Tampines Meridian Junior College
2024 H2 Mathematics (9758)
Chapter 8 Applications of Differentiation
Learning Package

Resources

- ☐ Core Concept Notes
- ☐ Discussion Questions
- ☐ Extra Practice Questions

SLS Resources

- ☐ Recordings on Core Concepts
- ☐ Quick Concept Checks
- ☐ Exploration Activity: Maxima/Minima

Reflection or Summary Page

H2 Mathematics (9758)
Chapter 8 Applications of Differentiation
Core Concept Notes

Success Criteria:

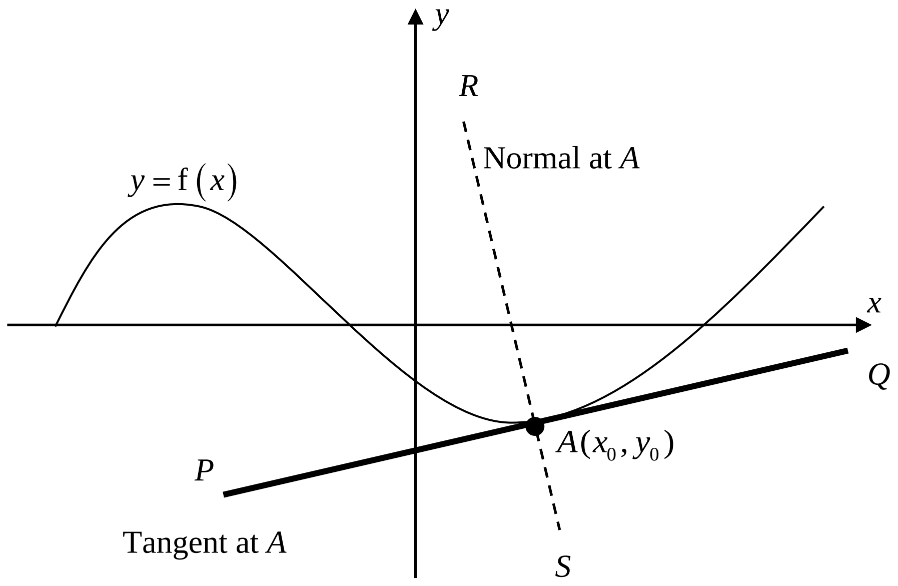
Surface Learning	**Deep Learning**	**Transfer Learning**
☐ Differentiate an equation of a curve defined implicitly to find gradient of tangents to a curve ☐ Differentiate parametric equations to find gradient of tangents to a curve ☐ Using relationship $\dfrac{-1}{\text{gradient of tangent}}$ to find gradient of normal to a curve	☐ Find equations of tangents and normal to a curve ☐ Use Chain Rule $\left(\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{\mathrm{d}y}{\mathrm{d}x}\times\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)$ to link up the derivatives and rate given	☐ Solve local maxima and minima problems ☐ Solve connected rates of change problems

§1 <u>Tangents and Normals</u>

Given $A\left(x_0, y_0\right)$ is a point on a curve, then

a) the line PQ touching the curve at A is called the **tangent** to the curve at A and

b) the line RS perpendicular to the tangent at A is called the **normal** to the curve at A.



Recall: Gradient of curve at A = Gradient of the tangent at A

$$= \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0}$$

The gradient of the tangent and normal at (x_0, y_0) are respectively given by:

- Gradient of tangent, $m = \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0}$
- Gradient of normal $= -\dfrac{1}{m}$

Recall: The product of gradients of perpendicular lines is -1

The equation of the tangent and normal at any point (x_0, y_0) on a curve $y = \mathrm{f}(x)$ is given by:

Equation of tangent	$y - y_0 = m(x - x_0)$	where $m = \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0}$
Equation of normal	$y - y_0 = -\dfrac{1}{m}(x - x_0)$	

Example 1

The equation of a curve is $y = x^2 + 5x - 1$. Find the equations of the

(i) tangent and

(ii) normal at the point (2, 13).

Solution:

(i) **<u>Non-GC approach:</u>**

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x + 5$

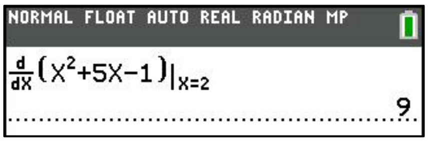
Gradient of Tangent = $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

$\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=2} = 9$

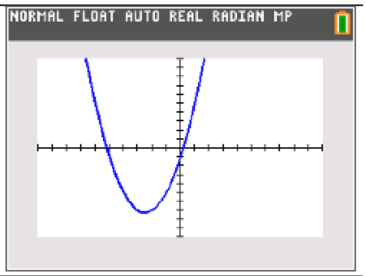
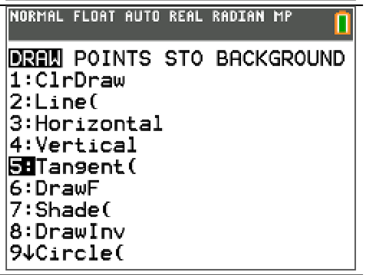
Equation of tangent: $y - 13 = 9(x - 2)$

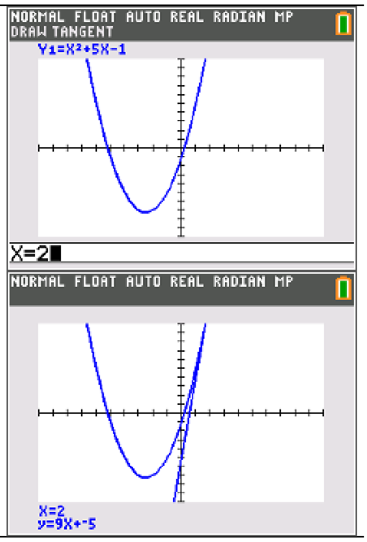
$y = 9x - 5$

Check using GC



<u>GC approach to get equation of tangent directly:</u>

<u>Step 1</u> Press [Y=]. Enter the equation of the graph $y = x^2 + 5x - 1$ under y_1 Press [GRAPH] to see the graph	
<u>Step 2</u> To draw/ obtain the tangent press [2nd][PRGM] and select **5: Tangent(**	

<u>**Step 3**</u> At the graph screen, key in the *x*-coordinate of the point required e.g. 2 and press $\boxed{\text{ENTER}}$	

Using GC,
Equation of tangent: $y = 9x - 5$

(ii) Gradient of normal at (2, 13) $= -\frac{1}{9}$

Equation of normal at (2, 13): $y - 13 = -\frac{1}{9}(x - 2)$

$$y = -\frac{1}{9}x + \frac{119}{9}$$

Gradient of Normal = $-\dfrac{1}{\frac{dy}{dx}}$

<u>**Note**</u>**:** Equation of normal cannot be found directly from GC

Conditions for using GC Approach:

1. There are no unknown constants in the equation of the curve.

2. *y* can be explicitly expressed in terms of *x*.

Example 2 (Curve is defined implicitly)
[2015 JJC Promo/5 (modified)]

A curve has equation $3x^2-4xy+2y^2-2=0$.

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3x-2y}{2x-2y}$. [3]

(ii) Find the equation of tangent to the curve at the point *P* with coordinates $(0,1)$. [2]

(iii) The normal to the curve at the point *P* meets the curve again at point *Q*. Find the area of triangle *OPQ*, where *O* is the origin. [4]

Solution:

(i)

$$3x^2-4xy+2y^2-2=0$$

Differentiate w.r.t. x

$$6x-4\left[x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right]+2\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)-0=0$$

$$3x-2x\frac{\mathrm{d}y}{\mathrm{d}x}-2y+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3x-2y}{2x-2y} \quad \text{(shown)}$$

(ii)

At $P(0,1)$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3(0)-2(1)}{2(0)-2(1)}=1$

Equation of tangent at *P*: $y-1=1(x-0)$

$$y=x+1$$

(iii)

At $P(0,1)$, gradient of normal $=-1$

Equation of normal at *P*: $y-1=-1(x-0)$

$$y=1-x$$

Since the normal to the curve at point *P* meets the curve again at point *Q*, substitute $y=1-x$ into equation of the curve:

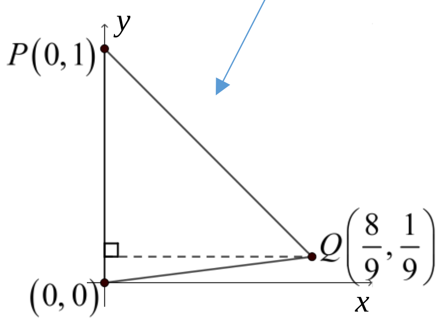
$$3x^2-4x(1-x)+2(1-x)^2-2=0$$

$$9x^2-8x=0$$

$$x(9x-8)=0$$

$$x=\frac{8}{9} \text{ or } x=0 \quad \text{(point } P\text{)}$$

Draw diagram to help with better visualisation.



