\cdot 3\cdot 5+1,\ \ldots$, i.e. $M_k = p_1 \cdot p_2 \cdot \ldots \cdot p_k + 1,$ where p_k denotes the k th prime number. Prove that none of the numbers M_k is a perfect square. [4]

(b) Let *p* be a prime number. Show that for any positive integers *x* and *y*,

$$(x+y)^p \equiv \left(x^p + y^p\right) \pmod{p}.$$ [4]

(c) Let *p* be a prime number larger than 7. It is a fact that infinitely many terms of the sequence

$$7, 77, 777, 7777, \ldots$$

are divisible by *p*.

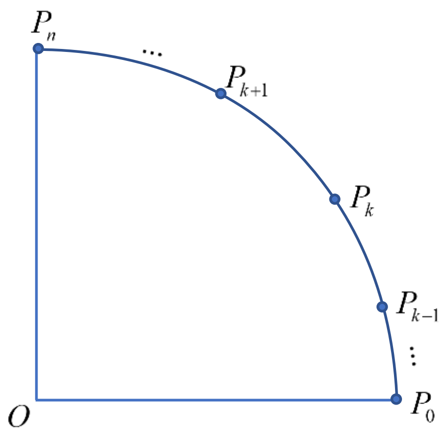
One way to prove this is to show that the following two statements are both true:

Statement A: At least one term of the sequence is divisible by *p*.

Statement B: If there exists one term of the sequence that is divisible by *p*, then there are infinitely many terms that are divisible by *p*.

Prove the two statements above. [7]

7 There are $n+1$ points $P_0, P_1, \ldots, P_n$ located on the arc of the unit circle in the first quadrant. P_0 is the point $(1, 0)$, P_n is the point $(0, 1)$, and the points $P_0, P_1, \ldots, P_n$ lie in anti-clockwise order, as shown in the diagram. Let $d\left(P_i, P_j\right)$ represent the straight-line distance between points P_i and P_j. Define S_n to be the sum $S_n = \sum_{i=1}^{n} d\left(P_{i-1}, P_i\right)$.



(a) **(i)** Suppose that points P_{k-1} and P_{k+1} are fixed. Show that the sum $d\left(P_{k-1}, P_k\right) + d\left(P_k, P_{k+1}\right)$ is maximised when P_k lies on the mid-point of the arc $P_{k-1}P_{k+1}$. [4]

(ii) Deduce the maximum value of S_n in terms of n. [3]

(iii) Hence by considering an appropriate limit, show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. [4]

(b) The derivative of a function F is the function f satisfying $\mathrm{f}(x) = \lim_{h \to 0} \frac{\mathrm{F}(x+h) - \mathrm{F}(x)}{h}$.

Show that the derivative of the function $\mathrm{F}: \mathbb{R} \to \mathbb{R}$, $\mathrm{F}(x) = \cos x$ is the function $\mathrm{f}: \mathbb{R} \to \mathbb{R}$, $\mathrm{f}(x) = -\sin x$. [4]

8 **(a)** Let a and b be distinct positive integers with $a\,|\,(2b+1)$ and $b\,|\,(2a+1)$.

(i) Prove that a and b are coprime. [2]

(ii) Show that $ab\,|\,(2a+2b+1)$. [2]

(iii) Explain why a and b are both odd. [1]

Given also that $a<b$,

(iv) Use part **(ii)** to show that $a^2<6a+3$. [2]

(v) Hence find all the possible solutions for a and b. [4]

(b) Given $n\geq 2,\ k\geq 1,$ and n has exactly k distinct prime factors $p_1,\ p_2,\ \ldots,\ p_k$.

(i) How many integers from the set $\{1,\ 2,\ \ldots,\ n\}$ are divisible by each of these k distinct primes? State your answer in terms of $n,\ p_1,\ p_2,\ \ldots,\ p_{k-1}$ and p_k. [1]

The Euler's totient function $\phi(t)$ counts the positive integers up to a given integer t that are relatively prime to t.

(ii) Given that $m=a^{\alpha}b^{\beta}c^{\gamma}$ where $a,\ b,\ c$ are distinct prime numbers and $\alpha,\ \beta,\ \gamma$ are positive integers, use the principle of inclusion and exclusion to show that

$$\phi(m)=m\left(\frac{a-1}{a}\right)\left(\frac{b-1}{b}\right)\left(\frac{c-1}{c}\right),$$

regardless of the value of $\alpha,\ \beta,\ \gamma$. [2]

(iii) Hence, deduce a formula for $\phi(n)$. [1]

(iv) Find the value of $\phi(2^3\times 5\times 7^4\times 11^2\times 47^2)$. [1]