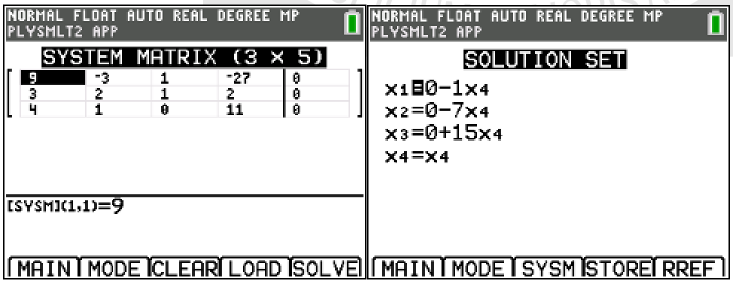
Raffles Institution
H2 Mathematics (9758)
Solution for 2019 A-Level Paper 1

Question 1

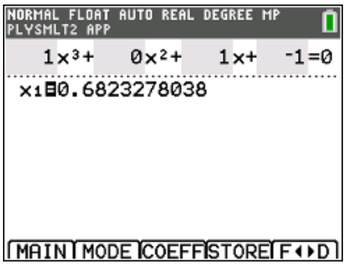
No.	Suggested Solution	Remarks for Student
	The coefficient of the cubic function are all real. Since $2+\mathrm{i}$ is a root of $\mathrm{f}(z) = 0$, then $(2+\mathrm{i})^* = 2-\mathrm{i}$ is also a root. $\mathrm{f}(z)=0$ $(z-(-3))(z-(2+\mathrm{i}))(z-(2-\mathrm{i}))=0$ $(z+3)(z^2-4z+5)=0$ $z^3-z^2-7z+15=0$ $az^3+bz^2+cz+d=a\left(z^3+\frac{b}{a}z^2+\frac{c}{a}z+\frac{d}{a}\right)$ $b=-a,\ c=-7a,\ d=15a$ Alternatively, With the given roots apply Factor Theorem. $\mathrm{f}(-3)=0 \Rightarrow -27a+9b-3c+d=0 \quad (1)$ $\mathrm{f}(2+\mathrm{i})=0 \Rightarrow (2+11\mathrm{i})a+(3+4\mathrm{i})b+(2+\mathrm{i})c+d=0$ $2a+3b+2c+d=0 \quad (2)$ $11a+4b+c=0 \quad (3)$ From (1), (2), (3) we form a system of linear equations with last column the coefficient of a.  As before, $b=-a,\ c=-7a,\ d=15a$	

Question 2

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td>(i)</td><td>

$x^3 + x - 1 = 0$

Using GC, Polysimult2

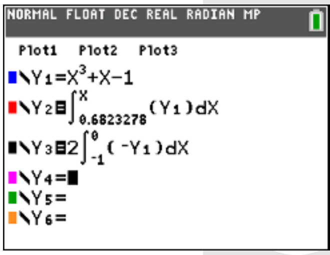


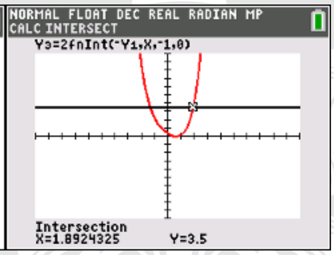
$0.68233 \approx 0.682$

</td><td>Do note the GC skills required in this question.</td></tr>
<tr><td>(ii)</td><td>

$\int_a^b \left(x^3 + x - 1\right) \mathrm{d}x = 2\int_{-1}^{0} 0 - \left(x^3 + x - 1\right) \mathrm{d}x$

We can solve with GC, noting that $b > a$.





