	Suggested Solution	Remarks
1i	$(2x+3y+6z)^2 \le (2^2+3^2+6^2)(x^2+y^2+z^2)=49$ $2x+3y+6z \le 7$	
1ii	$\begin{cases} 2x+3y+6z=7 \\ x^2+y^2+z^2=1 \end{cases}$ Since $2^2+3^2+6^2=7^2$, by observation, $x=\frac{2}{7}, y=\frac{3}{7}, z=\frac{6}{7}$.	
1iii	Suppose $\sum_{i=1}^{n} x_i^2 = 1$. $\left(\sum_{i=1}^{n} x_i\right)^2 = \left(\sum_{i=1}^{n} 1\cdot x_i\right)^2 \le n\sum_{i=1}^{n} x_i^2 = n \Rightarrow \sum_{i=1}^{n} x_i \le \sqrt{n}$ Since if we let $x_i = \frac{1}{\sqrt{n}}$ for all $1 \le i \le n$, we yield $\sum_{i=1}^{n} x_i^2 = n\left(\frac{1}{\sqrt{n}}\right)^2 = 1$ and $\sum_{i=1}^{n} x_i = n\left(\frac{1}{\sqrt{n}}\right) = \sqrt{n}$, the maximum possible value of $\sum_{i=1}^{n} x_i$ is $\sqrt{n}$.	
1iv	Suppose there are n squares of lengths x_i contained in the unit square. Their total area is $\sum_{i=1}^{n} x_i^2 \le 1$, and their total perimeter is $18 = \sum_{i=1}^{n} 4x_i$. By part (iii), $\frac{18}{4} = \sum_{i=1}^{n} x_i \le \sqrt{n} \Rightarrow n \ge 20.25$. Hence, there must be more than 20 such squares.	
2ia	Number of ways $= \binom{8+4-1}{4-1} = 165$ [Bijection with a string of eight 0's (objects) and three 1's (partitions). For example, the string 00100010100 would correspond to the combination of 2 A's, 3 C's, 1 G and 2 T's.]	
2ib	Number of ways $= \binom{4+4-1}{4-1} = 35$ [Place one 0 in each box and the remaining four 0's into the boxes.]	
2iia	Number of sequences $= 4^8 = 65536$	
2iib	Number of sequences $= 4 \times 3^7 = 8748$	
2iic	Number of sequences $= 4^8 - \binom{4}{3}3^8 + \binom{4}{2}2^8 - \binom{4}{1}1^8 = 40824$	Principle of inclusion / exclusion.
3ia	Given $x_1 = 1$ and $a \ge 0$. $x_{i+1} = \left(\frac{i+a}{i+1}\right)x_i \ge \left(\frac{i}{i+1}\right)x_i \ge \left(\frac{i}{i+1}\right)\left(\frac{i-1}{i}\right)x_{i-1} \ge \ldots \ge \left(\frac{i}{i+1}\right)\left(\frac{i-1}{i}\right)\ldots\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)x_1 = \frac{1}{i+1}$ $\therefore x_i \ge \frac{1}{i}$ for all $i \in \mathbb{Z}^+$	
3ib	From (3i(a)), $\sum_{i=n+1}^{2n} x_i \ge \sum_{i=n+1}^{2n} \frac{1}{i} \ge \sum_{i=n+1}^{2n} \frac{1}{2n} = \frac{1}{2}$.	

