1 In 2005, Andrew bought a painting. In 2008, the value of the painting is estimated to be \$3878.58. The value $V(t)$, in dollars, of the painting t years after being purchased is given by the function $V(t) = 2400\,e^{kt}$, where k is a constant.

(a) What was the value of the painting after 10 years? [4]

When $t = 3$,

$$3878.58 = 2400e^{k(3)}$$

$$e^{3k} \approx 1.616075$$

$$3k \approx \ln 1.616075$$

$$k \approx 0.16000$$

When $t = 10$,

$$V(10) = 2400e^{0.16000(10)}$$

$$\approx 11887.29248$$

Value $\approx \$11887.29$ (2 d.p.)

Or 11887.27782

≈ 11887.28.

(b) If Andrew intended to sell the painting after its value had at least doubled, which is the earliest year that he can sell it? [2]

When $t = 0$

$$V(0) = 2400e^{k(0)}$$

$$= 2400$$

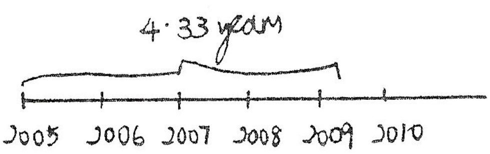
When $V(t) = 4800$

$$4800 = 2400e^{0.16000t}$$

$$2 = e^{0.16000t}$$

$$\ln 2 = 0.16000t$$

$$t \approx 4.33$$



$\therefore$ Earliest Year = 2009

2 (a) Prove that $3\cos^2 x + \sin^2 x = 2 + \cos 2x$. [3]

$LHS = 3\cos^2 x + \sin^2 x$

$= 3\cos^2 x + (1 - \cos^2 x)$

$= 2\cos^2 x + 1$

$= (\cos 2x + 1) + 1$

$= 2 + \cos 2x$

$= RHS$ (proved)

$2 + \cos 2x$

$= 2[\sin^2 x + \cos^2 x] + \cos^2 x - \sin^2 x$

$= 2\sin^2 x + 2\cos^2 x + \cos^2 x - \sin^2 x$

$= 3\cos^2 x + \sin^2 x$

$= LHS$ (proved)

$LHS = 3\cos^2 x + \sin^2 x$

$= 2\cos^2 x + \cos^2 x + \sin^2 x$

$= 2\cos^2 x + 1$

$= 2\cos^2 x - 1 + 1 + 1$

$= (2\cos^2 x - 1) + 2$

$= \cos 2x + 2$

Hence, using your result from part (a),

(b) (i) find the exact value of $3\cos^2 22.5° + \sin^2 22.5°$, [2]

$= 2 + \cos 2(22.5°)$

$= 2 + \cos 45°$

$= 2 + \frac{1}{\sqrt{2}}$

$\frac{4+\sqrt{2}}{2}$ or $2 + \frac{\sqrt{2}}{2}$

(ii) solve the equation $6\cos^2 x + 2\sin^2 x = 5$ for $0° \le x \le 360°$. [4]

$3\cos^2 x + \sin^2 x = \frac{5}{2}$

$2 + \cos 2x = \frac{5}{2}$

$\cos 2x = \frac{1}{2}$

Basic $\angle = 60°$

$2x = 60°, 300°, 420°, 660°$

$x = 30°, 150°, 210°, 330°$

3 It is given that $f(x) = x^3 + ax^2 + bx + 5$, where a and b are constants, has a factor of $x - 1$ and leaves a remainder of -1 when divided by $x - 2$.

(a) Find the values of a and b. [3]

$f(1) = 0$

$1^3 + a(1)^2 + b(1) + 5 = 0$

$6 + a + b = 0$ — ①

$f(2) = -1$

$2^3 + a(2)^2 + b(2) + 5 = -1$

$4a + 2b + 13 = -1$

$b = -7 - 2a$ — ②

Sub ② into ①

$6 + a + (-7 - 2a) = 0$

$-1 - a = 0$

$a = -1$

Sub $a = -1$ into ②,

$b = -7 - 2(-1)$

$= -5$

$\therefore a = -1,\ b = -5$

(b) Using these values of a and b, factorise $f(x)$ completely into three linear factors, using surds where necessary. [3]