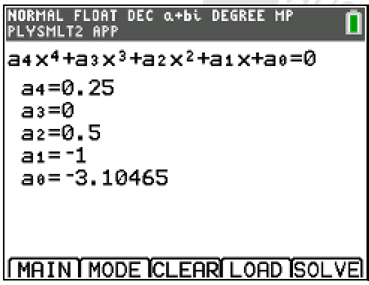
$b = 1.89243 \approx 1.892$

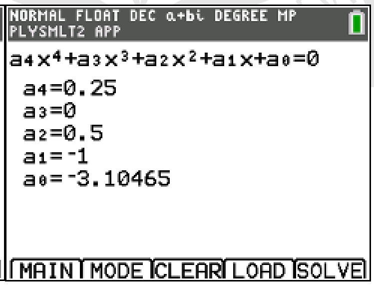
Alternatively,

$\int_a^b \left(x^3 + x - 1\right) \mathrm{d}x = 2\int_{-1}^{0} 0 - \left(x^3 + x - 1\right) \mathrm{d}x$

$\left(\frac{x^4}{4} + \frac{x^2}{2} - x\right) - \left(\frac{0.68233^4}{4} + \frac{0.68233^2}{2} - 0.68233\right) = 3.5$

$\frac{x^4}{4} + \frac{x^2}{2} - x - 3.10465 = 0$





$b = 1.89243 \approx 1.892$

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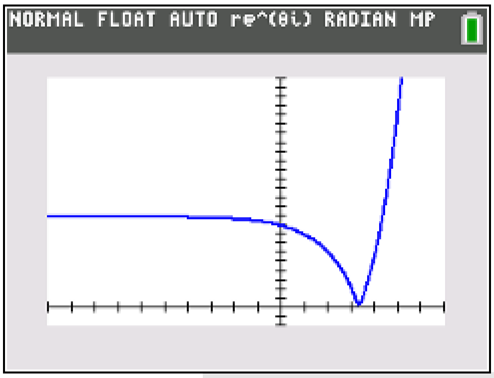
Question 3

No.	Suggested Solution	Remarks for Student
(i)	Note that $(x-1)^3 = x^3 - 3x^2 + 3x - 1$ $\mathrm{f}(x) = 2x^3 - 6x^2 + 6x - 12$ $= 2(x^3 - 3x^2 + 3x - 1) - 10$ $= 2(x-1)^3 - 10$ $= 2\{(x-1)^3 - 5\}$ $p = 2, q = -1, r = -5$	
(ii)	Sequence of 1. Translation of 1 unit parallel to the positive x-axis 2. Translation of 5 units parallel to the negative y-axis 3. Scale by factor 2 parallel to the y-axis.	

Question 4

No.	Suggested Solution	Remarks for Student
(i)	 $x=0,\ y=\left\|2^0-10\right\|=9$ $y=0,\ 2^x=10 \Rightarrow x=\dfrac{\ln 10}{\ln 2}$ or $\dfrac{1}{\lg 2}$	
(ii)	Solving $\left\|2^x-10\right\|=6$ $2^x-10=6$ or $2^x-10=-6$ $x=\dfrac{\ln 16}{\ln 2}$ $\quad$ $x=\dfrac{\ln 4}{\ln 2}=2$ $=\dfrac{4\ln 2}{\ln 2}$ $=4$ $\left\|2^x-10\right\|\le 6 \Rightarrow 2\le x\le 4$	

Question 5

No.	Suggested Solution	Remarks for Student
(i)	$\mathrm{f}(x)=\mathrm{e}^{2x}-4,\ x\in\mathbb{R}$ $\mathrm{g}(x)=x+2,\ x\in\mathbb{R}$ $y=\mathrm{e}^{2x}-4$ $\mathrm{e}^{2x}=y+4$ $2x=\ln(y+4)$ $x=\frac{1}{2}\ln(y+4)$ $\mathrm{f}^{-1}(x)=\frac{1}{2}\ln(x+4)$ Domain of f^{-1} = Range of f $=(-4,\infty)$	Note that we use "round brackets" here.
(ii)	$\mathrm{fg}(x)=5$ $\mathrm{f}^{-1}(\mathrm{fg}(x))=\mathrm{f}^{-1}(5)$ $\mathrm{g}(x)=\mathrm{f}^{-1}(5)$ $x+2=\frac{1}{2}\ln(5+4)$ $x=\ln 3-2$	

