National Junior College
2016 – 2017 H2 Further Mathematics
Topic F7: Matrices and Linear Spaces (Tutorial Set 3 Solutions)

Basic Mastery Questions

1 **(a)** $\begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 \\ 0 & 7 \\ 0 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. A basis for the row space is $\{(1 \quad 3),(0 \quad 1)\}$; a basis for the column space is $\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}\right\}$. rank $= 2$.

nullity $= 2 - 2 = 0$. The basis for the null space is $\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$.

(b) $\begin{pmatrix} 1 & 2 & -1 \\ -3 & -5 & 1 \\ 13 & 23 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & -3 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$. A basis for the row space is $\{(1 \quad 2 \quad -1),(0 \quad 1 \quad -2)\}$; a basis for the column space is $\left\{\begin{pmatrix} 1 \\ -3 \\ 13 \end{pmatrix},\begin{pmatrix} 2 \\ -5 \\ 23 \end{pmatrix}\right\}$.

rank $= 2$ nullity $= 3 - 2 = 1$. To find the null space, let $z = \lambda$, then $y = 2\lambda$ and $x = -3\lambda$, so a basis for the null space is $\left\{\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}\right\}$.

(c) $\begin{pmatrix} 2 & 1 & 3 & 3 \\ 0 & -3 & 1 & -2 \\ 4 & 5 & 5 & 8 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 3 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 3 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 3 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. A basis for the row space is $\{(2 \quad 1 \quad 3 \quad 3),(0 \quad -3 \quad 1 \quad -2)\}$; a basis for the column space is $\left\{\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix},\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}\right\}$., rank $= 2$, nullity $= 4 - 2 = 2$. To find the null space, let $z = 3\lambda, t = 6\mu$, then $y = \lambda - 4\mu$ and $x = -5\lambda - 7\mu$, so a basis for the null space is $\left\{\begin{pmatrix} -5 \\ 1 \\ 3 \\ 0 \end{pmatrix},\begin{pmatrix} -7 \\ -4 \\ 0 \\ 6 \end{pmatrix}\right\}$.

2 **(a)** Yes.

$$\mathrm{T}_1\left(\begin{pmatrix}x_1\\y_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\end{pmatrix}\right)=\mathrm{T}_1\begin{pmatrix}x_1+x_2\\y_1+y_2\end{pmatrix}=\begin{pmatrix}k(x_1+x_2)\\y_1+y_2\end{pmatrix}=\begin{pmatrix}kx_1\\y_1\end{pmatrix}+\begin{pmatrix}kx_2\\y_2\end{pmatrix}=\mathrm{T}_1\begin{pmatrix}x_1\\y_1\end{pmatrix}+\mathrm{T}_2\begin{pmatrix}x_2\\y_2\end{pmatrix}..$$

$$\mathrm{T}_1\left(m\begin{pmatrix}x_1\\y_1\end{pmatrix}\right)=\mathrm{T}_1\begin{pmatrix}mx_1\\my_1\end{pmatrix}=\begin{pmatrix}kmx_1\\my_2\end{pmatrix}=m\begin{pmatrix}kx_1\\y_1\end{pmatrix}=m\mathrm{T}_1\begin{pmatrix}x_1\\y_1\end{pmatrix}.$$

(b) No. $\mathrm{T}_2\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix},\mathrm{T}_2\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix},\mathrm{T}_2\left(\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right)=\mathrm{T}_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}\neq\mathrm{T}_2\begin{pmatrix}0\\1\end{pmatrix}+\mathrm{T}_2\begin{pmatrix}1\\1\end{pmatrix}.$

(c) Yes. $\mathrm{T}_3\left(\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}+\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}\right)=\mathrm{T}_3\begin{pmatrix}a_1+a_2&b_1+b_2\\c_1+c_2&d_1+d_2\end{pmatrix}=\begin{pmatrix}a_1+a_2+b_1+b_2\\c_1+c_2-d_1-d_2\end{pmatrix}$, and

$$\mathrm{T}_3\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}+\mathrm{T}_3\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}=\begin{pmatrix}a_1+b_1\\c_1-d_1\end{pmatrix}+\begin{pmatrix}a_2+b_2\\c_2-d_2\end{pmatrix}=\begin{pmatrix}a_1+a_2+b_1+b_2\\c_1+c_2-d_1-d_2\end{pmatrix}.$$

$$\mathrm{T}_3\left(m\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}\right)=\mathrm{T}_3\begin{pmatrix}ma_1&mb_1\\mc_1&md_1\end{pmatrix}=\begin{pmatrix}ma_1+mb_1\\mc_1-md_1\end{pmatrix}=m\begin{pmatrix}a_1+b_1\\c_1-d_1\end{pmatrix}=m\mathrm{T}_3\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}.$$

(d) No. $\mathrm{T}_4(x)=1, \mathrm{T}_4(2x)=4\neq 2\mathrm{T}_4(x)$.

(e) Yes. $\mathrm{T}_5(\mathbf{u}_1+\mathbf{u}_2)=\mathbf{0}=\mathbf{0}+\mathbf{0}=\mathrm{T}_5(\mathbf{u}_1)+\mathrm{T}_5(\mathbf{u}_2)$. $\mathrm{T}_5(m\mathbf{u}_1)=\mathbf{0}=m\mathbf{0}=m\mathrm{T}_5(\mathbf{u}_1)$.