$f(x) = x^3 - x^2 - 5x + 5$

$= (x - 1)(x^2 + bx - 5)$

Comparing x^2-terms,

$-x^2 = bx^2 - x^2$

$b = 0$

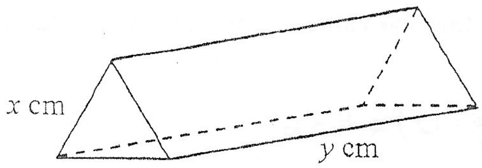
$\therefore f(x) = (x - 1)(x^2 - 5)$

$= (x - 1)(x + \sqrt{5})(x - \sqrt{5})$

OR Long Division

$$\begin{array}{r|l} & x^2 \quad\quad -5 \\ \hline x - 1 & x^3 - x^2 - 5x + 5 \\ & -(x^3 - x^2) \\ \hline & \quad\quad -5x + 5 \\ & \quad\quad -(-5x + 5) \\ \hline & \quad\quad\quad\quad 0 \end{array}$$

4



A miniature model in the shape of a triangular prism is shown above. A piece of wire, 36 cm in length, runs through all the edges of this triangular prism.

The cross-section of the prism is an equilateral triangle of side x cm and the length of the prism is y cm.

(i) Show that the total surface area A cm^2, of the prism is given by

$$A = \frac{\sqrt{3}-12}{2}x^2 + 36x\ .$$

[4]

$A = 2\left[\frac{1}{2}(x)(x)\sin 60^\circ\right] + 3xy$ ✓M1

$= x^2\left(\frac{\sqrt{3}}{2}\right) + 3xy$ — ①

$6x + 3y = 36$

$2x + y = 12$

$y = 12 - 2x$ — ②

Sub ② into ①,

$A = x^2\left(\frac{\sqrt{3}}{2}\right) + 3x(12 - 2x)$

$= \frac{\sqrt{3}}{2}x^2 + 36x - 6x^2$

$= \frac{\sqrt{3}}{2}x^2 - \frac{12}{2}x^2 + 36x$

$= \frac{\sqrt{3}-12}{2}x^2 + 36x$ (shown)

$\left(\frac{x}{2}\right)^2 + h^2 = x^2$

$\frac{x^2}{4} + h^2 = x^2$

$h^2 = \frac{3}{4}x^2$

$h = \frac{\sqrt{3}}{2}x$

Area of Δ = $\frac{\sqrt{3}}{2}x \times x = \frac{\sqrt{3}}{2}x^2$

(ii) Given that x can vary, find the value of x which gives a stationary value of A. [3]

$$\frac{dA}{dx} = (\sqrt{3}-12)x + 36$$

At stationary value of A, $\frac{dA}{dx} = 0$

$$(\sqrt{3}-12)x + 36 = 0$$

$$x = \frac{36}{12-\sqrt{3}}$$

$$\approx 3.5060$$

$$= 3.51 \text{ (3 s.f.)}$$

(iii) Justify if A is a maximum or a minimum for this value of x and find this stationary value of A. [3]

$$\frac{d^2A}{dx^2} = \sqrt{3} - 12$$

$$\approx -10.267 < 0 \text{ (maximum)}$$

Since $\frac{d^2A}{dx^2} < 0$, A is a maximum

When $x = 3.5060$,

$$A = \frac{\sqrt{3}-12}{2}(3.5060)^2 + 36(3.5060)$$

$$\approx 63.1089$$

$$= 63.1 \text{ (3s.f.)}$$

Value x	3.41	3.51	3.61
$\frac{dy}{dx}$	>0	0	<0
shape	/	—	\

A is a max.

5 (a) Without using a calculator, find the values of the integers a, b and c for which the solution to the equation $5x - \sqrt{3} = 4 + \sqrt{3}\,x$ is $\frac{a+b\sqrt{3}}{c}$. [4]

$$5x - \sqrt{3}x = 4 + \sqrt{3}$$

$$x(5 - \sqrt{3}) = 4 + \sqrt{3}$$

$$x = \frac{4+\sqrt{3}}{5-\sqrt{3}} \times \frac{5+\sqrt{3}}{5+\sqrt{3}}$$

$$= \frac{20 + 4\sqrt{3} + 5\sqrt{3} + 3}{5^2 - 3}$$

$$= \frac{23 + 9\sqrt{3}}{22}$$

$\therefore$ $a = 23$, $b = 9$, $c = 22$ $\leftarrow$ A few students did not specify the values of a, b and c.