Question 6

No.	Suggested Solution	Remarks for Student
(i)	$\frac{1}{4r^2-1}=\frac{1}{(2r-1)(2r+1)}=\frac{1}{2(2r-1)}-\frac{1}{2(2r+1)}$ $\sum_{r=1}^{n}\frac{1}{4r^2-1}=\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{(2r-1)}-\frac{1}{(2r+1)}\right)$ $=\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right.$ $+\frac{1}{3}-\frac{1}{5}$ $+\frac{1}{5}-\frac{1}{7}$ $\ldots$ $+\frac{1}{(2n-3)}-\frac{1}{(2n-1)}$ $\left.+\frac{1}{(2n-1)}-\frac{1}{(2n+1)}\right)$ $=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$	
(ii)	$\sum_{r=11}^{\infty}\frac{1}{4r^2-1}=\sum_{r=1}^{\infty}\frac{1}{4r^2-1}-\sum_{r=1}^{10}\frac{1}{4r^2-1}$ $=\frac{1}{2}-\frac{1}{2}\left(1-\frac{1}{2(10)+1}\right)$ $=\frac{1}{42}$	

Question 7

No.	Suggested Solution	Remarks for Student
(i)	$y = xe^{-x}$ $\frac{dy}{dx} = e^{-x} - xe^{-x}$ $x = 1,\ y = e^{-1},\ \frac{dy}{dx} = e^{-1} - e^{-1} = 0$ Equation of tangent: $y = e^{-1}$ $x = -1,\ y = -e,\ \frac{dy}{dx} = e + e = 2e$ Equation of tangent: $y + e = 2e(x+1)$ $y = 2ex + e$	
(ii)	$\tan\theta = 2e$ $\theta = 79.6^\circ$	Note that one of the tangents is a horizontal line. So the gradient of the other line allows us to get the angle directly.

Question 8

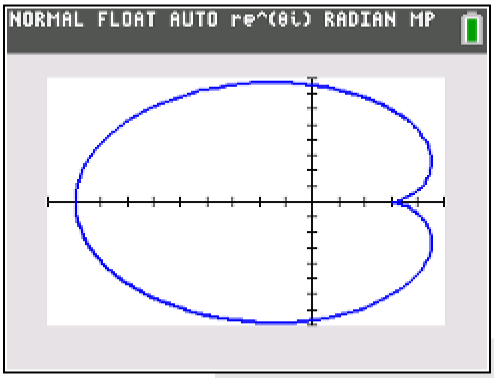
No.	Suggested Solution	Remarks for Student
(a)	$a\left(2^{k-1}\right)=\frac{64}{2}\left(2a+(64-1)(2a)\right)$ $2^{k-1}=32(2+126)$ $2^{k-1}=2^{5}\left(2^{7}\right)$ $k=13$	
(b)	Note that $f\neq 0$. If $r=1$, then sum of first 4 terms $=4f\neq 0$ Thus, $r\neq 1$. $\frac{f\left(1-r^{4}\right)}{1-r}=0$ $r^{4}=1\Rightarrow r=-1$ or $r=1$ (rejected) Thus, $r=-1$, $f\in\mathbb{R}\setminus\{0\}$ $S_n=\begin{cases}0, & n \text{ even}\\ f, & n \text{ odd}\end{cases}$	
(iii)	Let a be the first term. $\frac{4}{2}(2a+3d)=14$ $2a+3d=7$ ...(1) $a(a+d)(a+2d)(a+3d)=0$ $a=0$ or $-d$ or $-2d$ or $-3d$ Given $a<0$, Possible $d=-a$ or $-\frac{a}{2}$ or $-\frac{a}{3}$ Into (1), only $d=-a$ gives $a<0$ $a=-7, d=7$ 11th term $=a+10d=63$	