At point *Q*, $x=\dfrac{8}{9}$, $y=\dfrac{1}{9}$.

Area of triangle *OPQ* $=\dfrac{1}{2}(1)\left(\dfrac{8}{9}\right)$

$$=\frac{4}{9}\text{ units}^2$$

Example 3 (Curve is defined parametrically)
[2015 PJC Prelim/2/4 (modified)]
The parametric equations of a curve are

$$x = t^2, \quad y = t^2 - t.$$

(i) The point P on the curve has parameter p. Show that the equation of the tangent at P is $2py = (2p-1)x - p^2$. [3]

(ii) The tangent at P meets the x- and y- axes at the points Q and R respectively. Find, in terms of p, the coordinates of Q and R. [2]

(iii) Find the equation of the tangent at the point $(4,6)$ and determine if this tangent meets the curve again. [4]

Solution:

(i)

$x = t^2 \qquad\qquad y = t^2 - t$

$\dfrac{dx}{dt} = 2t \qquad\qquad \dfrac{dy}{dt} = 2t - 1$

$\dfrac{dy}{dx} = \dfrac{2t-1}{2t}$

Chain Rule:
$\dfrac{dy}{dx} = \dfrac{dy}{dt} \text{ x } \dfrac{dt}{dx} = \dfrac{dy}{dt} \text{ x } \dfrac{1}{\frac{dx}{dt}}$

At the point P, $t = p$

Remember to change parameter t to p at point P to find gradient and equation of tangent/normal at point P.

$$y - (p^2 - p) = \frac{2p-1}{2p}(x - p^2)$$

$$2py - (2p^3 - 2p^2) = (2p-1)(x - p^2)$$

$$2py - 2p^3 + 2p^2 = (2p-1)x - 2p^3 + p^2$$

$$2py = (2p-1)x - p^2 \text{ (shown)}$$

Note: No t.

(ii)

$2py = (2p-1)x - p^2$

At Q, $y = 0$

$0 = (2p-1)x - p^2$

$x = \dfrac{p^2}{2p-1}$

Coordinates of Q are $\left(\dfrac{p^2}{2p-1}, 0\right)$

At R, $x = 0$

$2py = -p^2$

$y = -\dfrac{p}{2} \qquad , p \neq 0$

Coordinates of R are $\left(0, -\dfrac{p}{2}\right)$

(iii) At (4, 6),

$x = 4 = t^2$

$t = 2$ or $t = -2$

$\therefore y = (2)^2 - 2$ $\qquad$ $\therefore y = (-2)^2 - (-2)$

$= 2$ $\qquad$ $= 6$

(rejected, $(4, 2)$ are not the given coordinates)

Hence $t = p = -2$.

From **(i)**, equation of tangent: $2py = (2p-1)x - p^2$

At $t = p = -2$,

Equation of tangent: $2(-2)y = (2(-2)-1)x - (-2)^2$

$$4y = 5x + 4$$

To determine if the tangent meets the curve again,

Substitute $x = t^2$, $y = t^2 - t$ into $4y = 5x + 4$

Tangent intersects curve again?

$4(t^2 - t) = 5t^2 + 4$

$t^2 + 4t + 4 = 0$

$(t+2)^2 = 0$

$t = -2$

Since there is only one solution for *t*, which is the given point $(4,6)$, the tangent at $(4,6)$ does not meet the curve again.

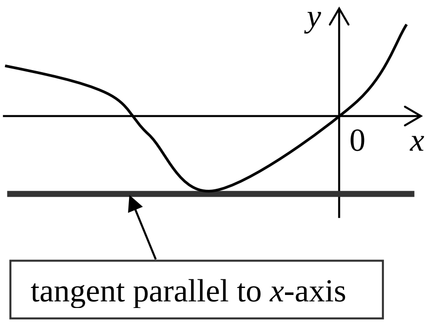
Discussion:

1. Write down the gradient of a tangent that is parallel to the *x*-axis. 0

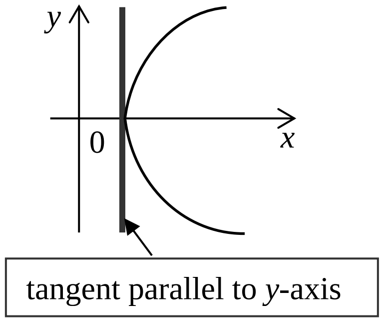
2. Given that the tangent is parallel to the *y*-axis, what can you say about the gradient of normal? 0

Note:

Tangent parallel to x-axis $\Rightarrow \dfrac{dy}{dx}=0$



Tangent parallel to y-axis $\Rightarrow \dfrac{dy}{dx}$ is undefined



Example 3 (Extension)

The parametric equations of a curve are

$$x=t^2,\quad y=t^2-t.$$

(iv) Find the equation of the tangent to the curve that is parallel to the x-axis.

(v) Find the equation of the tangent to the curve that is parallel to the y-axis.

Solution:

(iv) From **(i)**, $x=t^2,\quad y=t^2-t,\quad \dfrac{dy}{dx}=\dfrac{2t-1}{2t}$.

Since tangent is parallel to x-axis, $\dfrac{dy}{dx}=0$

$$\Rightarrow \frac{2t-1}{2t}=0$$

$$\Rightarrow 2t-1=0$$

For fraction $\dfrac{a}{b}=0$, $a=0, b\neq 0$.

$$\Rightarrow t=\frac{1}{2}.$$

When $t=\dfrac{1}{2}$, $y=\left(\dfrac{1}{2}\right)^2-\dfrac{1}{2}=-\dfrac{1}{4}$.

The equation of the tangent to the curve that is parallel to the x-axis is $y=-\dfrac{1}{4}$.

(v) Since tangent is parallel to y-axis, $\dfrac{dy}{dx}$ is undefined $\Rightarrow \dfrac{2t-1}{2t}$ is undefined

$$\Rightarrow 2t=0$$

$$\Rightarrow t=0.$$

For fraction $\dfrac{a}{b}$ to be undefined, $b=0, a\neq 0$.

When $t=0$, $x=0^2=0$.

The equation of the tangent to the curve that is parallel to the y-axis is $x=0$.

§2 Connected Rates of Change

Given a variable x, the **rate** of change of x with respect to time t refers to $\dfrac{dx}{dt}$.

In this section, we will use differentiation to study the rates of change of certain variables with respect to time and the relationships amongst them.

If two variables x and y are related, then their rates of change are also related.

Using the Chain Rule, we have

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}}$$

Steps to solve problems involving Rate of Change:

1. Identify the derivative to be found, say $\dfrac{dh}{dt}$.
2. Use Chain Rule to link up the derivatives to be found and the given rate, say $\dfrac{dV}{dt}$.

   Eg: $\dfrac{dh}{dt} = \dfrac{dh}{dV} \times \dfrac{dV}{dt}$.
3. Construct the equation(s) relating the variables of the derivative to be found, say an equation relating V and h, to find $\dfrac{dh}{dV}$.
4. Find the required derivative and answer the question.

Example 4 [2006(9233)/I/7]

A circular cone with semi-vertical angle 45° stands with its axis vertical and vertex pointing downwards. Water is poured in at a rate of $2\text{ cm}^3\text{s}^{-1}$.

(i) Find the rate of increase of the depth of water when the depth is 4 cm.

(ii) Find also the rate at which the area of the water surface is increasing at that instant.