3 **(i)** $\mathrm{T}\begin{pmatrix}1\\-7\end{pmatrix}=(-1)\mathrm{T}\begin{pmatrix}2\\3\end{pmatrix}+1\mathrm{T}\begin{pmatrix}3\\-4\end{pmatrix}=\begin{pmatrix}-11\\13\end{pmatrix}+\begin{pmatrix}8\\-11\end{pmatrix}=\begin{pmatrix}-3\\2\end{pmatrix}.$

(ii) $\mathrm{T}\begin{pmatrix}1\\0\end{pmatrix}=\frac{1}{17}\left[4\mathrm{T}\begin{pmatrix}2\\3\end{pmatrix}+3\mathrm{T}\begin{pmatrix}3\\-4\end{pmatrix}\right]=\frac{1}{17}\begin{pmatrix}44+24\\-52-33\end{pmatrix}=\begin{pmatrix}4\\-5\end{pmatrix},$

$\mathrm{T}\begin{pmatrix}0\\1\end{pmatrix}=\frac{1}{17}\left[3\mathrm{T}\begin{pmatrix}2\\3\end{pmatrix}-2\mathrm{T}\begin{pmatrix}3\\-4\end{pmatrix}\right]=\frac{1}{17}\begin{pmatrix}33-16\\-39+22\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$. Thus $\mathbf{A}=\begin{pmatrix}4&1\\-5&-1\end{pmatrix}$

4 **(i)** $\text{nullity}=3-2=1$.

$$\begin{pmatrix}1&2&-1\\2&a+1&-2\\3&2a&a^2-4a\end{pmatrix}\to\begin{pmatrix}1&2&-1\\0&a-3&0\\0&2a-6&a^2-4a+3\end{pmatrix}\to\begin{pmatrix}1&2&-1\\0&a-3&0\\0&0&(a-1)(a-3)\end{pmatrix}.$$

$a=1$.

(ii) The range space is the column space of $\begin{pmatrix}1&2&-1\\2&2&-2\\3&2&-3\end{pmatrix}$ with a basis $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right\}$.

5 **(a)** $\mathbf{e}_1=\begin{pmatrix}-4\\5\end{pmatrix}$ with $\lambda_1=7$, $\mathbf{e}_2=\begin{pmatrix}1\\1\end{pmatrix}$ with $\lambda_2=-2$. (working omitted)

(b) $\mathbf{e}_1=\begin{pmatrix}4\\-3\\-2\end{pmatrix}$ with $\lambda_1=-5$, $\mathbf{e}_2=\begin{pmatrix}1\\0\\-2\end{pmatrix}$ with $\lambda_2=10$, $\mathbf{e}_3=\begin{pmatrix}-1\\3\\1\end{pmatrix}$ with $\lambda_3=-10$.

(c) $0=\begin{vmatrix}\lambda-0 & -2 & -1\\ 1 & \lambda-3 & -1\\ -2 & 4 & \lambda+1\end{vmatrix}$

$=\lambda(\lambda-3)(\lambda+1)-4-4-2(\lambda-3)+2(\lambda+1)+4\lambda$

$=\lambda^3-2\lambda^2-3\lambda-8-2\lambda+6+2\lambda+2+4\lambda$

$=\lambda^3-2\lambda^2+\lambda$

$=\lambda(\lambda-1)^2$

Eigenvalues: $\lambda_1=\lambda_2=1$ and $\lambda_3=0$

When $\lambda=1$, we solve $\begin{pmatrix}1 & -2 & -1\\ 1 & -2 & -1\\ -2 & 4 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$, i.e. $\begin{pmatrix}1 & -2 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$.

Let $y=\alpha, z=\beta$, then $x=2\alpha+\beta$. $\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\alpha\begin{pmatrix}2\\ 1\\ 0\end{pmatrix}+\beta\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}$, so two corresponding vectors are $\mathbf{e}_1=\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}$ and $\mathbf{e}_2=\begin{pmatrix}2\\ 1\\ 0\end{pmatrix}$ with $\lambda_1=\lambda_2=1$

When $\lambda=0$, we solve $\begin{pmatrix}0 & -2 & -1\\ 1 & -3 & -1\\ -2 & 4 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$, i.e. $\begin{pmatrix}1 & -3 & -1\\ 0 & 2 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$.

Let $z=2\gamma$, then $y=-\gamma$ and $x=-\gamma$, a corresponding vector is $\mathbf{e}_3=\begin{pmatrix}1\\ 1\\ -2\end{pmatrix}$, $\lambda_3=0$

6 **(i)** The eigenvalue are –7 and 2 with eigenvectors $\begin{pmatrix}-4\\ 5\end{pmatrix}$ and $\begin{pmatrix}1\\ 1\end{pmatrix}$ respectively (working omiited). Thus, $\mathbf{P}=\begin{pmatrix}-4 & 1\\ 5 & 1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}-7 & 0\\ 0 & 2\end{pmatrix}$.

(ii) $\det(\mathbf{P})=-4-5=-9$, $\mathbf{P}^{-1}=-\frac{1}{9}\begin{pmatrix}1 & -1\\ -5 & -4\end{pmatrix}$. Then

$\mathbf{A}^5=\mathbf{PD}^5\mathbf{P}^{-1}=-\frac{1}{9}\begin{pmatrix}-4 & 1\\ 5 & 1\end{pmatrix}\begin{pmatrix}-16807 & 0\\ 0 & 32\end{pmatrix}\begin{pmatrix}1 & -1\\ -5 & -4\end{pmatrix}$

$=-\frac{1}{9}\begin{pmatrix}67228 & 32\\ -84035 & 32\end{pmatrix}\begin{pmatrix}1 & -1\\ -5 & -4\end{pmatrix}$

$=-\frac{1}{9}\begin{pmatrix}67068 & -67356\\ -84195 & 83907\end{pmatrix}$

$=\begin{pmatrix}7452 & -7484\\ -9355 & 9323\end{pmatrix}$

Practice Questions

8 **(i)** Reduce the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 5 \\ -2 & 5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -6 & -4 \\ 0 & 9 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, so $\dim(V) = 2$.

Reduce the matrix $\begin{pmatrix} 1 & 2 & 2 \\ 3 & 4 & 0 \\ -3 & -3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 \\ 0 & -2 & -6 \\ 0 & 3 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, so $\dim(W) = 2$.

(ii) A basis for *V* is $\left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} \right\}$. Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W$, then the rank of $\begin{pmatrix} 1 & 2 & x \\ 3 & 0 & y \\ -2 & 5 & z \end{pmatrix}$ must remain at 2. $\begin{pmatrix} 1 & 2 & x \\ 3 & 0 & y \\ -2 & 5 & z \end{pmatrix} \to \begin{pmatrix} 1 & 2 & x \\ 0 & -6 & y-3x \\ 0 & 9 & z+2x \end{pmatrix} \to \begin{pmatrix} 1 & 2 & x \\ 0 & -6 & y-3x \\ 0 & 0 & z+2x+1.5(y-3x) \end{pmatrix}$.