Question 9

No.	Suggested Solution	Remarks for Student
(i)	$w=\cos\theta+\mathrm{i}\sin\theta$	
(a)	$w+\dfrac{1}{w}=\cos\theta+\mathrm{i}\sin\theta+\dfrac{1}{\cos\theta+\mathrm{i}\sin\theta}$ $=\cos\theta+\mathrm{i}\sin\theta+\dfrac{\cos\theta-\mathrm{i}\sin\theta}{\cos^2\theta+\sin^2\theta}$ $=\cos\theta+\mathrm{i}\sin\theta+\cos\theta-\mathrm{i}\sin\theta$ $=2\cos\theta$ Alternatively, $w+\dfrac{1}{w}=\mathrm{e}^{\mathrm{i}\theta}+\dfrac{1}{\mathrm{e}^{\mathrm{i}\theta}}$ $=\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}$ $=(\cos\theta+\mathrm{i}\sin\theta)+(\cos\theta-\mathrm{i}\sin\theta)$ $=2\cos\theta$	
(b)	$\dfrac{w-1}{w+1}=\dfrac{w+1-2}{w+1}$ $=1-\dfrac{2}{1+\cos\theta+\mathrm{i}\sin\theta}$ $=1-\dfrac{2(1+\cos\theta-\mathrm{i}\sin\theta)}{(1+\cos\theta)^2+\sin^2\theta}$ $=1-\dfrac{2(1+\cos\theta)-\mathrm{i}(2\sin\theta)}{2+2\cos\theta}$ $=\dfrac{2+2\cos\theta-2(1+\cos\theta)+\mathrm{i}(2\sin\theta)}{2+2\cos\theta}$ $=\dfrac{\mathrm{i}(2\sin\theta)}{2+2\cos\theta}$ $=\dfrac{\mathrm{i}(\sin\theta)}{1+\cos\theta}$ $=\dfrac{\mathrm{i}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}{2\cos^2\frac{\theta}{2}}$ $=\mathrm{i}\tan\dfrac{\theta}{2}$ Alternatively, $\dfrac{w-1}{w+1}=\dfrac{\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}0}}{\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}0}}=\dfrac{\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}(\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}-\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}})}{\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}(\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}+\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}})}$	

	$=\dfrac{e^{\frac{i\theta}{2}}-e^{-\frac{i\theta}{2}}}{e^{\frac{i\theta}{2}}+e^{-\frac{i\theta}{2}}}=\dfrac{\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)-\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)+\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)}$ $=\dfrac{2i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$ $=i\tan\dfrac{\theta}{2}$	
(ii)	Let $z=a+ib$ Since $\|z\|=1\Rightarrow a^2+b^2=1$ $\|z-3i\|=\|a+ib-3i\|=\sqrt{a^2+(b-3)^2}=\sqrt{a^2+b^2-6b+9}=\sqrt{10-6b}$ $\|1+3iz\|=\|1+3i(a+ib)\|=\sqrt{(1-3b)^2+9a^2}=\sqrt{1-6b+9b^2+9a^2}=\sqrt{10-6b}$ $\therefore\left\|\dfrac{z-3i}{1+3iz}\right\|=1$ OR Since $\|z\|=1\Rightarrow z=\cos\theta+i\sin\theta$ $\|z-3i\|=\|\cos\theta+i\sin\theta-3i\|=\sqrt{\cos^2\theta+(\sin\theta-3)^2}=\sqrt{10-6\sin\theta}$ $\|1+3iz\|=\|1+3i\cos\theta-3\sin\theta\|=\sqrt{(1-3\sin\theta)^2+9\cos^2\theta}=\sqrt{10-6\sin\theta}$ $\therefore\left\|\dfrac{z-3i}{1+3iz}\right\|=1$ OR $\|z\|=1\Rightarrow\|z^*\|=1$ $\left\|\dfrac{z-3i}{1+3iz}\right\|=\left\|\dfrac{z-3i}{1+3iz}\right\|\|z^*\|=\left\|\dfrac{zz^*-3iz^*}{1+3iz}\right\|=\left\|\dfrac{1-3iz^*}{1+3iz}\right\|$ Note that $(1+3iz)^*=1^*+(3iz)^*=1+(3i)^*z^*=1-3iz^*$ $\therefore\left\|\dfrac{z-3i}{1+3iz}\right\|=\left\|\dfrac{(1+3iz)^*}{1+3iz}\right\|=1$	