Solution:

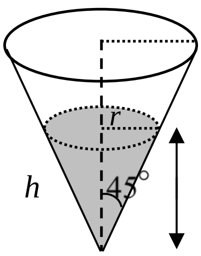
(i) Let the volume and depth of the water in the cone at the time t be V cm^3 and h cm respectively. Let the radius of the water surface at the time t be r cm.

Want to find: $\dfrac{dh}{dt}$

Given: $\dfrac{dV}{dt}$

Using Chain Rule, $\dfrac{dh}{dt} = \dfrac{dh}{dV} \times \dfrac{dV}{dt}$

Need to find: $\dfrac{dh}{dV}$ [Find an equation relating V and h.]



Note: $V=\frac{1}{3}\pi r^2 h$

And: $\tan 45^\circ=\frac{r}{h}\Rightarrow r=h$

Therefore, $V=\frac{1}{3}\pi h^3$

Express the quantity you need to differentiate in terms of a single variable.

$$\frac{\mathrm{d}V}{\mathrm{d}h}=\pi h^2$$

When h = 4 cm,

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{\frac{\mathrm{d}V}{\mathrm{d}h}}\times\frac{\mathrm{d}V}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{\pi(4)^2}(2)=\frac{1}{8\pi}$$

Note: $\frac{\mathrm{d}h}{\mathrm{d}V}=\frac{1}{\frac{\mathrm{d}V}{\mathrm{d}h}}$ (reciprocal relationship)

Thus, the depth of the water is increasing at the rate of $\frac{1}{8\pi}$ cm s^{-1} when h = 4 cm.

(ii) Let the area of the water surface at the time t be A cm^2.

Want to find: $\frac{\mathrm{d}A}{\mathrm{d}t}$

Given $\frac{\mathrm{d}V}{\mathrm{d}t}=2\text{ cm}^3\text{ s}^{-1}$ and $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{8\pi}$ when h = 4 cm

Using Chain Rule, $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}$

Need to find: $\frac{\mathrm{d}A}{\mathrm{d}h}$ [Find an equation relating A and h.]

$$A=\pi r^2=\pi h^2 \qquad (\text{Since } r=h)$$

$$\frac{\mathrm{d}A}{\mathrm{d}h}=2\pi h$$

When h = 4 cm,

$$\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}$$

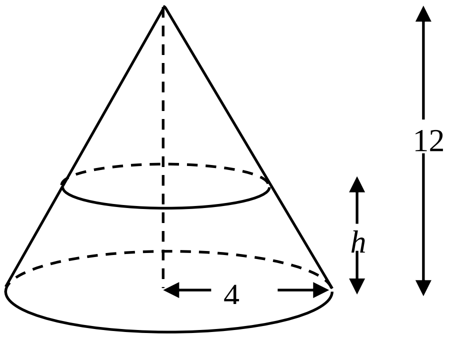
$$\frac{\mathrm{d}A}{\mathrm{d}t}=2\pi(4)\left(\frac{1}{8\pi}\right)=1$$

Rate at which Area is increasing $\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t}>0$ (positive)

The area of the water surface is increasing at the rate of $1\text{ cm}^2\text{ s}^{-1}$ when h = 4 cm.

Example 5

An upright cone, with a closed circular base, is shown in the diagram on the right. It has a circular base radius of 4 cm and height 12 cm and is initially full of water. Water is leaking from the circular base of the cone at a rate of 2π cm^3 min^{-1}. If h is the depth of water in the cone at time t minutes, show that the volume of water remaining, V cm^3, in the cone at time t minutes is given by $V = 64\pi - \frac{\pi}{27}(12-h)^3$.

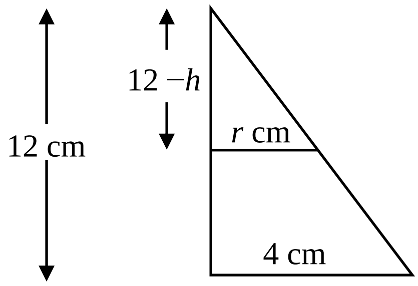


Hence find the rate of change of the depth of water when the depth of water is 6 cm. [4]

Solution:

Let the radius of the water surface at some time t minutes be r cm.

$$V = \frac{1}{3}\pi(4)^2(12) - \frac{1}{3}\pi(r)^2(12-h)$$



By similar triangles,

$$\frac{r}{4} = \frac{12-h}{12}$$

$$r = \frac{1}{3}(12-h)$$

$$V = \frac{1}{3}\pi(4)^2(12) - \frac{1}{3}\pi(r)^2(12-h)$$

$$V = 64\pi - \frac{1}{3}\pi\left[\frac{1}{3}(12-h)\right]^2(12-h)$$

$$V = 64\pi - \frac{\pi}{27}(12-h)^3 \qquad \text{(shown)}$$

Express the quantity you need to differentiate in terms of a single variable.

Want to find: $\frac{\mathrm{d}h}{\mathrm{d}t}$

Given: $\frac{\mathrm{d}V}{\mathrm{d}t}$

Using Chain Rule, $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$

Need to find: $\frac{\mathrm{d}h}{\mathrm{d}V}$ [Find an equation relating V and h.]

$$V = 64\pi - \frac{\pi}{27}(12-h)^3$$

Express the quantity you need to differentiate in terms of a single variable.

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{\pi}{9}(12-h)^2$$

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{\frac{\mathrm{d}V}{\mathrm{dh}}}\times\frac{\mathrm{d}V}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{\frac{\pi}{9}(12-h)^2}\times(-2\pi)$$

$$=\frac{-18}{(12-h)^2}\text{ cm/min}$$

Rate at which Depth is decreasing $\Rightarrow \frac{\mathrm{d}h}{\mathrm{d}t}<0$ (negative)

When $h=6$ cm, $\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{1}{2}$ cm/min

The depth of water is decreasing at the rate of $\frac{1}{2}$ cm/min when h = 6 cm.

Write positive value when answering the question for rate of decrease of depth.

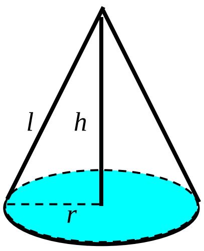
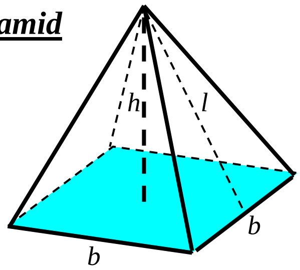
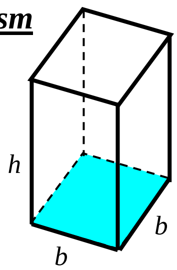
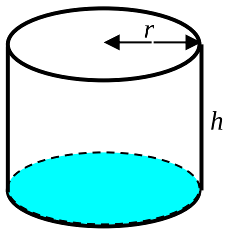
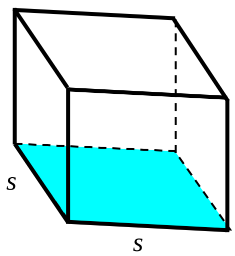
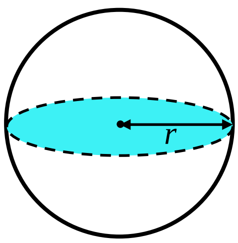
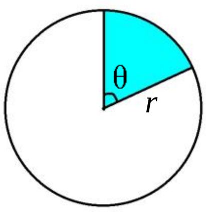
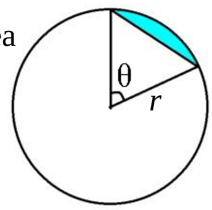
§3 Maxima/Minima

Steps to solve Maxima/Minima problems:

1. Identify the quantity (weight, volume, radius, length etc.) is to be maximised /minimised, say V.
2. Express quantity to be maximised/minimised in terms of a **single** variable, say x (if there are 2 variables, express 1 in terms of another). Drawing a clear diagram with all the given information included helps in visualization.
3. Using differentiation, solve $\frac{\mathrm{d}V}{\mathrm{d}x}=0$ to find the stationary point(s).
4. Use 1st or 2nd derivative test to determine/prove nature of the stationary point.
5. Answer the question.