3ic	$\sum_{i=1}^{2^n} x_i = \sum_{i=2^{n-1}+1}^{2^n} x_i + \sum_{i=2^{n-2}+1}^{2^{n-1}} x_i + \ldots + \sum_{i=2+1}^{2^2} x_i + (x_2 + x_1) \geq n \times \frac{1}{2} = \frac{n}{2}$ $\sum_{i=1}^{\infty} x_i = \lim_{n\to\infty} \sum_{i=1}^{2^n} x_i \geq \lim_{n\to\infty} \frac{n}{2}$ is unbounded.	
3iia	Since $(i+1)x_{i+1} - ix_i = (i+1)\frac{i+a}{i+1}x_i - ix_i = ax_i$, summing from $i=m$ to $i=n$ yields the following. $a\sum_{i=m}^{n} x_i = \sum_{i=m}^{n} (i+1)x_{i+1} - ix_i = \left[(m+1)x_{m+1} - mx_m\right] + \left[(m+2)x_{m+2} - (m+1)x_{m+1}\right]$ $+\ldots+\left[(n+1)x_{n+1} - nx_n\right] = (n+1)x_{n+1} - mx_m$ **<u>Alternative Solution</u>** Let P$_n$ be the statement $a\sum_{i=1}^{n} x_i = (n+1)x_{n+1} - 1$. RHS of $P_1 = 2x_2 - 1 = 2\left(\frac{1+a}{2}\right) - 1 = a = ax_1 =$ LHS of P_1 $\therefore P_1$ is true. Assume P$_k$ is true for some $k \in \mathbb{Z}^+$. LHS of $P_{k+1} = a\sum_{i=1}^{k+1} x_i = a\sum_{i=1}^{k} x_i + ax_{k+1}$ $= (k+1)x_{k+1} - 1 + ax_{k+1}$ $= (k+1+a)x_{k+1} - 1$ $= (k+2)x_{k+2} - 1 \quad \left[\because x_{k+2} = \left(\frac{k+1+a}{k+2}\right)x_{k+1}\right]$ $=$ RHS of P_1 $\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true. Since P$_1$ is true and $(P_k$ is true $\Rightarrow P_{k+1}$ is true$)$, P$_n$ is true for all $n \in \mathbb{Z}^+$. Hence, $\sum_{i=m}^{n} x_i = \begin{cases} \sum_{i=1}^{n} x_i = (n+1)x_{n+1} - 1 & \text{if } m = 1 \\ \sum_{i=1}^{n} x_i - \sum_{i=1}^{m-1} x_i = \left[(n+1)x_{n+1} - 1\right] - \left[mx_m - 1\right] & \text{if } m > 1 \end{cases}$ $= (n+1)x_{n+1} - mx_m$.	**<u>Alt. Sol.</u>** We first ask ourselves what is the partial sum from 1 to n, and from there we would be able to find the sum from m to n. By observing the equation that we need to show, we may let $m = 1$ so that we obtain P$_n$ which we will then prove.
3iib	Let $M = \lceil -a \rceil$. Then $x_{M+1} = \left(\frac{M+a}{M+1}\right)x_M$. *<u>Case 1</u>*: $x_M \geq 0$. Since $\left(\frac{M+a}{M+1}\right) \geq 0,\ x_{M+1} \geq 0.$ Consequently, for all $n > M$, since $\left(\frac{n+a}{n+1}\right) \geq 0,\ x_{n+1} = \left(\frac{n+a}{n+1}\right)x_n \geq 0.$ $\therefore x_n x_m \geq 0$ for all $n, m > M$.	