Question 10

No.	Suggested Solution	Remarks for Student
(i)	Taking a as positive.  Label x-intercepts, $(-3a, 0)$ and $(a, 0)$ Cartesian equation of line of symmetry: $y = 0$	Based on the marks allocation, it should be 1 mark for the curve. Use your GC to help you get the shape out by letting a to be a positive number such as 2, 3 or 4.
(ii)	$y=0 \;\Rightarrow a\left(2\sin\theta-\sin 2\theta\right)=0$ $2\sin\theta-2\sin\theta\cos\theta=0$ $2\sin\theta\left(1-\cos\theta\right)=0$ $\sin\theta=0$ or $\cos\theta=1$ $\theta=0,\pi,2\pi$ or $\theta=0,2\pi$ $\therefore\theta=0,\pi,2\pi$	
(iii)	Area of region bounded by x-axis and part of C above x-axis $=\int_{\pi}^{0} y\dfrac{\mathrm{d}x}{\mathrm{d}\theta}\ \mathrm{d}\theta$ $=\int_{\pi}^{0} a\left(2\sin\theta-\sin 2\theta\right)a\left(-2\sin\theta+2\sin 2\theta\right)\ \mathrm{d}\theta$ $=\int_{\pi}^{0} -a^2\left(4\sin^2\theta-2\sin\theta\sin 2\theta-4\sin\theta\sin 2\theta+2\sin^2 2\theta\right)\ \mathrm{d}\theta$ $=\int_{0}^{\pi} a^2\left(4\sin^2\theta-6\sin\theta\sin 2\theta+2\sin^2 2\theta\right)\ \mathrm{d}\theta$	From (ii) "lower limit" $\theta=\pi\Rightarrow x=-3a$ "upper limit" $\theta=0\Rightarrow x=a$
(iv)	$\int_{0}^{\pi} a^2\left(4\sin^2\theta-6\sin\theta\sin 2\theta+2\sin^2 2\theta\right)\ \mathrm{d}\theta$ $=a^2\int_0^{\pi}4\sin^2\theta\ \mathrm{d}\theta-a^2\int_0^{\pi}6\sin\theta\sin 2\theta\ \mathrm{d}\theta+a^2\int_0^{\pi}2\sin^2 2\theta\ \mathrm{d}\theta$ $=2a^2\int_0^{\pi}\left(1-\cos 2\theta\right)\ \mathrm{d}\theta-a^2\int_0^{\pi}12\sin^2\theta\cos\theta\ \mathrm{d}\theta+a^2\int_0^{\pi}\left(1-\cos 4\theta\right)\ \mathrm{d}\theta$ $=2a^2\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\pi}-4a^2\left[\sin^3\theta\right]_0^{\pi}+a^2\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\pi}$ $=2a^2\pi-0+a^2\pi$ $=3a^2\pi$ $\therefore$ Total area $=2\times 3a^2\pi=6a^2\pi$ from (i) by symmetry.	

Question 11

No.	Suggested Solution	Remarks for Student
(i) **(a)**	$\dfrac{d\theta}{dt} = -k(\theta - 16), k > 0$ $\displaystyle\int \frac{1}{\theta - 16}\ d\theta = \int -k\ dt$ $\ln(\theta - 16) = -kt + c,$ where c is an arbitrary constant and $\theta > 16$ $\theta - 16 = Ae^{-kt},\ A = e^{c}$ $\theta = 16 + Ae^{-kt}$ $t = 0, \theta = 80 \Rightarrow A = 64$ $\therefore \theta = 16 + 64e^{-kt}$ $t = 30, \theta = 32 \Rightarrow 32 = 16 + 64e^{-30k}$ $e^{-30k} = \dfrac{1}{4}$ $e^{-k} = \left(\dfrac{1}{4}\right)^{\frac{1}{30}}$ $\theta = 16 + 64\left(e^{-k}\right)^{t} = 16 + 64\left(\dfrac{1}{4}\right)^{\frac{t}{30}}$	
(b)	$\theta = 16 + 64\left(\dfrac{1}{4}\right)^{\frac{45}{30}} = 24$	
(ii)	$\dfrac{dT}{dt} = \dfrac{\alpha}{T}, \alpha > 0$ $\displaystyle\int T\ dT = \int \alpha\ dt$ $\dfrac{1}{2}T^2 = \alpha t + b,$ where b is an arbitrary constant. $t = 0, T = 0 \Rightarrow b = 0$ $t = 60, T = 1 \Rightarrow \alpha = \dfrac{1}{120}$ $\therefore \dfrac{1}{2}T^2 = \dfrac{t}{120}$, that is, $T^2 = \dfrac{t}{60}$ We want $T \geq 3 \Rightarrow T^2 \geq 9$ $\dfrac{t}{60} \geq 9$ $t \geq 540$ Thus, it would be 540 min (9 hours) after freezing commences.	