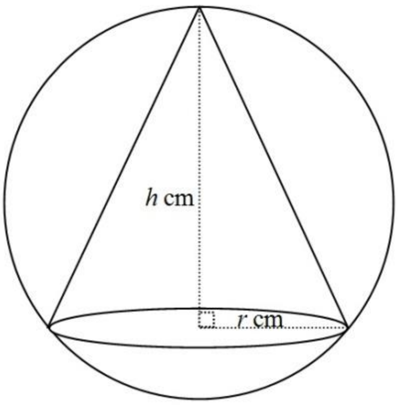
Useful Formulae

Note: V denotes volume and A denotes total surface area.

Cone $V=\frac{1}{3}\pi r^2 h$ $A=\pi rl+\pi r^2$ 	***Right Square Pyramid*** $V=\frac{1}{3}b^2 h$ $A=b^2+2bl$ 
Regular Square Prism $V=b^2 h$ $A=2b^2+4bh$ 	***Cylinder*** $V=\pi r^2 h$ $A=2\pi r^2+2\pi rh$ 
Cube $V=s^3$ $A=6s^2$ 	***Sphere*** $V=\frac{4}{3}\pi r^3$ $A=4\pi r^2$ 
Circle Sector (in radians) $A=\frac{1}{2}r^2\theta$ 	***Circle Segment (in radians)*** $A=$ sector area $-\ \Delta$ area $=\frac{1}{2}r^2\theta-\frac{1}{2}r^2\sin\theta$ $=\frac{1}{2}r^2(\theta-\sin\theta)$ 

Example 6

The following diagram shows the model of a display set that a museum is designing for a special exhibit. The exhibit is to be placed in the right circular cone of base area radius r and height h inscribed in a sphere of radius 6 metres. The height of the cone must be greater than the radius of the sphere.



(i) Show that $h = 6 + \left(36 - r^2\right)^{\frac{1}{2}}$.

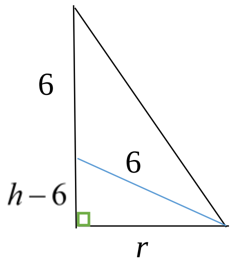
(ii) Given that $r = r_1$ is the value of r which gives the maximum volume of the cone, show that r_1 satisfies the equation $r^4 - 32r^2 = 0$. Hence find the maximum volume of the cone.

(iii) Sketch a graph showing the volume of the inscribed cone as its radius varies.

[Volume of Cone = $\frac{1}{3}\pi r^2 h$]

Solution:

(i) By Pythagoras Theorem,



$$6^2 = (h-6)^2 + r^2$$

$$h - 6 = \pm\sqrt{36 - r^2}$$

$$h = 6 + \left(36 - r^2\right)^{\frac{1}{2}} \text{ since } h > 6.$$

(ii) **Aim:** Maximise volume of cone

Volume of Cone, $V = \frac{1}{3}\pi r^2 h$

V is in terms of **2 variables** ***r*** and h.

$$= \frac{1}{3}\pi r^2\left(6 + \sqrt{36 - r^2}\right)$$

Express V in a **single variable**, in this case, V in terms of ***r***.

$$\frac{dV}{dr} = \frac{1}{3}\pi r^2\left[\frac{1}{2}\left(36 - r^2\right)^{-\frac{1}{2}}(-2r)\right] + \frac{2}{3}\pi r\left[6 + \left(36 - r^2\right)^{\frac{1}{2}}\right]$$

Differentiate V in terms of the **single variable**, in this case, find $\frac{dV}{dr}$.

At maximum volume, $\dfrac{\mathrm{d}V}{\mathrm{d}r}=0$.

$$\frac{1}{3}\pi r^2\left[\frac{1}{2}\left(36-r^2\right)^{-\frac{1}{2}}\left(-2r\right)\right]+\frac{2}{3}\pi r\left[6+\left(36-r^2\right)^{\frac{1}{2}}\right]=0$$

$$\left(36-r^2\right)^{-\frac{1}{2}}\left(-r^2\right)+2\left[6+\left(36-r^2\right)^{\frac{1}{2}}\right]=0$$

$$-r^2+12\left(36-r^2\right)^{\frac{1}{2}}+2\left(36-r^2\right)=0$$

$$12\left(36-r^2\right)^{\frac{1}{2}}+72-3r^2=0$$

$$4\left(36-r^2\right)^{\frac{1}{2}}=r^2-24$$

$$16\left(36-r^2\right)=\left(r^2-24\right)^2$$

$$576-16r^2=r^4-48r^2+576$$

$r^4-32r^2=0$ (shown)

$$r^2\left(r^2-32\right)=0$$

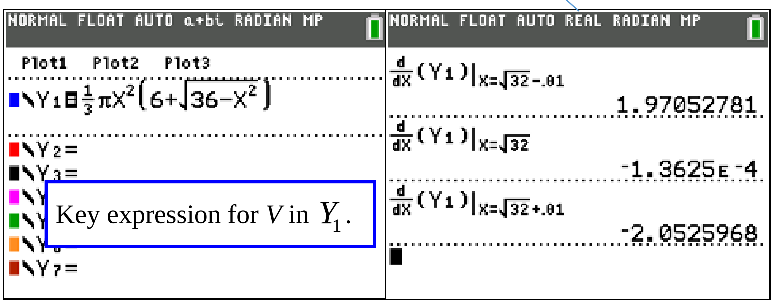
$r=0$ (rejected since $r>0$) or $\sqrt{32}$ or $-\sqrt{32}$ (rejected since $r>0$)

Remember to prove V is maximum when $r=\sqrt{32}$.

Using 1st Derivative Test,

r	$\sqrt{32}^-$	$\sqrt{32}$	$\sqrt{32}^+$
$\dfrac{\mathrm{d}V}{\mathrm{d}r}$	positive	0	negative

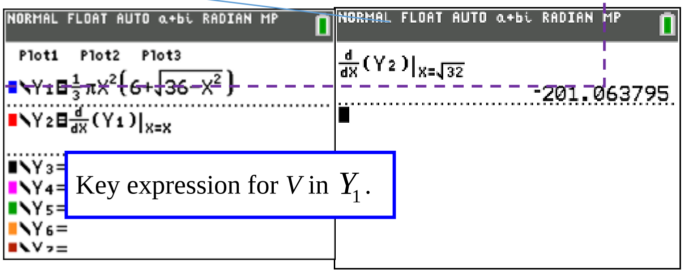
$\therefore V$ is maximum when $r=\sqrt{32}$.



Alternative method to prove that V is maximum: (using 2nd derivative test)

By GC, when $r=\sqrt{32}$, $\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=-201(3sf)<0$

V is maximum when $r=\sqrt{32}$.



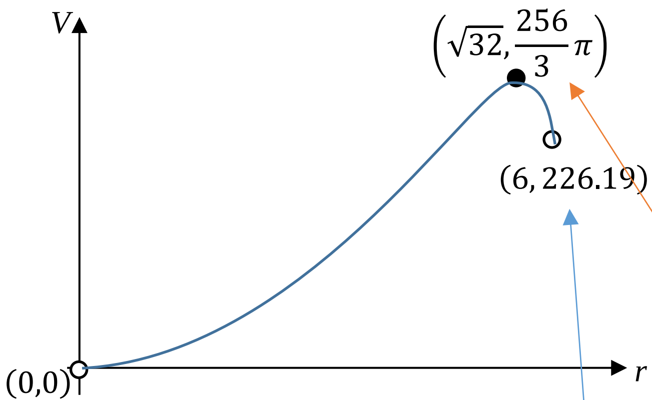
Key expression for V in Y_1.

Maximum volume $=\frac{1}{3}\pi\sqrt{32}^2\left(6+\sqrt{36-\sqrt{32}^2}\right)$

Answer the question.

$=\frac{256}{3}\pi\ m^3$

(iii)

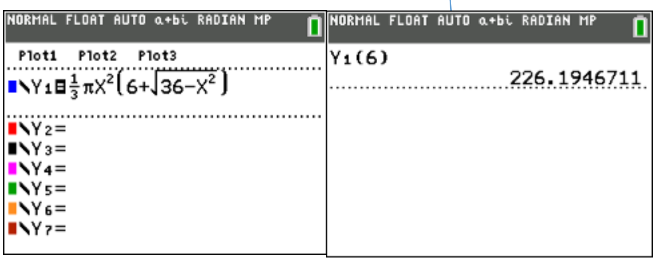


Note: In order for the right circular cone of base radius r and height h to be inscribed in a sphere of radius 6 metres, $\therefore\ 0<r<6$.