Therefore $z + 2x + 1.5(y-3x) = 0$, i.e. $5x - 3y - 2z = 0$.

(iii) A basis for *W* is $\left\{ \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} \right\}$, which is the null space of **A**, so the rank of **A** must be 1.

Let $\mathbf{A} = \begin{pmatrix} a & b & c \end{pmatrix}$, substituting the vectors in this basis into $\mathbf{AX} = \mathbf{0}$, we have

$$\begin{cases} a + 3b - 3c = 0 \\ 2a + 4b - 3c = 0 \end{cases} \Rightarrow \begin{cases} a + 3b - 3c = 0 \\ -2b + 3c = 0 \end{cases}.$$

We need a non-trivial solution. We may $c = -2$, then $b = -3$ and $a = 3$, so **A** can be $\begin{pmatrix} 3 & -3 & -2 \end{pmatrix}$.

(iv) For a vector **u** to be in $V \cap W$, i.e. in both *V* and *W*,

$$\mathbf{u} = \lambda \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 3 & 0 & -3 & -4 \\ -2 & 5 & 3 & 3 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 3 & 0 & -3 & -4 \\ -2 & 5 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -6 & 0 & 2 \\ 0 & 9 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -6 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$. Let $\beta = t$, then $\alpha = -2t$. Thus $\mathbf{u} = -2t \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} = t \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$, so a basis for $V \cap W$ is $\left\{ \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \right\}$.

9 **(a)** We can say $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is in the column space of $\begin{pmatrix} 3 & -2 & 1 \\ 2 & -2 & -2 \\ 1 & 1 & 7 \end{pmatrix}$, consider

$$\begin{pmatrix} 3 & -2 & 1 & a \\ 2 & -2 & -2 & b \\ 1 & 1 & 7 & c \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 7 & c \\ 2 & -2 & -2 & b \\ 3 & -2 & 1 & a \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 7 & c \\ 0 & -4 & -16 & b-2c \\ 0 & -5 & -20 & a-3c \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 7 & c \\ 0 & -4 & -16 & b-2c \\ 0 & 0 & 0 & a-3c-1.25(b-2c) \end{pmatrix}$$

Thus $a-3c-1.25(b-2c)=0$, i.e. $4a-5b-2c=0$.

(b) **(i)** For any vector $\mathbf{b} \in$ column space of $\mathbf{PQ}$, then $\mathbf{PQx}=\mathbf{b}$ has a solution. Now $\mathbf{Py}=\mathbf{b}$ also has a solution, $\mathbf{y}=\mathbf{Qx}$, so $\mathbf{b} \in$ column space of $\mathbf{P}$. Therefor the column space of **PQ** is a subspace of the column space of **P**.

(ii) The row space of **PQ** is a subspace of the row space of **Q**. (Since the column space of $(\mathbf{PQ})^{\mathrm{T}} = \mathbf{Q}^{\mathbf{T}}\mathbf{P}^{\mathbf{T}}$ is a subspace of the column space of $\mathbf{Q}^{\mathbf{T}}$).

(iii) From **(i)**, we know the rank of **PQ** cannot exceed the rank of **P**; from **(ii)**, we know the rank of **PQ** cannot exceed the rank of **Q**. Therefore, $\text{rank}(\mathbf{PQ})$ cannot exceed the smaller of $\text{rank}(\mathbf{P})$ and $\text{rank}(\mathbf{Q})$.

10 The elements of the matrices **A** and **B** are given by

$$\mathbf{AB} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & .. & .. \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & .. & .. \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & .. & .. \end{pmatrix}$$

Its first column $\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{32} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{32} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{32} \end{pmatrix} = b_{11}\mathbf{c}_1 + b_{21}\mathbf{c}_2 + b_{31}\mathbf{c}_3$. Similarly, its second and third columns are $b_{12}\mathbf{c}_1 + b_{22}\mathbf{c}_2 + b_{32}\mathbf{c}_3$ and $b_{13}\mathbf{c}_1 + b_{23}\mathbf{c}_2 + b_{33}\mathbf{c}_3$ respectively.

The rank of **AB** is the dimension of the column space of **AB**, consider any vector in this column space:

$\mathbf{u} = k_1(b_{11}\mathbf{c}_1 + b_{21}\mathbf{c}_2 + b_{31}\mathbf{c}_3) + k_2(b_{12}\mathbf{c}_1 + b_{22}\mathbf{c}_2 + b_{32}\mathbf{c}_3) + k_3(b_{13}\mathbf{c}_1 + b_{23}\mathbf{c}_2 + b_{33}\mathbf{c}_3), k_1, k_2, k_3 \in \mathbb{R}$

Since **u** is a linear combination of $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$, it must be in the column space of **A**. Therefore, The column space of **AB** is a subspace of the column space of **A**, and $\text{rank}(\mathbf{AB}) \le \text{rank}(\mathbf{A})$.

(i) $\begin{pmatrix} 1 & \alpha & \beta \\ 2 & 2\alpha+\beta-1 & \alpha+2\beta \\ 5 & 5\alpha+3\beta-3 & 3\alpha+5\beta \end{pmatrix} \to \begin{pmatrix} 1 & \alpha & \beta \\ 0 & \beta-1 & \alpha \\ 0 & 3\beta-3 & 3\alpha \end{pmatrix} \to \begin{pmatrix} 1 & \alpha & \beta \\ 0 & \beta-1 & \alpha \\ 0 & 0 & 0 \end{pmatrix}$. The row space has at most two vectors in its basis, thus, its rank cannot exceed 2.

(ii) When $\alpha = 0$ and $\beta = 1$, the 2nd row in the row-echelon form is 0's. $\text{rank}(\mathbf{A}) = 1$.

For all 3×3 matrix **B**, $\text{rank}(\mathbf{AB}) \le \text{rank}(\mathbf{A}) = 1$.

Since $\text{rank}(\mathbf{AB}) + \text{nullity}(\mathbf{AB}) = 3$, $\text{nullity}(\mathbf{AB}) \ge 2$. There are at least two vectors in its null space. Therefore, there are at least two linearly independent solutions for **x** of the equation $\mathbf{ABx} = \mathbf{0}$.