	Case 2: $x_M < 0$. Since $\left(\dfrac{M+a}{M+1}\right) \geq 0,\ x_{M+1} \leq 0.$ Consequently, for all $n > M$, since $\left(\dfrac{n+a}{n+1}\right) \geq 0,\ x_{n+1} = \left(\dfrac{n+a}{n+1}\right) x_n \leq 0.$ $\therefore x_n x_m \geq 0$ for all $n, m > M$.	
4ia	Let W_n be the number of such n-digit numbers having first digit 3. Then by symmetry, $W_n = Y_n$ for all $n \in \mathbb{Z}^+$. If the first digit is 2, then the next digit can only be 1 or 3. Hence, $Y_n = X_{n-1} + W_{n-1} = X_{n-1} + Y_{n-1}$.	
4ib	If the first digit is 1, then the next digit can be either 1, 2 or 3. $X_n = X_{n-1} + Y_{n-1} + W_{n-1}$ $= X_{n-1} + 2Y_{n-1}$	
4ic	$X_{n+1} = X_n + 2Y_n$ (by 4i(b)) $= X_n + 2\left(X_{n-1} + Y_{n-1}\right)$ (by 4i(a)) $= X_n + X_{n-1} + X_n$ (by 4i(b)) $= 2X_n + X_{n-1}$	
4ii	Let P_n be the statement $X_n \equiv n^2 - n + 1 \pmod 4$. LHS of $P_1 = X_1 = 1 \pmod 4$ LHS of $P_2 = X_2 = X_1 + Y_1 + W_1 = 3 \pmod 4$ RHS of $P_1 = 1^2 - 1 + 1 = 1 \pmod 4$ RHS of $P_2 = 2^2 - 2 + 1 = 3 \pmod 4$ $\therefore P_1$ and P_2 are true. Assume P_k and P_{k-1} are true for some $k \in \mathbb{Z}^+$, $k \geq 2$. LHS of $P_{k+1} = X_{k+1} = 2X_k + X_{k-1}$ $= 2k^2 - 2k + 2 + (k-1)^2 - (k-1) + 1$ $\equiv 3k^2 - k + 1$ $\equiv \begin{cases} 1 \text{ (mod 4), if } k \equiv 0 \text{ or } k \equiv 3 \text{ (mod 4)} \\ 3 \text{ (mod 4), if } k \equiv 1 \text{ or } k \equiv 2 \text{ (mod 4)} \end{cases}$ RHS of $P_{k+1} = (k+1)^2 - (k+1) + 1$ $= k^2 + k + 1$ $\equiv \begin{cases} 1 \text{ (mod 4), if } k \equiv 0 \text{ or } k \equiv 3 \text{ (mod 4)} \\ 3 \text{ (mod 4), if } k \equiv 1 \text{ or } k \equiv 2 \text{ (mod 4)} \end{cases}$ $\therefore P_{k-1}$ and P_k are true $\Rightarrow P_{k+1}$ is true. Since P_1 and P_2 are true, and $\left(P_{k-1} \text{ and } P_k \text{ are true} \Rightarrow P_{k+1} \text{ is true}\right)$, P_n is true for all $n \in \mathbb{Z}^+$.	
4iii	$T_n = X_n + Y_n + W_n = X_{n+1} \equiv (n+1)^2 - (n+1) + 1 \equiv n^2 + n + 1 \pmod 4$	

5i	$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}=\frac{\mathrm{d}u}{\mathrm{d}x}$ $\frac{\mathrm{d}t}{\mathrm{d}x}=t\Rightarrow\int\frac{1}{t}\,\mathrm{d}t=\int 1\,\mathrm{d}x$ $\ln\lvert t\rvert=x+C$ $\frac{\mathrm{d}u}{\mathrm{d}x}=t=A\mathrm{e}^x$ $u=\int A\mathrm{e}^x\,\mathrm{d}x=A\mathrm{e}^x+B$	
5ii	$u=\mathrm{e}^{-\int \mathrm{f}(x)y\,\mathrm{d}x}$ $\frac{\mathrm{d}u}{\mathrm{d}x}=-\mathrm{f}(x)yu\Rightarrow y=-\frac{1}{\mathrm{f}(x)u}\frac{\mathrm{d}u}{\mathrm{d}x}$ $\frac{\mathrm{d}^2u}{\mathrm{d}x^2}=\left(-\mathrm{f}'(x)y-\mathrm{f}(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right)u-\mathrm{f}(x)y\frac{\mathrm{d}u}{\mathrm{d}x}$ $=-\mathrm{f}'(x)yu-\mathrm{f}(x)u\left(\mathrm{f}(x)y^2+\mathrm{g}(x)y\right)-\mathrm{f}(x)y\frac{\mathrm{d}u}{\mathrm{d}x}$ $=\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{1}{u}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2+\mathrm{g}(x)\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{1}{u}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2$ $=\left(\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}+\mathrm{g}(x)\right)\frac{\mathrm{d}u}{\mathrm{d}x}$ $\therefore \mathrm{f}(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-\left(\mathrm{f}'(x)+\mathrm{f}(x)\mathrm{g}(x)\right)\frac{\mathrm{d}u}{\mathrm{d}x}=0$ (shown)	
5iii	$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{-2x}y^2+3y$ By (5ii), $\mathrm{e}^{-2x}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-\left(-2\mathrm{e}^{-2x}+\mathrm{e}^{-2x}\cdot 3\right)\frac{\mathrm{d}u}{\mathrm{d}x}=0$, where $u=\mathrm{e}^{-\int \mathrm{e}^{-2x}y\,\mathrm{d}x}$ $\mathrm{e}^{-2x}\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-\frac{\mathrm{d}u}{\mathrm{d}x}\right)=0\Rightarrow\frac{\mathrm{d}^2u}{\mathrm{d}x^2}=\frac{\mathrm{d}u}{\mathrm{d}x}$ By (5i), $u=A\mathrm{e}^x+B$ $\mathrm{e}^{-\int \mathrm{e}^{-2x}y\,\mathrm{d}x}=A\mathrm{e}^x+B$ $\int\mathrm{e}^{-2x}y\,\mathrm{d}x=-\ln\left(A\mathrm{e}^x+B\right)\Rightarrow\mathrm{e}^{-2x}y=-\frac{A\mathrm{e}^x}{A\mathrm{e}^x+B}$ -------(1) When $x=0,\ y=-\frac{1}{4}$. So, $-\frac{1}{4}=-\frac{A}{A+B}\Rightarrow B=3A.$ By (1), $\mathrm{e}^{-2x}y=-\frac{A\mathrm{e}^x}{A\mathrm{e}^x+3A}$ $y=-\frac{\mathrm{e}^{3x}}{\mathrm{e}^x+3}$	