So graph exists for $0<r<6$ with end points clearly indicated.

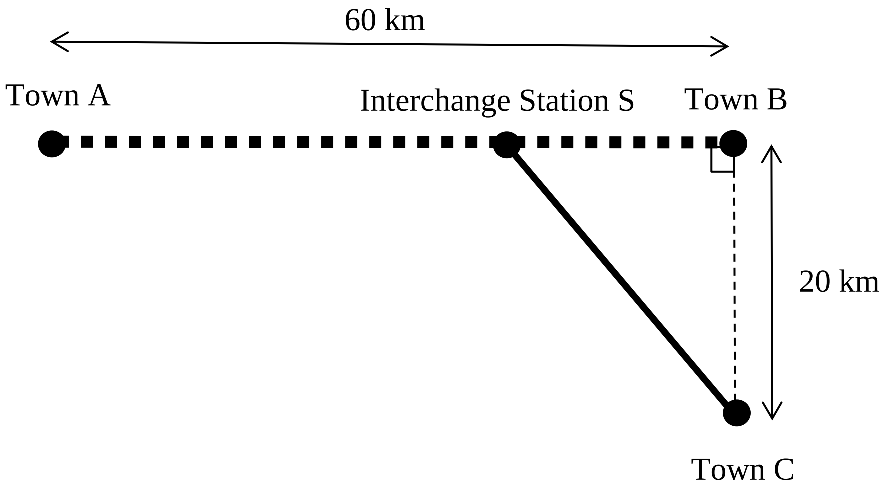
Note: From (ii), $r=\sqrt{32}$ gives Maximum $V=\frac{256}{3}\pi$

Indicate this on graph.



Example 7 [MI PU2 Promo 9758/2019/02/Q1] [Self-Reading]

Town A, Town B and Town C are located in Wakandi Country. The distance between Town A and Town B is 60 km and the distance between Town B and Town C is 20 km. A railroad connects Town A to Town B (see diagram).



A manufacturer plans to deliver a certain number of containers of its goods daily from Town A to Town C. To support this plan, the Wakandi government decides to build Interchange Station S and a road connecting this station to Town C (see diagram). Once the road is built, the goods manufacturer can deliver its containers from Town A to Town C by a combination of rail and road via Interchange Station S.

The cost to deliver the containers daily by rail is \$200 per km and the cost to deliver the containers daily by road is \$300 per km.

(i) Show that the daily total delivery cost, \$$T$, of the containers from Town A to Town C is given by

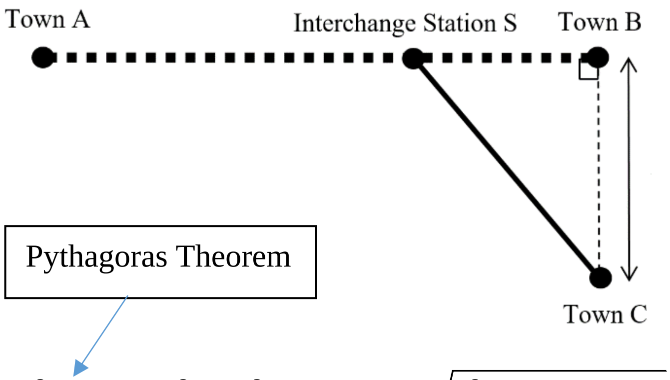
$$T = 200x + 300\sqrt{x^2 - 120x + 4000},$$

where x km is the distance between Town A and Interchange Station S. [2]

(ii) Hence use differentiation to find the value of x that gives a stationary value of T, giving your answer correct to 2 decimal places. Show that T is a minimum for this value of x. [4]

Solution:

(i)



$SC^2 = (60-x)^2 + 20^2 \quad \Rightarrow \quad SC = \sqrt{x^2 - 120x + 4000}$

$T = 200x + 300\sqrt{x^2 - 120x + 4000}$ (shown)

(ii)

$$T = 200x + 300\left[x^2 - 120x + 4000\right]^{\frac{1}{2}}$$

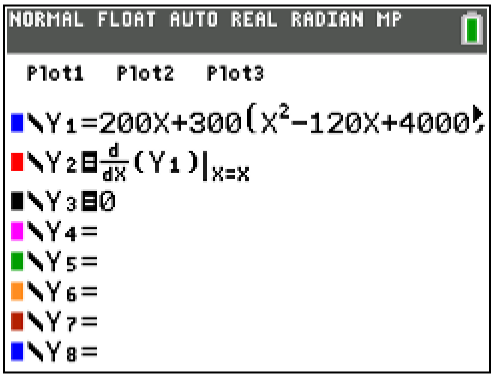
$$\frac{\mathrm{d}T}{\mathrm{d}x} = 200 + \frac{300}{2}\left[x^2 - 120x + 4000\right]^{-\frac{1}{2}}\left[2x - 120\right]$$

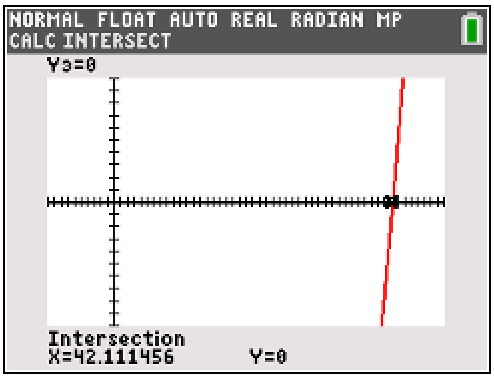
$$= 200 - \frac{300(60-x)}{\left[x^2 - 120x + 4000\right]^{\frac{1}{2}}}$$

For stationary values of T, $\frac{\mathrm{d}T}{\mathrm{d}x} = 0.$

$$200 - \frac{300(60-x)}{\left[x^2 - 120x + 4000\right]^{\frac{1}{2}}} = 0$$

Using GC, $x = 42.111 = 42.11$ (2 d.p)





Using the First Derivative Test:

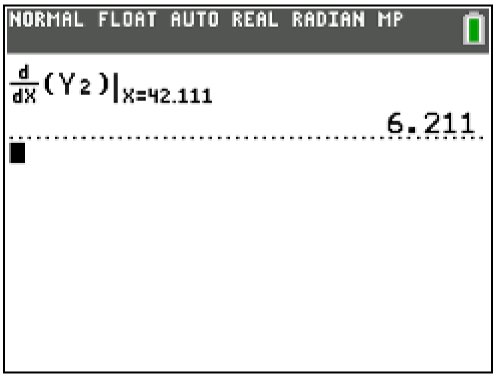
x	42.111^-	42.111	42.111^+
$\dfrac{\mathrm{d}T}{\mathrm{d}x}$	negative	0	positive
Shape	\	—	/

$\Rightarrow T$ is minimum when $x = 42.111$

Alternatively, using the second derivative test.

$$\left.\frac{\mathrm{d}^2T}{\mathrm{d}x^2}\right|_{x=42.111} = 6.211 > 0$$

$\Rightarrow T$ is minimum when $x = 42.111$

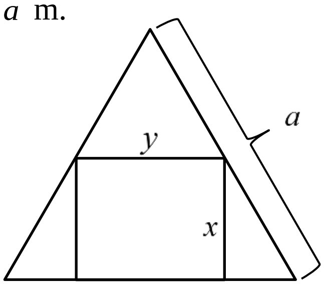


H2 Mathematics (9758)
Chapter 8 Applications of Differentiation
Discussion Questions

<u>Level 1</u>

1 2012 ACJC JC1 Promo/9 (modified)

The diagram below shows a rectangle of height x m and width y m inscribed in an equilateral triangle of side a m.



(i) Show that $x = \frac{\sqrt{3}}{2}(a - y)$.

(ii) Hence find the maximum area of the rectangle.

2 2018(9758)/I/7

A curve C has equation $\frac{x^2 - 4y^2}{x^2 + xy^2} = \frac{1}{2}$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x - y^2}{2xy + 16y}$. [3]

The points P and Q on C each have x-coordinate 1. The tangents to C at P and Q meet at the point N.

