1. With respect to the origin as the pole, a curve C has polar equation of the form

$$r = a + b\sin 2\theta \quad \text{for } -\pi < \theta \le \pi,$$

where a and b are constants, and b is non-zero.

(a) Find the range of values of $\frac{a}{b}$ so that there are tangents to C at the pole. [2]

(b) For the case where $a = 1$ and $b = 2$, sketch C, stating clearly the equations of the tangents to C at the pole, and the equations of the line of symmetry of C. [4]

2. Consider the equation $\mathrm{f}(x) = 0$, where $\mathrm{f}(x) = \cos x - x - 1 - \beta$ and β is very close to zero.

(a) By sketching the curve $y = \cos x - x - 1$, show that $\mathrm{f}(x) = 0$ has a single root α, and that α is very close to 0. [3]

(b) Use two iterations of the Newton-Raphson method, with initial approximation $x_0 = 0$, to show that $\alpha \approx -\beta + \frac{1}{2}\frac{\beta^2}{\beta - 1}$. [You may assume for θ small, $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$.] [5]

3. ~~Points F and G lie on the x-axis at the points $(-c, 0)$ and $(c, 0)$ respectively, with $c > 0$. The curve L is defined as the locus of points P for which the total distance $FP + PG = 2a$, where $a > c$. It is given that $FP = r$ and $\angle GFP = \theta$.~~

~~**(a)** Show that $r = \frac{l}{1 - k\cos\theta}$, giving the constants l and k in terms of a and c. [4]~~

~~A, B, C, D are points on L such that the chords AB and CD both pass through F, and AB is perpendicular to CD.~~

~~**(b)** Find $\frac{1}{|FA|} + \frac{1}{|FB|}$ in terms of a and c. [2]~~

~~**(c)** Find $\frac{1}{|AB|} + \frac{1}{|CD|}$ in terms of a and c. [3]~~

4. ~~A sequence~~ $\cancel{u_1,\ u_2,\ u_3,\ \ldots}$ ~~is such that~~ $\cancel{u_1 = \frac{2}{3}}$ ~~and~~ $\cancel{u_n = u_{n-1} - \frac{1}{4n^2-1}}$, ~~for all~~ $\cancel{n \geq 2}$.

(a) ~~Find the values of~~ $\cancel{u_2,\ u_3}$ ~~and~~ $\cancel{u_4}$~~, giving each answer as a fraction in its lowest term. Make a conjecture for~~ $\cancel{u_n}$ ~~in terms of~~ $\cancel{n}$. ~~[2]~~

(b) ~~Use Mathematical Induction to prove your conjecture in **(i)** for all positive integers~~ $\cancel{n}$. ~~[4]~~

(c) ~~Deduce the value of~~ $\cancel{\sum_{n=2}^{\infty} \frac{1}{4n^2-1}}$. ~~[3]~~

5. A function f is defined as $\mathrm{f}(x) = x^3 - x^{-1}$, $x \in \mathbb{R}$, $x > 1$.

(a) Using the trapezium rule with 3 ordinates, the value for $\int_2^a \mathrm{f}(x)\,\mathrm{d}x$, where $a \in \mathbb{R}$, $a > 2$, is approximately 5.56. Find the value of a correct to 1 decimal place. [3]

An approximate value for $\int_2^{2.5} \mathrm{f}(x)\,\mathrm{d}x$ is to be found using Simpson's rule.

(b) Explain why it is not possible to use Simpson's rule with 3 strips to find an approximate value for $\int_2^{2.5} \mathrm{f}(x)\,\mathrm{d}x$. [2]

(c) Estimate $\int_2^{2.5} \mathrm{f}(x)\,\mathrm{d}x$ using Simpson's rule with 4 strips. Leave your answer correct to 1 decimal place. [2]

(d) A student made the following claim:

'Since Simpson's rule will give an exact value for $\int_2^{2.5} \mathrm{f}(x)\,\mathrm{d}x$ if $\mathrm{f}(x)$ is of degree 3 or less, the value for $\int_2^{2.5} \mathrm{f}(x)\,\mathrm{d}x$ calculated in part **(c)** should be taken instead of the value for $\int_2^{2.5} \mathrm{f}(x)\,\mathrm{d}x$ given in part **(a)**.'

Comment on the student's claim. [2]

6. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a function given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y + 5z + 2k \\ x + 3y + 6z - k \\ y + z + 3k \end{pmatrix},$$

where k is a constant.

(a) State the value of k for which T is a linear transformation, justifying your answer. [2]

For the rest of this question, use the value of k found in part **(a)**.

(b) Find the standard matrix representing T. [1]

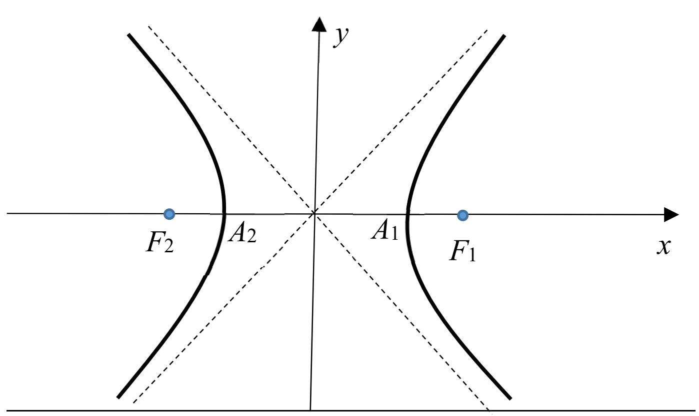
(c) Find a basis for the nullspace of T and hence deduce the nullity and rank of T. [3]

(d) Find a basis for the range of T. [2]

(e) Find the values of a and b for which the set of vectors in $\mathbb{R}^3$ mapped to $\begin{pmatrix} 2a \\ b+1 \\ 0 \end{pmatrix}$ by T is a subspace of $\mathbb{R}^3$. [2]

7. The diagram shows a hyperbola H with equation $x^2 - y^2 = 2$. Its foci are at $F_1(c, 0)$ and $F_2(-c, 0)$ respectively, and its vertices are at A_1 and A_2 respectively.