11 Observe that $\mathbf{A}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\3\\4\\11\end{pmatrix}$, $\mathbf{A}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\\4\\9\end{pmatrix}$ and $\mathbf{A}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\3\\9\\17\end{pmatrix}$, so a particular solution

$\mathbf{Ax} = p\begin{pmatrix}1\\3\\4\\11\end{pmatrix} + q\begin{pmatrix}1\\2\\4\\9\end{pmatrix} + r\begin{pmatrix}2\\3\\9\\17\end{pmatrix}$ is $\mathbf{x}_p = p\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + q\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + r\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}p\\q\\r\\0\end{pmatrix}$.

For the complementary solution, we need to solve $\mathbf{Ax} = \mathbf{0}$.

$$\begin{pmatrix}1 & 1 & 2 & 3\\3 & 2 & 3 & 13\\4 & 4 & 9 & 7\\11 & 9 & 17 & 36\end{pmatrix} \to \begin{pmatrix}1 & 1 & 2 & 3\\0 & -1 & -3 & 4\\0 & 0 & 1 & -5\\0 & -2 & -5 & 3\end{pmatrix} \to \begin{pmatrix}1 & 1 & 2 & 3\\0 & 1 & 3 & -4\\0 & 0 & 1 & -5\\0 & 0 & 1 & -5\end{pmatrix} \to \begin{pmatrix}1 & 1 & 2 & 3\\0 & 1 & 3 & -4\\0 & 0 & 1 & -5\\0 & 0 & 0 & 0\end{pmatrix}.$$

Let $x_4 = \lambda$, then $x_3 = 5\lambda, x_2 = 4x_4 - 3x_3 = -11\lambda, x_1 = -x_2 - 2x_3 - 3x_4 = -2\lambda$.

Thus $\mathbf{x}_c = \begin{pmatrix}-2\lambda\\-11\lambda\\5\lambda\\\lambda\end{pmatrix}$ and $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_c = \begin{pmatrix}p-2\lambda\\q-11\lambda\\r+5\lambda\\\lambda\end{pmatrix}$.

(i) Comparing $\mathbf{Ax} = p\begin{pmatrix}1\\3\\4\\11\end{pmatrix} + q\begin{pmatrix}1\\2\\4\\9\end{pmatrix} + r\begin{pmatrix}2\\3\\9\\17\end{pmatrix}$ with $\mathbf{Ax} = \begin{pmatrix}4\\8\\17\\37\end{pmatrix}$, we have $p = q = r = 1$.

Thus the solution is $\mathbf{x} = \begin{pmatrix}1-2\lambda\\1-11\lambda\\1+5\lambda\\\lambda\end{pmatrix}$. When $1 - 2\lambda = 0$, $\lambda = 0.5$ and

$\mathbf{x} = \begin{pmatrix}0 & -4.5 & 3.5 & 0.5\end{pmatrix}^T$

(ii) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (1-2\lambda)^2 + (1-11\lambda)^2 + (1+5\lambda)^2 + \lambda^2$

$= 151\lambda^2 - 16\lambda + 3$

$= 64\lambda^2 - 16\lambda + 1 + 87\lambda^2 + 2$

$= (8\lambda - 1)^2 + 87\lambda^2 + 2$

≥ 2

So there is not x satisfying $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$.

12 **(i)(ii)** By deducing $\mathbf{M}_1$ to a row-echelon, we can identify a basis for its column space, $\left\{\begin{pmatrix}1\\0\\-1\\0\end{pmatrix},\begin{pmatrix}8\\-5\\-2\\0\end{pmatrix},\begin{pmatrix}-5\\1\\0\\1\end{pmatrix}\right\}$ which is also a basis for the range space for T_1. Using the row-echelon matrix to solve $\mathbf{M}_1\mathbf{x}=\mathbf{0}$, we get a basis for the null space $\left\{\begin{pmatrix}1\\-1\\1\\1\end{pmatrix}\right\}$.

(Working omitted)

(iii) By solving $\mathbf{M}_2\mathbf{x}=\mathbf{0}$, we can obtain a basis for the null space of T_2, K_2 $\left\{\begin{pmatrix}-5\\1\\0\\1\end{pmatrix},\begin{pmatrix}8\\-5\\-2\\0\end{pmatrix}\right\}$, which is obviously a subset of the basis for R_1. (Working omitted)

*In case the vectors look different, show that each of them is a linear combination of the basis for R_1.

(iv) For the null space of T_3, we use the equation $\mathbf{M}_2\mathbf{M}_1\mathbf{x}=\mathbf{0}$. This implies $\mathbf{M}_1\mathbf{x}\in K_2$.

When $\mathbf{M}_1\mathbf{x}=\mathbf{0}$, a solution is $\mathbf{x}=\begin{pmatrix}1\\-1\\1\\1\end{pmatrix}$; when $\mathbf{M}_1\mathbf{x}=\begin{pmatrix}-5\\1\\0\\1\end{pmatrix}$, a solution is $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$;

when $\mathbf{M}_1\mathbf{x}=\begin{pmatrix}8\\-5\\-2\\0\end{pmatrix}$, a solution is $\mathbf{x}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$. These are 3 such independent vectors.

13 **(i)** (Detailed working omitted)

Step 1. Explaining K is a subset of $\mathbb{R}^4$

Step 2. Explaining K is non-empty by verifying the zero vector is contained

Step 3. Showing K is closed under vector addition

Step 4. Showing K is closed under real scalar multiplication of vector

(ii) Assuming S is non-empty, and vector $\mathbf{u}\in S$, i.e. $\mathbf{Au}=\mathbf{b}$. It is obvious that $2\mathbf{u}\notin S$ since $\mathbf{A}(2\mathbf{u})=2(\mathbf{Au})=2\mathbf{b}\neq\mathbf{b}$, so S is not a vector space.