(ii) Find the exact coordinates of N. [6]

3 Specimen Paper (9758)/I/1

A circular ink-blot is expanding such that the rate of change of its diameter D with respect to time t is 0.25cm/s. Find the rate of change of both the circumference and the area of the circle with respect to t when the radius of the circle is 1.5cm. Give your answers correct to 4 decimal places.

Level 2

4 2009(9740)/II/1 (modified)

A curve C has parametric equations

$$x = t^2 + 4t, \quad y = t^3 + t^2.$$

(i) Find the coordinates of the x-intercepts and sketch the curve for $-2 \le t \le 1$. You do not need to label the coordinates of the turning point(s). [3]

The tangent to the curve at the point P where $t = 2$ is denoted by l.

(ii) Find the Cartesian equation of l. [3]

(iii) The tangent l meets C again at the point Q. Use a non-calculator method to find the coordinates of Q. [4]

(iv) Determine the acute angle between the tangent l and the line $y = x + 3$. [2]

5 2017(9758)/II/1 (modified)

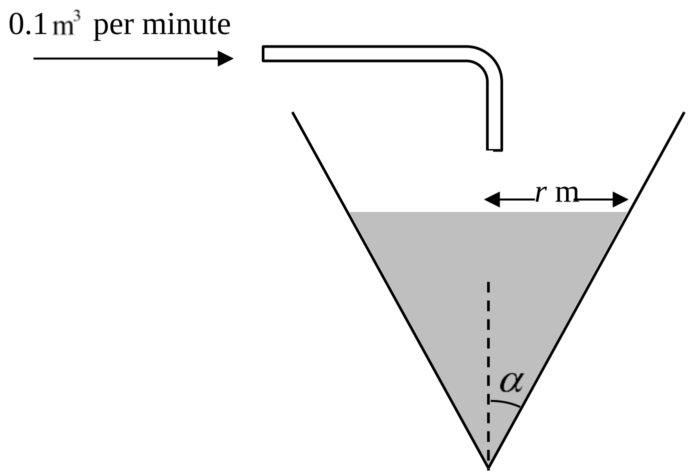
A curve C has parametric equations

$$x = \frac{3}{t}, \; y = 2t.$$

(i) The tangent at the point $P\left(\frac{3}{p}, 2p\right)$ on C meets the x-axis at D and the y-axis at E. The point F is the midpoint of DE. Find a cartesian equation of the curve traced by F as p varies. [5]

(ii) Show that the area of triangle ODE is independent of p, where O is the origin.

6 2016(9740)/II/1



Water is poured at a rate of $0.1\,\text{m}^3$ per minute into a container in the form of an open cone. The semi-vertical angle of the cone is α, where $\tan\alpha = 0.5$. At time t minutes after the start, the radius of the water surface is r m (see diagram). Find the rate of increase of the depth of water when the volume of water in the container is $3\,\text{m}^3$. [7]

[The volume of a cone of base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.]

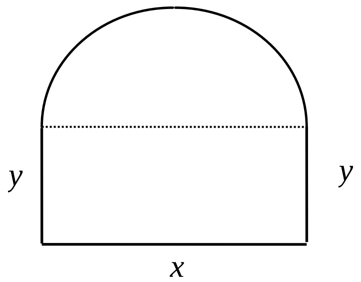
7 **1981(9205)/I/6**

Two variables u and v are connected by the relation $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$, where f is a constant.

Given that u and v both vary with time, t, find an equation connecting $\frac{\mathrm{d}u}{\mathrm{d}t},\frac{\mathrm{d}v}{\mathrm{d}t},u$ and v.

Given also that u is decreasing at a rate of 2 cm per second and that f = 10 cm, calculate the rate of increase of v when u = 50 cm.

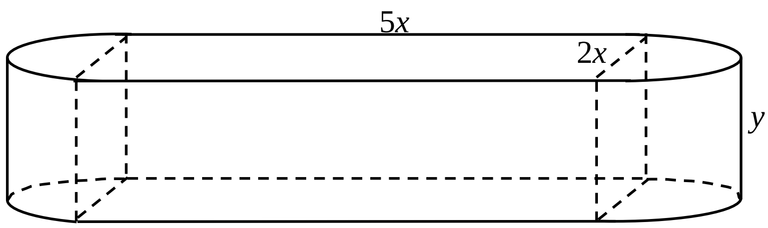
8 **2008(9740)/I/7**

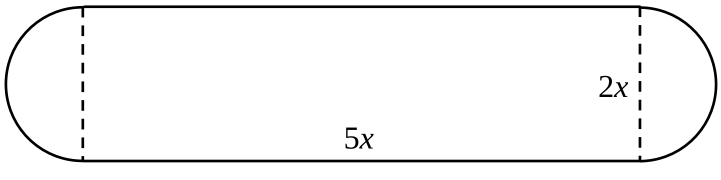


A new flower-bed is being designed for a large garden. The flower-bed will occupy a rectangle x m by y m together with a semicircle of diameter x m, as shown in the diagram. A low wall will be built around the flower-bed. The time needed to build the wall will be 3 hours per metre for the straight parts and 9 hours per metre for the semicircular part. Given that a total time of 180 hours is taken to build the wall, find, using differentiation, the values of x and y which give a flower-bed of maximum area. [10]

9 **RI JC1 Promo 9758/2019/Q6**

An **open** tin box of negligible thickness is to be made. The design of the box and its horizontal base are shown below.





The middle portion of the horizontal base of the box is a rectangle of length $5x$ cm and width $2x$ cm while the two ends are semicircles of radius x cm. The box has a depth of y cm and its volume is 800 cm^3.

Show that the total external surface area, A in cm^2, of the box is given by

$$A=(\pi+10)x^2+\frac{1600(\pi+5)}{(\pi+10)x}.$$

Use differentiation to find the value of x which minimizes A. [6]

Level 3

10 2012(9740)/I/11(modified)

A curve C has parametric equations

$$x=\theta-\sin\theta, \qquad y=1-\cos\theta$$

where $0\le\theta\le 2\pi$.

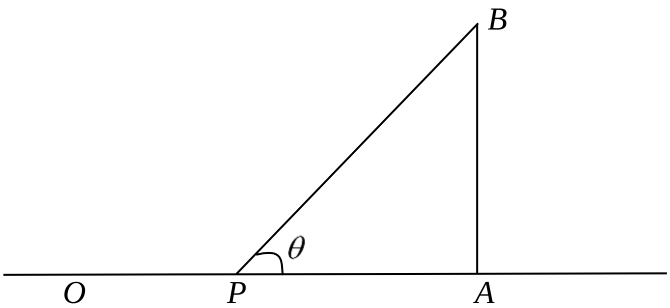
(i) Show that $\dfrac{dy}{dx}=\cot\dfrac{1}{2}\theta$ and find the gradient of C at the point where $\theta=\pi$. What can be said about the tangents to C as $\theta\to 0$ and $\theta\to 2\pi$? [5]

(ii) Sketch C, showing clearly the features of the curve at the points where $\theta=0$, π and 2π . [3]

(iii) A point P on C has parameter p, where $0<p<\dfrac{1}{2}\pi$. Show that the normal to C at P crosses the x-axis at the point with coordinates $(p,0)$. [5]

(iv) Given that θ is increasing at a rate of 2 radians per second, find the rate of change of $\dfrac{dy}{dx}$ at $\theta=\dfrac{\pi}{3}$. [3]

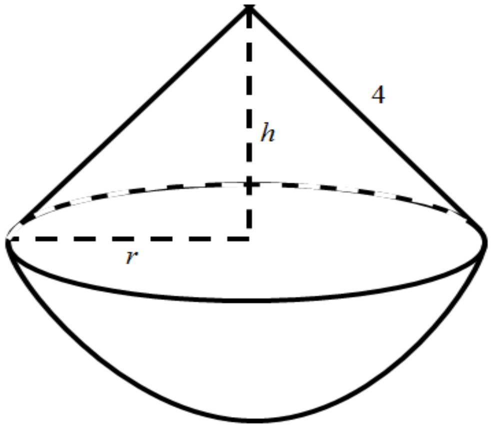
11 1987(9202)/I/17



In the diagram, O and A are fixed points 1000m apart on horizontal ground. The point B is vertically above A, and represents a balloon which is ascending at a steady rate of 2ms^{-1}. The balloon is being observed from a moving point P on the line OA. At time $t=0$, the balloon is at A and the observer is at O. The observation point P moves towards A with steady speed 6ms^{-1}. At time t, the angle APB is θ radians. Show that $\dfrac{d\theta}{dt}=\dfrac{500}{t^2+(500-3t)^2}$.