6i	Let $x_1, x_2, ..., x_{2n}$ be the positions of the $(+1)$'s and (-1)'s. With this arrangement, we are able to find a consecutive pair of $+1$ and -1 in the clockwise direction (i.e. in the clockwise direction, the $+1$ precedes the -1). Remove this pair of numbers and there will be $2n-2$ numbers left arranged in the circle. We will repeat this procedure by removing consecutive pairs of $+1$ and -1 in the clockwise direction until there is a final pair of $+1$ and -1 left. Let x_k be the position of this final $+1$. We claim that x_k is the starting position for which T_i is never negative. Since the pairs of $+1$'s and -1's removed were consecutive, and within each pair, the $+1$ preceded the -1, there will always be an increase of the partial sum T_k prior to a decrease. Hence, T_i is never negative for all $1 \le i \le 2n$. **<u>Alternative Solution</u>** Let $x_1, x_2, ..., x_{2n}$ be the positions of the $(+1)$'s and (-1)'s, and let x_1 be the starting position. As we evaluate T_i for $1 \le i \le 2n$, there exists a k such that T_k is minimum. We then claim that x_{k+1} is a starting position for which T_i is never negative for all $1 \le i \le 2n$. Relabel $x_{k+1}, ..., x_{2n}, x_1, ..., x_k$ as $y_1, y_2, ..., y_{2n}$ respectively, so $y_1 = x_{k+1}$ is the starting position. To avoid confusion, we shall let S_i be the new partial sum from position y_1 to y_i. Since T_k is a minimum, $S_i = T_{i+k} - T_k \ge 0$ for $1 \le i \le 2n-k$. With equal number of $(+1)$'s and (-1)'s, $T_{2n} = 0$. Hence, $S_{2n-k} = -T_k$. Furthermore, $T_j \ge T_k \Rightarrow -T_k + T_j \ge 0$ for all $1 \le j \le k$, we have that $S_i = S_{2n-k} + T_{i-2n+k} = -T_k + T_{i-2n+k} \ge 0$ for $2n-k+1 \le i \le 2n$.	
6ii	Note that $T_i \equiv i \pmod 2$ due to the following. Enumerate T_i starting with the index $i=1$. We have $T_1 \equiv 1 \pmod 2$ regardless of the first value, $+1$ or -1. Subsequently, as the index i increases each time by 1, we add $+1$ or -1 to the value of T_i, changing its parity. Hence, $n + \sum_{i=1}^{2n} T_i \equiv n + \sum_{i=1}^{2n} i \equiv n + n = 2n \equiv 0 \pmod 2$; i.e. $n + \sum_{i=1}^{2n} T_i$ is even.	
7i	$a > c\cos\theta + d\sin\theta$ and $b > c\sin\theta + d\cos\theta$	
7ii	"$\Rightarrow$" Suppose on the contrary that $d \ge b$, then $a > c \ge d \ge b$ and $d\cos\theta + c\sin\theta \ge b(\cos\theta + \sin\theta) = b\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) > b$ since $0 < \theta < \frac{\pi}{2} \Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) > \frac{1}{\sqrt{2}}$. "$\Leftarrow$" Choose θ small enough such that $c\sin\theta < \varepsilon := \min(a-c, b-d)$. Then $c\cos\theta + d\sin\theta \le c\cos\theta + c\sin\theta < c + \varepsilon \le a$ and $d\cos\theta + c\sin\theta < d + \varepsilon \le b$.	
7iii	Let θ_0 be the angle for which the $c \times d$ rectangle is strictly contained in the $a \times b$ rectangle. By (i), we must have $c\cos\theta_0 + d\sin\theta_0 < a$ and $c\sin\theta_0 + d\cos\theta_0 < b \Rightarrow c\cos\left(\frac{\pi}{2} - \theta_0\right) + d\sin\left(\frac{\pi}{2} - \theta_0\right) < b \le a$.	