(a) Find the value of c and the coordinates of A_1 and A_2. [2]



(b) ~~With respect to the two-dimensional cartesian coordinate system, prove that if the position vector~~ $\cancel{\begin{pmatrix} V_x' \\ V_y' \end{pmatrix}}$ ~~is obtained by rotating the position vector~~ $\cancel{\begin{pmatrix} V_x \\ V_y \end{pmatrix} \theta}$ ~~radian anticlockwise about the origin, then~~

$$\cancel{\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix}.}$$ ~~[3]~~

(c) ~~H is rotated $\frac{\pi}{3}$ radians anticlockwise about the origin to form the curve H'.~~

~~Find the cartesian equation of H'.~~ ~~[3]~~

(d) ~~Given that for any point P on H', $|BP - B'P|$ is a constant for points B and B'.~~

~~Find the coordinates of B and B'.~~ ~~[2]~~

8. The population of pigeons in a town increases at a rate of x% per month, where $x > 0$. After numerous complaints from the town residents about the problems caused by the increasing number of pigeons, the town council takes action and sends a team to cull k pigeons at the end of every month starting from January 2023, with $k > 0$.

Let P_n be the pigeon population in the town n months after 31 December 2022. It is given that the pigeon population in the town on 31 December 2022 was 400.

(a) If $x = 2.5$, write down a recurrence relation to model the pigeon population n months after 31 December 2022, where $n \geq 1$. Solve the recurrence relation. [4]

(b) Using the result from part **(a)**, find the range of values of k so that the town council is able to resolve the complaints from residents regarding the pigeons. [2]

(c) State an assumption required for the model in part **(a)** to work. [1]

An environmentalist claimed that although the complaints from residents had to be addressed, the town also cannot exterminate all the pigeons.

(d) Find P_n in terms of x, k and n. Hence determine the relationship between x and k so that both the complaints from residents and the environmentalist's concern can be addressed. [4]

(e) State a possible reason why it is not advisable to exterminate all pigeons in the town. [1]

9. **(a)** Show that for a 3×3 upper triangular matrix

$$\mathbf{T}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

its eigenvalues are 3 of its entries. Hence state its eigenvalues in the form of a_{ij}. [2]

It is given that $\mathbf{A}=\begin{pmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{pmatrix}$, $a,b,c\in\mathbb{R}$.

(b) Write down the eigenvalues of **A** and obtain their corresponding eigenvectors. [4]

It is given that **E** is a non-zero scalar matrix, i.e., $\mathbf{E}=\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$, $k\neq0$.

(c) Write down $\mathbf{E}^{-1}$. [1]

The matrix **B** is given as $\mathbf{B}=\mathbf{E}^{-1}\mathbf{A}^{\mathrm{T}}$, where $\mathbf{A}^{\mathrm{T}}$ is the transpose of **A**.

(d) Show that if $\mathbf{e}_1$ is an eigenvector of **A**, then $\mathbf{e}_1$ is also an eigenvector of $\mathbf{B}^{\mathrm{T}}$. [2]

(e) Hence express $\mathbf{B}^{\mathrm{T}}$ in the form of

$$\mathbf{PDP}^{-1},$$

where **D** is a diagonal matrix, **P** and $\mathbf{P}^{-1}$ are invertible matrices to be determined. [4]

10. A curve C is defined by the parametric equations

$$x = \sin t - t\cos t,\ y = \cos t + t\sin t\ ,$$

for $\frac{\pi}{6} \le t \le \pi a$, where a is a positive constant.

(a) It is given that C has a length of $\frac{\pi^2}{9}$ units. Show that $a = \frac{1}{2}$. [5]

Let $a = \frac{1}{2}$ for the rest of the question.

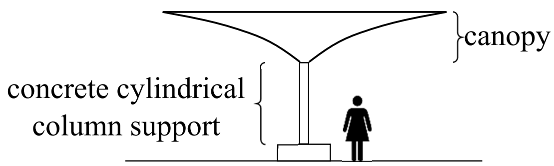


Figure 1

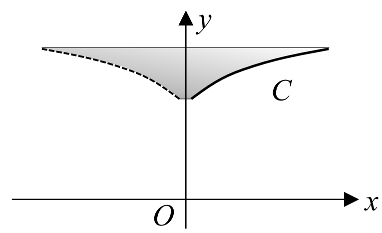


Figure 2

An architect designs a shelter for a public park. The shelter consists of a canopy supported by a concrete cylindrical column, with the diameter of the column the same as the width of the bottom of the canopy as shown in Figure 1. The canopy is formed (in suitable units) by rotating C completely about the y-axis. Coordinate axes have been superimposed onto C for ease of reference as shown in Figure 2.

(b) Determine the exact value of the diameter of the concrete cylindrical column. [2]

(c) The bottom surface area of the canopy, denoted by A, is to be coated with heat insulation paint. Show, with full working, that the exact value of A can be written in the form $p\pi^3 + q\pi^2 + r\pi$, where p , q and r are exact real numbers to be determined. [6]

(d) A reviewer of the design noted that the flat upper surface of the canopy (roof) may not be practical, even though it is easy to construct and cost less. Explain why he may think so. [1]

END OF PAPER