12 N2014/I/11

[It is given that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]



A toy manufacturer makes a toy which consists of a hemisphere of radius r cm joined to a circular cone of base radius r cm and height h cm (see diagram). The manufacturer determines that the length of the slant edge of the cone must be 4 cm and that the total volume of the toy, V cm^3, should be as large as possible.

(i) Find a formula for V in terms of r. Given that $r = r_1$ is the value of r which gives the maximum value of V, show that r_1 satisfies the equation $45r^4 - 768r^2 + 1024 = 0$. [6]

(ii) Find the two solutions to the equation in part **(i)** for which $r > 0$, giving your answers correct to 3 decimal places. [2]

(iii) Show that one of the solutions found in part **(ii)** does not give a stationary value of V. Hence write down the value of r_1 and find the corresponding value of h. [3]

(iv) Sketch the graph showing the volume of the toy as the radius of the hemisphere varies. [3]

13 2013(9740)/II/2

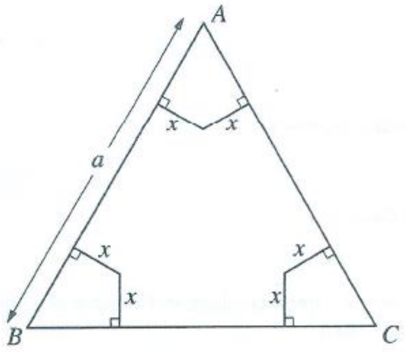


Fig. 1

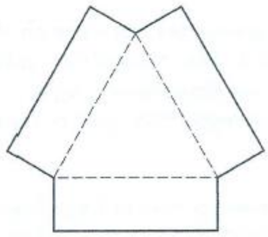


Fig. 2

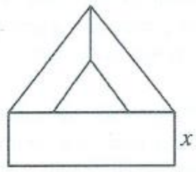


Fig. 3

Fig. 1 shows a piece of card, ABC, in the form of an equilateral triangle of side a. A kite shape is cut from each corner, to give the shape shown in Fig. 2. The remaining card shown in Fig. 2 is folded along the dotted lines, to form the open triangular prism of height x shown in Fig. 3.

(i) Show that the volume V of the prism is given by $V = \frac{1}{4}x\sqrt{3}\left(a - 2x\sqrt{3}\right)^2$. [3]

(ii) Use differentiation to find, in terms of a, the maximum value of V, proving that it is a maximum. [6]

Answer Key	
1(ii)	$\frac{\sqrt{3}}{8}a^2\ \text{m}^2$
2(ii)	$\left(-\frac{1}{17}, 0\right)$
3	0.7854cm/s; 1.1781cm^2/s
4(ii) **4(iii)**	$y = 2x - 12$ $(-3, -18)$
5(i)	$y = \frac{6}{x}$
6	0.0251 m per minute
7	$\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}t} = 0$ 0.125 cm/s
8	$x = 6.09$; $y = 12.6$
9	$x = 3.35$
10(iv)	-4
12(i) **12(ii)** **12(iii)**	$V = \frac{1}{3}\pi r^2 h + \frac{2}{3}\pi r^3$ $r = 3.951$ or $r = 1.207$ $r_1 = 3.951$; $h = 0.625$
13	$x = \frac{a\sqrt{3}}{6}$

H2 Mathematics (9758)
Chapter 8 Applications of Differentiation
Extra Practice Questions

1 **2018/MI Promo/1/5(a)**

Given a function $\mathrm{f}(x) = x^2 \mathrm{e}^{x^2}$, for $x \in \mathbb{R}$.

(i) By differentiation, find the range of values of x for which the function is increasing. [4]

(ii) Hence find the equation of the tangent to the curve, $\mathrm{f}(x) = x^2 \mathrm{e}^{x^2}$ for $x \in \mathbb{R}$, at the point where $x = 1$, giving your answer in terms of e. [2]

2 **2018/DHS Prelim/2/5(a)**

A curve has the equation $(x+y)^2 = 4\mathrm{e}^{xy}$.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [2]

(ii) Given that the curve cuts the positive y-axis at point A, find the equation of the tangent to the curve at A. [2]

(iii) The tangent to the curve at A meets the curve at another point B. Find the coordinates of B. [3]

3 **2018/HCI Prelim/1/6**

A curve C has parametric equations

$$x = 2\sin t, \qquad y = 1 + \cos t, \qquad 0 < t < \pi.$$

(i) Show that the equation of the tangent to C at the point $P(2\sin p, 1+\cos p)$ is $2y + x\tan p = 2(1+\sec p)$. [4]

(ii) The tangent at P meets the x-axis at the point A and the y-axis at the point B. The point M is the midpoint of AB. Find the Cartesian equation of the curve traced by M as p varies. [5]

4 **2018/MJC Promo/1/6**

A curve C has parametric equations

$$x = 2t + 1, \quad y = \frac{4}{t}.$$

(i) Show that the equation of the normal to C at the point M with coordinates $(3, 4)$ is $2y = x + 5$. [3]

(ii) The normal at M meets the curve again at the point N. Find the coordinates of the point N. [3]

(iii) The tangent at the point $P\left(2p+1, \frac{4}{p}\right)$ on C meets the x- and y- axes at the points Q and R respectively. Find the area of the triangle OQR in terms of p. [5]

5 **2018/CJC Promo/1/7**

A curve C has equation $kx^2+2xy-3y^2=5$ where k is a non-zero constant.

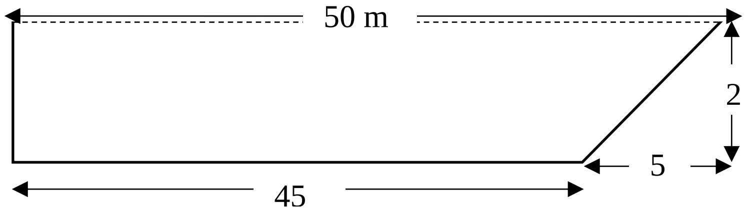
(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{kx+y}{3y-x}$. [2]

(ii) Find the range of values of k such that tangents to the curve C are parallel to the x–axis. [4]

(iii) For the case where $k=13$, a point $P\left(x,y\right)$ moves along the curve C in such a way that its x–coordinate is increasing at a constant rate of 5 units per second. Find the rate of change of its y–coordinate at the instant when $x=1$ and $y=2$. [2]

6 **2018/NJC Promo/1/12**

The diagram below (not drawn to scale) shows a cross-section of an empty swimming pool measuring 50 m long, 20 m wide and 2 m deep.



Suppose water is being pumped into the pool at a rate of 100 m^3 per min. How fast is the water level in the pool rising when the depth of water is 1.6 m? [4]

7 **2020/SAJC Promo/1/11**

In geometry, a spherical cap is the **smaller** part of a sphere when the sphere is cut horizontally as shown in Figure 1.

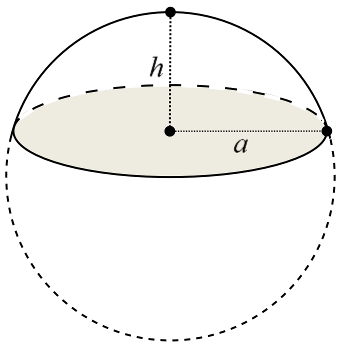


Figure 1

It is given that a spherical cap with height h and base radius of the cap a has curved surface area $\pi\left(a^2+h^2\right)$ and volume $\dfrac{1}{6}\pi h\left(3a^2+h^2\right)$.

A pharmaceutical company wants to manufacture a capsule shell to contain their new supplement as shown in Figure 2 below. The capsule shell consists of two identical spherical caps each joined to one end of a cylinder. The spherical caps have base radius r cm and height $\dfrac{1}{2}r$ cm while the cylinder has radius r cm and height h cm. The capsule shell has a fixed volume of v cm^3 and is assumed to have a negligible thickness.
[A cylinder with radius r and height h has curved surface area $2\pi rh$ and volume $\pi r^2 h$.]