(iii) Since $\dim(K)=2$, the two linearly independent vectors in K will form a basis $\{\mathbf{e}_1,\mathbf{e}_2\}$. Since $\mathbf{A}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}1-2+1+3\\2+1+5-7\\4-3+7-1\\3+14+15-43\end{pmatrix}=\begin{pmatrix}3\\1\\7\\-11\end{pmatrix}$, $\mathbf{x}_p=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ is a particular solution, and the general solution is in the form $\mathbf{x}_p+\lambda\mathbf{e}_1+\mu\mathbf{e}_2$.

14 (Verifying independence or spanning by reducing the corresponding matrix to row-echelon form, working omitted)

For null space, $\mathrm{L}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y-z \\ x+z \\ x+y \\ 2x+y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Let $z=\lambda$, then $y=\lambda$ and $x=-\lambda$. Thus the null space is $\left\{ \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$, and its dimension is 1.

$$\mathrm{L}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-0 \\ 1+0 \\ 1+1 \\ 2+1+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} y-z \\ x+z \\ x+y \\ 2x+y+z \end{pmatrix} = (x+y)\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -x-z \\ x+z \\ 0 \\ 2x+y+z \end{pmatrix} = (x+y)\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - (x+z)\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2x+2z \\ 0 \\ 2x+y+z \end{pmatrix}$$

$$= (x+y)\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - (x+z)\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (2x+2z)\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + (y-z)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus $\mathrm{L}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and $\mathrm{L}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

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15 The linear transformation $\mathrm{T}:\mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix

$$\begin{pmatrix} 1 & 2 & -3 & -4 \\ 2 & 5 & -4 & -5 \\ 3 & a^2+5 & 2a-7 & 3a-9 \\ 6 & a^2+12 & 2a-14 & 3a-18 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & a^2-1 & 2a+2 & 3a+3 \\ 0 & a^2 & 2a+4 & 3a+6 \end{pmatrix}$$

$\to \begin{pmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & (a+1)(a-1) & 2(a+1) & 3(a+1) \\ 0 & 0 & 0 & 0 \end{pmatrix}$. When $a \neq -1$, it can be further deduced to

$\begin{pmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & a-1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 4-2a & 6-3a \\ 0 & 0 & 0 & 0 \end{pmatrix}$. When $a \neq 2$, the third row does not

contain only 0's,, then its row space has 3 vectors in a basis. The rank of **A** is 3.
Thus the range space of **T** is 3 when $a \neq -1$ and $a \neq 2$.

(i) When $a = 2$, the third row contains only 0's, the rank of **A** becomes 2. Since $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = 4$, the nullity of **A** (or T or the dimension of *K*) is 2.

(ii) To find a basis for *K*, we solve $\begin{pmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = 0$. Let $z = \lambda$ and $t = \mu$,

then $y = -2\lambda - 3\mu$ and $x = 10\lambda + 13\mu$. Thus the vectors in *K* can be written as $\begin{pmatrix} 10\lambda + 13\mu \\ -2\lambda - 3\mu \\ \lambda \\ \mu \end{pmatrix} = \lambda\begin{pmatrix} 10 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 13 \\ -3 \\ 0 \\ 1 \end{pmatrix}$, and a basis can be $\left\{\begin{pmatrix} 10 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 13 \\ -3 \\ 0 \\ 1 \end{pmatrix}\right\}$.

(iii) By observation, $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

16 Since $\mathbf{Ax} = \lambda\mathbf{x}$ and $\mathbf{Bx} = \mu\mathbf{x}$, $(\mathbf{AB})\mathbf{x} = \mathbf{A}(\mathbf{Bx}) = \mathbf{A}(\mu\mathbf{x}) = \mu(\mathbf{Ax}) = \lambda\mu\mathbf{x}$. Therefore, **x** is an eigenvector of **AB** with eigenvalue $\lambda\mu$.
(Working omitted)

C has eigenvalues -1, 3, 5 with corresponding eigenvectors $\begin{pmatrix} -17 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

D has eigenvalues -3, -2, -1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

CD has an eigenvector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ with corresponding eigenvalue $3 \times (-3) = -9$.

17 $\begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, a basis for the range space is $\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$.

By letting $z = \lambda$, we can find a basis for the null space $\left\{\begin{pmatrix} 2 & -2 & 1 \end{pmatrix}^{\mathrm{T}}\right\}$.

(i) $\begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 + 2\lambda \\ 2 - 2\lambda \\ -2 \end{pmatrix} = \begin{pmatrix} 5 + 2\lambda \\ 3 \\ -2 + 2\lambda \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

(ii) Let $\begin{pmatrix}1 & 0 & -2\\ 1 & 1 & 0\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}5\\ 3\\ -2\end{pmatrix}$ and perform row operations on its augmented matrix,

$\begin{pmatrix}1 & 0 & -2 & 5\\ 1 & 1 & 0 & 3\\ 0 & 1 & 2 & -2\end{pmatrix}\to\begin{pmatrix}1 & 0 & -2 & 5\\ 0 & 1 & 2 & -2\\ 0 & 1 & 2 & -2\end{pmatrix}\to\begin{pmatrix}1 & 0 & -2 & 5\\ 0 & 1 & 2 & -2\\ 0 & 0 & 0 & 0\end{pmatrix}$. Let $z=\mu$, then

$y=-2-2\mu$ and $x=5+2\mu$. Therefore the line is $\mathbf{r}=\begin{pmatrix}5\\ -2\\ 0\end{pmatrix}+\mu\begin{pmatrix}2\\ -2\\ 1\end{pmatrix},\mu\in\mathbb{R}$.

(iii) We first convert $x-y+z=2$ to a parametric form by letting $y=\alpha$ and $z=\beta$.

The plane has equation $\mathbf{r}=\begin{pmatrix}2+\alpha-\beta\\ \alpha\\ \beta\end{pmatrix}$. Under L, it becomes

$$\begin{pmatrix}1 & 0 & -2\\ 1 & 1 & 0\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}2+\alpha-\beta\\ \alpha\\ \beta\end{pmatrix}=\begin{pmatrix}2+\alpha-3\beta\\ 2+2\alpha-\beta\\ \alpha+2\beta\end{pmatrix}=\begin{pmatrix}2\\ 2\\ 0\end{pmatrix}+\alpha\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}+\beta\begin{pmatrix}-3\\ -1\\ 2\end{pmatrix}.$$

Its normal vector can be $\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}\times\begin{pmatrix}-3\\ -1\\ 2\end{pmatrix}=\begin{pmatrix}5\\ -5\\ 5\end{pmatrix}$ or $\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}$, therefore a Cartesian equation for the plane is $x-y+z=2-2+0=0$.