Let $\mathrm{f}(\theta)=c\cos\theta+d\sin\theta$, $\theta\in\left[0,\frac{\pi}{2}\right]$, $\theta_1=\min\left(\theta_0,\frac{\pi}{2}-\theta_0\right)$ and $\theta_2=\max\left(\theta_0,\frac{\pi}{2}-\theta_0\right)$. Then we must have $\mathrm{f}(\theta_1)<a$, $\mathrm{f}(\theta_2)<a$ and $\frac{\pi}{4}\in[\theta_1,\theta_2]$.

If we can prove that f is decreasing on $[\theta_1,\theta_2]$, then $a>\mathrm{f}\left(\frac{\pi}{4}\right)=\frac{c+d}{\sqrt{2}}$ and we are done.

Since $\mathrm{f}(\theta)>0$ and $\mathrm{f}''(\theta)=-c\cos\theta-d\sin\theta<0$ for $\theta\in\left[0,\frac{\pi}{2}\right]$, f has a maximum at $\theta_{\max}$, where $\mathrm{f}'(\theta_{\max})=-c\sin\theta_{\max}+d\cos\theta_{\max}=0\Rightarrow\theta_{\max}=\tan^{-1}\frac{d}{c}\le\frac{\pi}{4}$.

Since $\mathrm{f}(0)=c\ge a$ and f is increasing on $[0,\theta_{\max}]$, we must have $\theta_{\max}<\theta_1$. Since f is decreasing on $\left[\theta_{\max},\frac{\pi}{2}\right]$, it is also decreasing on $[\theta_1,\theta_2]$ and we are done.

7iv

A $c\times d$ rectangle (with $c\ge d$) can be strictly contained in an $a\times a$ square if and only if $a>c$ or $a\sqrt{2}>c+d$.

<u>Proof</u>

$(\Rightarrow)$: Suppose that a $c\times d$ rectangle (with $c\ge d$) can be strictly contained in an $a\times a$ square. If $a>c$, we are done. Otherwise, if $a\le c$, then by (iii), we have $a\sqrt{2}>c+d$.

$(\Leftarrow)$: Suppose $a>c$ or $a\sqrt{2}>c+d$.

If $a>c$, then by (ii), a $c\times d$ rectangle can be strictly contained in an $a\times a$ square iff $a>d$. But since $a>c\ge d$, the rectangle can always be contained in the square.

If $a\sqrt{2}>c+d$, substituting $\theta=\frac{\pi}{4}$ into the inequalities in (i) yields $c\cos\frac{\pi}{4}+d\sin\frac{\pi}{4}=\frac{c+d}{\sqrt{2}}<a$ and $c\sin\frac{\pi}{4}+d\cos\frac{\pi}{4}=\frac{c+d}{\sqrt{2}}<a$. This shows that the rectangle can be contained in the $a\times a$ square.

8i

n (mod 12)	0	1	2	3	4	5	6	7	8	9	10	11
n^2 (mod 12)	0	1	4	9	4	1	0	1	4	9	4	1

By exhaustion, the elements of S(12) are all square numbers.

8ii

Let $N=9$. Since $4^2=16\equiv 7 \pmod 9$, $7\in\mathrm{S}(9)$ and 7 is not a square number.

Answer is not unique.

8iii

Since $(N+r)^2\equiv r^2 \pmod N$ for all $r\in\mathbb{Z}$ with $0\le r<\sqrt{N}$, there are at least $\sqrt{N}$ unique r's.

Hence, $\left\{0,1,4,\ldots,\left(\left\lfloor\sqrt{N-1}\right\rfloor\right)^2\right\}\subseteq\mathrm{S}(N)$ and $\mathrm{S}(N)$ has at least $\sqrt{N}$ elements.

NB *We use* $\left\lfloor\sqrt{N-1}\right\rfloor$ *instead of* $\left\lfloor\sqrt{N}\right\rfloor$ *to be precise, since if N is a square number, then* $\left\lfloor\sqrt{N}\right\rfloor^2=N$ *is not an element of* $S(N)$.