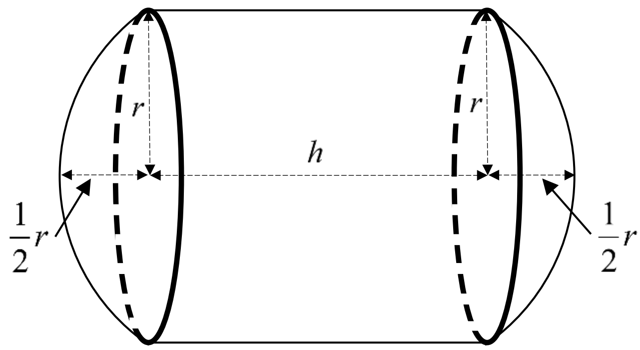


Figure 2

(i) Show that the external surface area of the capsule shell, $A\,\text{cm}^2$, is given by $A = \dfrac{17}{12}\pi r^2 + \dfrac{2v}{r}$. [4]

(ii) The pharmaceutical company wants to reduce the external surface area of the capsule shell in order to lower manufacturing cost. Using differentiation, find, in terms of v, the exact value of r that gives the minimum surface area of the capsule shell. [5]

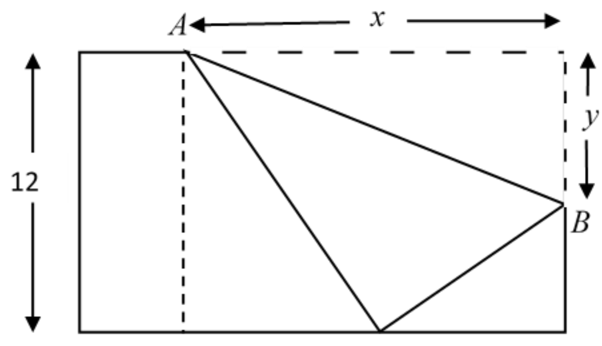
Given $v = 2$,

(iii) find the range of values of r such that $h > 0$; [1]

(iv) sketch the graph showing the surface area of the capsule as the radius varies. [3]

8 **2018/AJC Promo/1/9(b)**

A rectangular piece of paper is 24 cm long and 12 cm wide. The upper right-hand corner is folded down to reach the bottom edge of the paper at C as shown below.



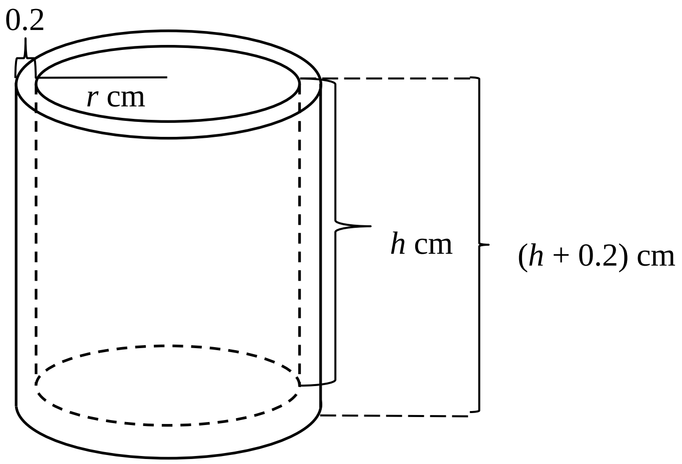
Show that $\triangle APC$ and $\triangle CQB$ are similar triangles and hence show that

$$x = \frac{\sqrt{6}y}{\sqrt{y-6}}.$$ [4]

By using differentiation, find the minimum length of the resulting crease AB. [5]

9 **2018/MI Promo/1/11**

The diagram shows an empty double-walled drinking glass. Both the inner and outer walls are made of ceramic glass in the shape of an open cylinder with a rim of 0.2cm, which is also made of ceramic glass and there is a vacuum in between the two walls. The inner cylindrical wall is of height h cm and radius r cm and the outer cylindrical wall is of height 0.2 cm *taller* than that of the inner wall. Its total inner capacity is 150π cm^3. It is also given that the ceramic glass is of negligible thickness.



(i) Show that the surface area of the rim is equal to $\frac{\pi}{25}(10r+1)$. [1]

(ii) Show the total surface area of a drinking glass A is given by

$$A = 2\pi\left[r^2 + \frac{3}{5}r + \frac{2}{25} + \frac{300}{r} + \frac{30}{r^2}\right].$$ [3]

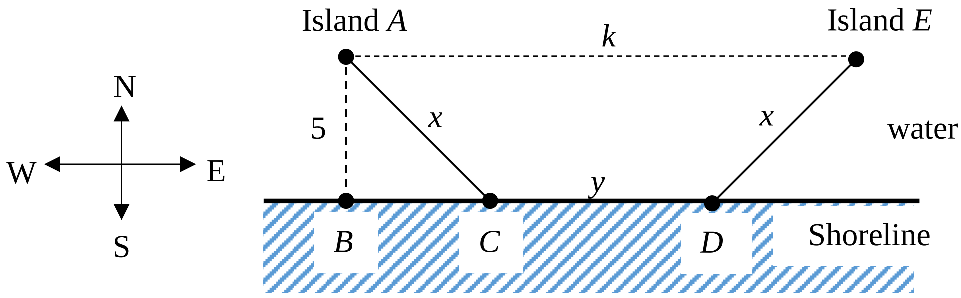
(iii) Use differentiation to find the minimum value of A. [4]

It is now given that $h = 2r$. One of such drinking glasses is filled to the brim with water and it is found to be leaking from the bottom of the glass. At time t seconds, the depth of water in the glass is x cm. It is assumed that the volume of water in the glass is decreasing at a constant rate of 0.5 cm^3 per second.

(i) Calculate the rate at which the depth of water is decreasing. [4]

10 2016/MJC Promo/1/10

Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that the birds travel at a slower speed over water than land because hot air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island A that is 5 km north of point B on a straight shoreline, flies to a point C on the shoreline which is due east of point B, and then flies east along the shoreline to a point D before flying to island E which is at a fixed distance k km due east of island A. It is known that island A and point C are x km apart, points C and D are y km apart and point D and island E are x km apart.



It is given that the bird travels at a speed of 65 km/h over water and travels at a speed of 90 km/h along the shoreline.

Using differentiation, determine the value of x such that the total time taken for the bird to fly from island A to island E via the route $ACDE$ is the minimum.

Hence determine the minimum value of k, to the nearest integer, in order for the bird to choose the route $ACDE$ over flying from island A to island E directly. [11]

$\left[\text{Time taken is defined by Time} = \dfrac{\text{Distance}}{\text{Speed}}\right]$

Answer Key

No	Year	JC/CI	Answers
1	**2018**	**MI**	(i) $x > 0$ (ii) $y = 4ex - 3e$
2	**2018**	**DHS**	(a)(i) $\dfrac{2ye^{xy} - y - x}{y + x - 2xe^{xy}}$ (a)(ii) $y = x + 2$ (a)(iii) $(-2, 0)$
3	**2018**	**HCI**	(ii) $y = \dfrac{x^2}{x^2 - 1}$
4	**2018**	**MJC**	(ii) $(-7, -1)$ (iii) $\left(\dfrac{4p+1}{p}\right)^2$
5	**2018**	**CJC**	(ii) $-\dfrac{1}{3} < k < 0$ (iii) 15 units per second
6	**2018**	**NJC**	$\dfrac{5}{49}$ m/min
7	**2020**	**SAJC**	(ii) A is minimum when $r = \sqrt[3]{\dfrac{12v}{17\pi}}$ (iii) $0 < r < 1.06$
8	**2018**	**AJC**	(a) $y = 6x - 32$; (4,-8) $L = \sqrt{243}(\text{since } L > 0) = 15.6\text{cm}$
9	**2018**	**MI**	(iii) 559 (iv) $0.00895\ \text{cms}^{-1}$
10	**2016**	**MJC**	x = 7.23, min k = 25