(In gerenal, to find a Cartesian equation $k_1x+k_2y+k_3z=D$, we can let the eq

$$k_1(2+\alpha-3\beta)+k_2(2+2\alpha-\beta)+k_3(\alpha+2\beta)=D,$$

and solve for the values of k_1, k_2, k_3, D that makes α and β disappear.)

18 **(i)(ii)** By reducing **M** to its row-echelon form (working omitted), we can obtain a basis for *V*, $\left\{\begin{pmatrix}1\\ -1\\ 1\end{pmatrix},\begin{pmatrix}1\\ 4\\ -3\end{pmatrix}\right\}$, and its dimension is 2.

(iii) Let $\begin{pmatrix}-1\\ 4\\ 5\end{pmatrix}=\alpha\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}+\beta\begin{pmatrix}1\\ 4\\ -3\end{pmatrix}$. From the *x*- and *y*- components, we get $\alpha=-1.6$ and $\beta=0.6$. Check the *z*-component, $\text{RHS}=-1.6-3(0.6)=-3.4\neq 5=\text{LHS}$. Thus the vector cannot be expressed as a linear combination of the vectors in the basis, so this vector is not in *V*.

(iv)(a) $\begin{pmatrix}1 & 1 & 2\\ -1 & 4 & 3\\ 1 & -3 & -2\end{pmatrix}\to\begin{pmatrix}1 & 1 & 2\\ 0 & 5 & 5\\ 0 & -4 & -4\end{pmatrix}\to\begin{pmatrix}1 & 1 & 2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}$, so $\begin{pmatrix}2\\ 3\\ -2\end{pmatrix}$ is in *V* and it has a solution.

(iv)(b) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 4 & 7 \\ 1 & -3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 5 & 8 \\ 0 & -4 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1.6 \\ 0 & 0 & 1 \end{pmatrix}$, so $\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \notin V$ and it has no solution.

19 $\begin{pmatrix} 1 & -2 & -3 & a \\ -1 & 3 & a+3 & -a+1 \\ 1 & -1 & a-3 & a+1 \\ 2 & -3 & a-6 & 2a+1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -3 & a \\ 0 & 1 & a & 1 \\ 0 & 1 & a & 1 \\ 0 & 1 & a & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -3 & a \\ 0 & 1 & a & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so its rank is 2.

(iv) Given that $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 0 \\ -1 \end{pmatrix}$,

find the values of a for which the equation $\mathbf{Mx} = \mathbf{b}$ has a solution of the form $\begin{pmatrix} v \\ v^{-1} \\ 1 \\ 1 \end{pmatrix}$.

(i) Since $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for the null space, $\mathbf{Me}_1 = \mathbf{0}, \mathbf{Me}_2 = \mathbf{0}$.

$\mathbf{Mx} = \mathbf{M}(\mathbf{x}_0 + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2) = \mathbf{Mx}_0 + \lambda\mathbf{Me}_1 + \mu\mathbf{Me}_2 = \mathbf{b} + \mathbf{0} + \mathbf{0} = \mathbf{b}$

(ii) If $\mathbf{Mx} = \mathbf{b}$, $\mathbf{M}(\mathbf{x} - \mathbf{x}_0) = \mathbf{Mx} - \mathbf{Mx}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}$, then $\mathbf{x} - \mathbf{x}_0 \in K$. Thus $\mathbf{x} - \mathbf{x}_0$ is a linear combination of vectors in the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, so $\mathbf{x} = \mathbf{x}_0 + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$.

(iii) Vectors $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in K$ satisfy $\begin{cases} x - 2y - 3z + at = 0 \\ y + az + t = 0 \end{cases}$. Let $z = \alpha$ and $t = \beta$, then

$y = -a\alpha - \beta$ and $x = -2a\alpha - 2\beta + 3\alpha - a\beta = \alpha(3 - 2a) + \beta(-2 - a)$.

Therefore so $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \alpha \begin{pmatrix} 3-2a \\ -a \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2-a \\ -1 \\ 0 \\ 1 \end{pmatrix}$. $\mathbf{e}_1 = \begin{pmatrix} 3-2a \\ -a \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} -2-a \\ -1 \\ 0 \\ 1 \end{pmatrix}$.

(iv) $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 3-2a \\ -a \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2-a \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} v \\ v^{-1} \\ 1 \\ 1 \end{pmatrix}$. By comparing all the components, we have

$\alpha = \beta = 1$, $v = 2 - 3a$ and $v^{-1} = -a$, so $(2 - 3a)(-a) = 1$. Solving this equation, we obtain $a = -\frac{1}{3}$ or $a = 1$.

20 When $\mathbf{Ae} = \lambda\mathbf{e}$, $\mathbf{A}^2\mathbf{e} = \mathbf{A}(\mathbf{Ae}) = \mathbf{A}(\lambda\mathbf{e}) = \lambda(\mathbf{Ae}) = \lambda\lambda\mathbf{e} = \lambda^2\mathbf{e}$, so $\mathbf{e}$ is an eigenvector of a square matrix $\mathbf{A}^2$ with corresponding eigenvalue λ^2.

(Working omitted)

B has eigenvalues are 1, 3, -4 with corresponding eigenvectors $\begin{pmatrix} 11 \\ -5 \\ -6 \end{pmatrix}, \begin{pmatrix} 5 \\ -7 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

$\mathbf{Q} = \begin{pmatrix} 11 & 5 & 1 \\ -5 & -7 & 0 \\ -6 & -5 & -1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{pmatrix}$.

21 **(i)** If **A** is non-singular, $\det(0\mathbf{I} - \mathbf{A}) = \det(-\mathbf{A}) = (-1)^3 \det(\mathbf{A}) \neq 0$, so 0 is not an eigenvalue, i.e. $\lambda \neq 0$.