8iv	Suppose $\exists x,n,\lambda\in\mathbb{Z}$ such that $x^2=17+2^n\lambda$, with $n\ge 5$. If λ is even, then $x^2=17+2^{n+1}\left(\frac{\lambda}{2}\right)\Rightarrow x^2\equiv 17 \pmod{2^{n+1}}$, i.e. $17\in \mathrm{S}\left(2^{n+1}\right)$. If λ is odd, let $\mu=\frac{\lambda+x+2^{n-2}}{2}$. Note that $\mu\in\mathbb{Z}$ since λ, x are odd and $n\ge 5$. Then $2^{n+1}\mu+17=2^n\left(\lambda+x+2^{n-2}\right)+17$ $=x^2+2^n x+2^{2n-2}$ $=\left(x+2^{n-1}\right)^2$. Hence, $\left(x+2^{n-1}\right)^2\equiv 17 \pmod{2^{n+1}}$, i.e. $17\in\mathrm{S}\left(2^{n+1}\right)$.	Think: When is $x^2\equiv 17$ (mod 2^{n+1})? Get that λ is even. Then consider when λ is odd, and try to find a square number by working backwards.
8v	Let P_n be the statement: $17\in\mathrm{S}\left(2^n\right), n\ge 5$. For $n=5$, $7^2=49\equiv 17 \pmod{2^5}$. Hence, P_5 is true. By (iv), we have shown that $17\in\mathrm{S}\left(2^n\right)\Rightarrow 17\in\mathrm{S}\left(2^{n+1}\right)$, i.e., $\mathrm{P}_n\Rightarrow\mathrm{P}_{n+1}$ for $n\ge 5$. By induction, P_n is true for all $n\ge 5$. From (iii), we have shown that there are at least $\sqrt{N}$ numbers in $\mathrm{S}(N)$, where all these elements are square numbers. Since 17 is not a square number, and is also in $\mathrm{S}\left(2^n\right)$ for $n\ge 5$, then we must have at least $1+\sqrt{2^n}$ elements in $\mathrm{S}\left(2^n\right)$. **<u>Alternative Solution</u>** Case 1: n is even $\left(2^{\frac{n}{2}}+1\right)^2=2^n+2^{\frac{n}{2}+1}+1\equiv 2^{\frac{n}{2}+1}+1 \pmod{2^n}$ Then $\left\{1^2,2^2,\ldots,0\equiv\left(2^{\frac{n}{2}}\right)^2,2^{\frac{n}{2}+1}+1\right\}\subseteq\mathrm{S}\left(2^n\right)$, and this shows that $\mathrm{S}\left(2^n\right)$ has at least $1+\sqrt{2^n}$ elements. Case 2: n is odd $\left(2^{\frac{n+1}{2}}+1\right)^2=2^{n+1}+2^{\frac{n+3}{2}}+1\equiv 2^{\frac{n+3}{2}}+1 \pmod{2^n}$ Then $\left\{0^2,1^2,2^2,\ldots,\left(\left\lfloor 2^{\frac{n}{2}}\right\rfloor\right)^2,2^{\frac{n+3}{2}}+1\right\}\subseteq\mathrm{S}\left(2^n\right)$, and this shows that $\mathrm{S}\left(2^n\right)$ has at least $1+\sqrt{2^n}$ elements. Note that in both cases above, $2^{\frac{n}{2}+1}+1$ and $2^{\frac{n+3}{2}}+1$ are not square numbers. Suppose that $p^2=2^q+1$ for some $p,q\in\mathbb{Z}^+$, then $(p+1)(p-1)=2^q$. This implies that $p+1=2^k$ and $p-1=2^l$ for some $k,l\in\mathbb{Z}^+$ with $k+l=q$. Hence, $2^k-1=p=2^l+1\Rightarrow 2\left(2^{l-1}+1\right)=2^k\Rightarrow l=1,k=2\Rightarrow q=3$.	

	But $n \geq 5 \Rightarrow \frac{n}{2}+1, \frac{n+3}{2} \geq 4$, which says that $2^{\frac{n}{2}+1}+1$ and $2^{\frac{n+3}{2}}+1$ are not square numbers.	