Anglo-Chinese School

(Independent)

THE BEST IS YET TO BE

PRELIMINARY EXAMINATION 2022

YEAR 6 IB DIPLOMA PROGRAMME

MATHEMATICS

HIGHER LEVEL

PAPER 3

Thursday, 15 September 2022

1 hour

Candidate Session Number

0	0	2	3	2	9				

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**

Question	Marks
1	
2	
Total	/ 55

This question paper consists of 4 printed pages including this cover page.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 25]

This question explores some interesting relationships between the Golden Ratio Φ, the arguments of complex roots on an Argand diagram and trigonometric functions.

The Golden Ratio Φ is a very special number. It can be seen almost everywhere, from nature to the human body. It has a numerical value 1.618(3sf) approximately.

Consider the equation $z^{10} = 1$, where $z \in \mathbb{C}$.

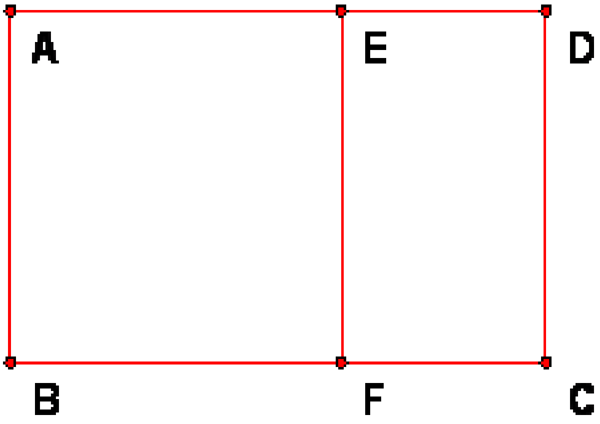
(a) Show that $\omega = e^{i\frac{\pi}{5}}$ is a root. [2]

(b) Given that every integer n can be expressed as $n = 10p + k$, where $p \in \mathbb{Z}$ and $k = 0, 1, 2, \ldots, 9,$ show that ω^n is also a root of $z^{10} = 1$ and **hence** explain clearly why there are 10 distinct roots. [3]

(c) For $0 \le k \le 5$, mark the coordinates of the roots of $z^{10} = 1$ on a unit circle in an Argand Diagram. Label each root as R_k. Explain why α, the angle between adjacent pair of roots, is equal to $\frac{\pi}{5}$. [3]

(d) Using your Argand diagram, show that $\sin 4\alpha = \sin \alpha$, $\cos 4\alpha = -\cos \alpha$, $\sin 3\alpha = \sin 2\alpha$ and hence $\tan 3\alpha \tan 4\alpha = \tan \alpha \tan 2\alpha$. [5]

This part investigates the derivation of the exact value of the golden ratio Φ.



Consider the rectangle ABCD, with width AB $<$ length BC. E is a point on segment AD and F is a point on segment BC such that BFEA is a square and CDEF is a rectangle.

ABCD is defined as a **golden rectangle** if CDEF and DABC are similar, such that $\frac{AD}{AB} = \frac{DC}{DE} = \Phi$.

(e) By letting AB = 1, express ED in terms of Φ. [1]

(f) Hence prove that the exact value for Φ is $\frac{1+\sqrt{5}}{2}$. [4]

This part investigates how the golden ratio Φ can be expressed as a trigonometric function of a fixed angle in radians.

(g) Starting with $\sin 3\alpha = \sin 2\alpha$, show that $4\cos^2\alpha - 2\cos\alpha - 1 = 0$. [4]

(h) Hence show that the angle $\frac{\pi}{5}$ is related to the golden ratio Φ by the relation $\cos\frac{\pi}{5} = \frac{\Phi}{2}$. [3]

2. [Maximum mark: 30]

This question explores finding the solutions of second order differential equations. Consider the system of linear differential equations of the form:

$\frac{dy}{dx} = u$ and $\frac{du}{dx} = -u + 2y,$ where y and u are functions of x.

(a) Show that $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0.$ [3]

For the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$, it is given that $y = e^{rx}$, where r is a constant, is a solution to this differential equation.

(b) Show that $r^2 + r - 2 = 0.$ [3]

(c) Hence find the two values or r, r_1 and r_2, where $r_1 < r_2$. [2]

(d) Verify that $y = Ae^{r_1 x} + Be^{r_2 x}$ is also a solution to $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$, where A and B are arbitrary constants. [4]

For another differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$, it is given that $y = e^{kx}$, where k is a constant, is a solution to this differential equation.

(e) Deduce the possible value(s) of k. [3]

(f) Verify that $y = (A + Bx)e^{kx}$ is also a solution to $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$, where A and B are arbitrary constants. [4]

(g) Given $A = 1,\ B = 2$ and the values of r_1, r_2, k found earlier, sketch the graph of $y = \frac{(A + Bx)e^{kx}}{Ae^{r_1 x} + Be^{r_2 x}}$, showing clearly any asymptote(s), intercept(s) and stationary point(s). [4]

(h) With the aid of the diagram in (g), find the oblique asymptote of the graph in the form $y = mx + c$, justifying your conclusions. [3]

(i) Determine the general solution to $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0.$ [4]