(ii) $\mathbf{x} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{x} = \mathbf{A}^{-1}(\mathbf{Ax}) = \mathbf{A}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{A}^{-1}\mathbf{x}$. Since $\lambda \neq 0$, $\mathbf{A}^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$, so **x** is an eigenvector of the matrix $\mathbf{A}^{-1}$ with corresponding eigenvalue λ^{-1}.

A has eigenvalues 1, 2, -3 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 8 \\ -5 \end{pmatrix}$.

B has eigenvalues $\frac{1}{6}, \frac{1}{7}, \frac{1}{2}$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 8 \\ -5 \end{pmatrix}$.

22 When $\mathbf{e}_i$ is a common eigenvector of **A** and B with corresponding eigenvalues λ_i and μ_i, $\mathbf{Ae}_i = \lambda_i\mathbf{e}_i$, $\mathbf{Be}_i = \mu_i\mathbf{e}_i$ and $(\mathbf{A}+\mathbf{B})\mathbf{e}_i = \mathbf{Ae}_i + \mathbf{Be}_i = \lambda_i\mathbf{e}_i + \mu_i\mathbf{e}_i = (\lambda_i + \mu_i)\mathbf{e}_i$.

Therefore, $\mathbf{A}+\mathbf{B}$ has eigenvalues $\lambda_1 + \mu_1, \lambda_2 + \mu_2, \lambda_3 + \mu_3$ with the corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

$\begin{pmatrix} 0 & -1 & 0 \\ -4 & -9 & -6 \\ 5 & 11 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, the corresponding eigenvalue is 1.

$\begin{pmatrix} 0 & -1 & 0 \\ -4 & -9 & -6 \\ 5 & 11 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, the corresponding value is –1.

$\begin{pmatrix} 0 & -1 & 0 \\ -4 & -9 & -6 \\ 5 & 11 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 6 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$, the corresponding value is –2.

(i) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & -2 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -4 & -3 \\ 1 & 3 & 2 \end{pmatrix}$.

(ii) $\mathbf{A}+\mathbf{B}$ has the same 3 eigenvectors with the corresponding eigenvalues 2, 1, –5. Thus $\mathbf{M} = \mathbf{RER}^{-1}$ and $\mathbf{M}^5 = \mathbf{RE}^5\mathbf{S} = \mathbf{RDS}$ where

$$\mathbf{R}=\begin{pmatrix}1&1&1\\-1&1&2\\1&-2&-3\end{pmatrix},\mathbf{S}=\mathbf{R}^{-1}=\begin{pmatrix}1&1&1\\-1&-4&-3\\1&3&2\end{pmatrix}\text{ and }\mathbf{D}=\mathbf{E}^5=\begin{pmatrix}32&0&0\\0&1&0\\0&0&-3125\end{pmatrix}.$$

23 When $\mathbf{Ax}=\lambda\mathbf{x}$,

$\mathbf{B}(\mathbf{Ex})=\mathbf{EAE}^{-1}(\mathbf{Ex})=\mathbf{EA}(\mathbf{E}^{-1}\mathbf{E})\mathbf{x}=\mathbf{EAx}=\mathbf{E}(\mathbf{Ax})=\mathbf{E}(\lambda\mathbf{x})=\lambda(\mathbf{Ex})$, so $\mathbf{Ex}$ is an eigenvector of the matrix **B** and that λ is the corresponding eigenvalue.

(i) (working omitted).

A has eigenvalues 1, -2, 3 with corresponding eigenvectors $\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}-a\\3\\0\end{pmatrix},\begin{pmatrix}ac+5b\\2c\\10\end{pmatrix}$.

(ii) **B** has the same eigenvalues, the corresponding vectors are

$$\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}-a\\3\\0\end{pmatrix}=\begin{pmatrix}-a\\3\\-a\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}ac+5b\\2c\\10\end{pmatrix}=\begin{pmatrix}ac+5b\\2c\\ac+5b+10\end{pmatrix}$$

(iii) $\mathbf{Q}=\begin{pmatrix}1&-a&ac+5b\\0&3&2c\\1&-a&ac+5b+10\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}1&0&0\\0&(-2)^n&0\\0&0&3^n\end{pmatrix}$.

24 **(i)** Eigenvalues are 1, 2, -1 with corresponding eigenvectors $\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}54\\7\\44\end{pmatrix},\begin{pmatrix}-3\\1\\-4\end{pmatrix}$.

(ii) If μ is an eigenvalue for **B**, $\det(\mu\mathbf{I}-\mathbf{B})=\det(\mu\mathbf{I}-\mathbf{A}-k\mathbf{I})=\det((\mu-k)\mathbf{I}-\mathbf{A})$ is 0 if $\mu-k$ is an eigenvalue for **A**. Thus the eigenvalues are $1+k, 2+k, -1+k$.

Since $\mathbf{Ax}=\lambda\mathbf{x}$, $\mathbf{Bx}=(\mathbf{A}+k\mathbf{I})\mathbf{x}=\mathbf{Ax}+k\mathbf{Ix}=\lambda\mathbf{x}+k\mathbf{x}=(\lambda+k)\mathbf{x}=\mu\mathbf{x}$, so an eigenvector for **A** remains as an eigenvector for **B**, independent of *k*.

(iii) $\mathbf{Q}=\begin{pmatrix}1&54&-3\\0&7&1\\1&44&-4\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}(1+k)^2&0&0\\0&(2+k)^2&0\\0&0&(-1+k)^2\end{pmatrix}$.

25 By GC (or Factor Theorem), $\lambda-a=\frac{1}{4},\frac{1}{2},\frac{3}{4}$, so the eigenvalues are $a+\frac{1}{4},a+\frac{1}{2},a+\frac{3}{4}$.

The eigenvectors for **M** are the same as the eigenvalues for $\mathbf{M}-a\mathbf{I}$, i.e $\begin{pmatrix}\frac{3}{8}&-\frac{1}{8}&\frac{1}{8}\\-\frac{1}{4}&\frac{1}{2}&\frac{1}{4}\\-\frac{1}{8}&\frac{1}{8}&\frac{5}{8}\end{pmatrix}$,

and are the same as the eigenvalues for $\begin{pmatrix}3&-1&1\\-2&4&2\\-1&1&5\end{pmatrix}$, with corresponding eigenvalues

2, 4 and 6. The corresponding eigenvectors are $\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\1\end{pmatrix}$ (Working omitted).