Question 12

No.	Suggested Solution	Remarks for Student
(i)	$\overrightarrow{PQ}=\lambda\begin{pmatrix}-2\\-3\\-6\end{pmatrix}, \lambda>0$ and Q lies on $\underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=1$ $\overrightarrow{OQ}=\lambda\begin{pmatrix}-2\\-3\\-6\end{pmatrix}+\begin{pmatrix}2\\2\\4\end{pmatrix}$ $\overrightarrow{OQ}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=1\Rightarrow -2\lambda+2-3\lambda+2-6\lambda+4=1\Rightarrow \lambda=\frac{7}{11}$ Q is $\left(2-\frac{14}{11},2-\frac{21}{11},4-\frac{42}{11}\right)=\left(\frac{8}{11},\frac{1}{11},\frac{2}{11}\right)$ $\overrightarrow{RS}=\mu\begin{pmatrix}-2\\-3\\-6\end{pmatrix}, \mu>0$ and R lies on $\underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=-9$ $\overrightarrow{OR}=\begin{pmatrix}-5\\-6\\-7\end{pmatrix}-\mu\begin{pmatrix}-2\\-3\\-6\end{pmatrix}$ $\overrightarrow{OR}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=-9\Rightarrow -5+2\mu-6+3\mu-7+6\mu=-9\Rightarrow \mu=\frac{9}{11}$ R is $\left(-5+\frac{18}{11},-6+\frac{27}{11},-7+\frac{54}{11}\right)=\left(-\frac{37}{11},-\frac{39}{11},-\frac{23}{11}\right)$	
(ii)	$\cos\theta=\frac{\left\lvert\begin{pmatrix}-2\\-3\\-6\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right\rvert}{\sqrt{4+9+36}\sqrt{3}}=\frac{11}{7\sqrt{3}}$ $\overrightarrow{RQ}=\frac{1}{11}\begin{pmatrix}8\\1\\2\end{pmatrix}-\frac{1}{11}\begin{pmatrix}-37\\-39\\-23\end{pmatrix}=\frac{1}{11}\begin{pmatrix}45\\40\\25\end{pmatrix}$ $\cos\beta=\frac{\left\lvert\begin{pmatrix}45\\40\\25\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right\rvert}{\sqrt{45^2+40^2+25^2}\sqrt{3}}=\frac{110}{\sqrt{12750}}=\frac{22}{\sqrt{510}}$	

(iii)	$\dfrac{\left\lvert \overrightarrow{RQ}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right\rvert}{\sqrt{3}}=\dfrac{\frac{1}{11}(110)}{\sqrt{3}}=\dfrac{10}{\sqrt{3}}$	
(iv)	$k=\dfrac{\sin\theta}{\sin\beta}=\sqrt{\dfrac{1-\cos^2\theta}{1-\cos^2\beta}}=\sqrt{\dfrac{1-\left(\frac{11}{7\sqrt{3}}\right)^2}{1-\left(\frac{22}{\sqrt{510}}\right)^2}}=\dfrac{\sqrt{170}}{7}=1.86\ \text{(3 s.f.)}$	
(v)	Since $0^\circ<\theta<\beta<90^\circ$, we have $0<\sin\theta<\sin\beta<1$. Thus, $0<\dfrac{\sin\theta}{\sin\beta}<1$, that is, $0<k<1$.	