Therefore, $\mathbf{Q}=\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}a+\frac{1}{4}&0&0\\0&a+\frac{1}{2}&0\\0&0&a+\frac{3}{4}\end{pmatrix}$.

For $\mathbf{D}\to\mathbf{O}$, $-1<a+\frac{1}{4}<a+\frac{1}{2}<a+\frac{3}{4}<1$,the set of values is $\left\{a\in\mathbb{R}:-\frac{5}{4}<a<\frac{1}{4}\right\}$.

26 $(k\mathbf{M})\mathbf{e}=k(\mathbf{Me})=k\lambda\mathbf{e}$ so **e** is an eigenvector of the matrix $k\mathbf{M}$ with the corresponding eigenvalue $k\lambda$.

$$2\mathbf{I}-\mathbf{A}=\begin{pmatrix}1-a&1-a&a-1\\-1-a&3-a&1+a\\-2&2&2\end{pmatrix}\to\begin{pmatrix}1&1&-1\\-2&2&2\\-2&2&2\end{pmatrix}\to\begin{pmatrix}1&1&-1\\0&1&0\\0&0&0\end{pmatrix}.$$

To find a corresponding eigenvector, solve $\begin{cases}x+y-z=0\\y=0\end{cases}$. An eigenvector is $\begin{pmatrix}1\\0\\1\end{pmatrix}$.

$$-2\mathbf{I}-\mathbf{A}=\begin{pmatrix}-3-a&1-a&a-1\\-1-a&-1-a&1+a\\-2&2&-2\end{pmatrix}\to\begin{pmatrix}-1&1&-1\\0&-2-2a&2+2a\\0&4&-4\end{pmatrix}\to\begin{pmatrix}1&-1&1\\0&1&-1\\0&0&0\end{pmatrix}.$$

To find a corresponding eigenvector, solve $\begin{cases}x-y+z=0\\y-z=0\end{cases}$. An eigenvector is $\begin{pmatrix}0\\1\\1\end{pmatrix}$.

$$2a\mathbf{I}-\mathbf{A}=\begin{pmatrix}a-1&1-a&a-1\\-1-a&1+a&1+a\\-2&2&2a\end{pmatrix}\to\begin{pmatrix}1&-1&1\\1&-1&1\\1&-1&-a\end{pmatrix}\to\begin{pmatrix}1&-1&1\\0&0&1\\0&0&0\end{pmatrix}.$$

To find a corresponding eigenvector, solve $\begin{cases}x-y+z=0\\z=0\end{cases}$. An eigenvector is $\begin{pmatrix}1\\1\\0\end{pmatrix}$.

Therefore, $\mathbf{Q}=\begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}-2&0&0\\0&2&0\\0&0&2a\end{pmatrix}$.

(i) $a=100$, so $\mathbf{D}=\begin{pmatrix}-2&0&0\\0&2&0\\0&0&200\end{pmatrix}$, Q remains as $\begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}$.

(ii) $a=500$. $\mathbf{A}=\begin{pmatrix}501&499&499\\501&499&501\\2&2&0\end{pmatrix}$, $0.001\mathbf{A}=\begin{pmatrix}0.501&0.499&-0.499\\0.501&0.499&-0.501\\0.002&-0.002&0\end{pmatrix}$. Thus

$\mathbf{D}=0.001\begin{pmatrix}2&0&0\\0&-2&0\\0&0&1000\end{pmatrix}=\begin{pmatrix}0.002&0&0\\0&-0.002&0\\0&0&1\end{pmatrix}$ and Q remains as $\begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}$.)

27 $\lambda_i = -5, -3, -2$ with $\mathbf{e}_i = \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix}$. (Working omitted).

If **e** is an eigenvector of **M**, with corresponding eigenvalue λ, then $k_1(\mathbf{M} + k_2\mathbf{I})\mathbf{e} = k_1\mathbf{Me} + k_1k_2\mathbf{e} = k_1\lambda\mathbf{e} + k_1k_2\mathbf{e} = k_1(\lambda + k_2)\mathbf{e}$, i.e. **e** is an eigenvector of the matrix $k_1(\mathbf{M} + k_2\mathbf{I})$, with corresponding eigenvalue $k_1(\lambda + k_2)$.

Let μ_i be the eigenvalues of **B** with corresponding eigenvectors $\mathbf{e}_i$. We observe that $\mu_i = \frac{1}{3}(\lambda_i + 5)$, so $\mathbf{B} = \frac{1}{3}(\mathbf{A} + 5\mathbf{I}) = \frac{1}{3}\begin{pmatrix} 3 & 1 & -1 \\ -4 & -2 & 4 \\ 0 & -1 & 4 \end{pmatrix}$.

Arranging the eigenvalues from largest to smallest, the corresponding eigenvectors form the matrix $\mathbf{P} = \begin{pmatrix} -1 & 1 & 1 \\ 4 & -2 & -4 \\ 1 & -1 & -1 \end{pmatrix}$, and $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{2}{3}\right)^n & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Solve $\left(\frac{2}{3}\right)^n < 0.001$, we have least $n = 18$ by GC.

Application Problems

28 **(a)** **(i)** Reflect Q about the x-axis; $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(ii) Scaling Q parallel to the y-axis by a factor 3; $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

(b) **(i)** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. **(ii)** $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(d) Rotate Q through an angle θ about the origin in anticlockwise direction.

29 **(ii)** $\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 100 \end{pmatrix}$, $\theta = \tan^{-1}\left(-\frac{3}{4}\right)$. **(iii)** $100y'^2 + 25x' + 25 = 0$.

(iv) Rotate S about O through an angle $\tan^{-1}\frac{3}{4}$ in anticlockwise direction.

(v) $\left(-\frac{17}{16}, 0\right)$, $x' = -\frac{15}{16}$; $\left(-\frac{17}{20}, \frac{51}{80}\right)$, $64x - 48y + 75 = 0$.