(b) It is given that $\sin A = \frac{3}{5}$ where A is acute, $\tan B = \frac{5}{12}$ and that A and B are in different quadrants. Evaluate, without using a calculator, the values of

(i) $\sin\left(\frac{\pi}{2} - B\right)$, [2]

Many forgot about $\sin\left(\frac{\pi}{2}-B\right) = \cos B$

$= \cos B$

$= -\frac{12}{13}$

Common Mistake

① $\cos B \neq \frac{12}{13}$

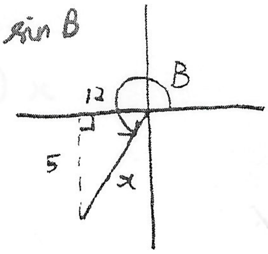
② $\sin\left(\frac{\pi}{2}-B\right) \neq \sin\frac{\pi}{2} - \sin B$

OR (Many used this)

$= \sin\frac{\pi}{2}\cos B - \cos\frac{\pi}{2}\sin B$

$= (1)\cos B - 0$

$= -\frac{12}{13}$



$x = \sqrt{12^2+5^2}$

$= 13$

(ii) $\tan 2A$. [2]

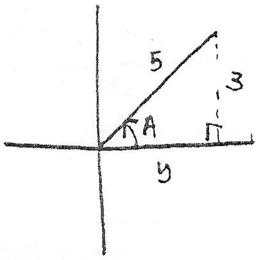
$= \frac{2\tan A}{1-\tan^2 A}$

$= \frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2}$

$= \frac{3}{2} \div \left(1 - \frac{9}{16}\right)$

$= \frac{3}{2} \times \frac{16}{7}$

$= \frac{24}{7}$



$y = \sqrt{5^2-3^2}$

$= 4$

Common Mistake

$\tan A = -\frac{3}{4}$

6 (a) Solve the equation $6^z(6^z - 1) = 12$. [4]

$6^{2z} - 6^z - 12 = 0$

Let $y = 6^z$

$y^2 - y - 12 = 0$

$(y-4)(y+3) = 0$

$y = 4$ or $y = -3$

$6^z = 4$ $\quad$ $6^z = -3$ (N/A)

$z \ln 6 = \ln 4$

$z \approx 0.7737$

$= 0.774$ (3s.f.) $\quad$ OR $z = \frac{\lg 4}{\lg 6}$

Common Mistake

① $6^z(6^z) \neq 36^{2z}$

② $6^{2z} - 6^z = 12$

$\lg 6^{2z} - \lg 6^z = \lg 12$

(b) A curve has the equation $y = xe^{3x}$.

(i) Find $\frac{dy}{dx}$. [2]

$\frac{dy}{dx} = (1)e^{3x} + x(3e^{3x})$

$= e^{3x}(1+3x)$ $\quad$ or $\quad$ $e^{3x} + 3xe^{3x}$

(ii) Find the value of k for which $\frac{d^2y}{dx^2} = ke^{3x}(2+3x)$. [3]

$\frac{d^2y}{dx^2} = 3e^{3x}(1+3x) + (3)(e^{3x})$ ✓ m1

← Many did not differentiate completely.

$= 3e^{3x} + 9xe^{3x} + 3e^{3x}$

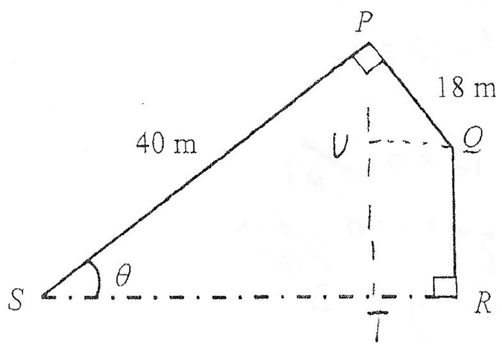
$= 6e^{3x} + 9xe^{3x}$

$= 3e^{3x}(2+3x)$

$\therefore k = 3$

Some students did not specify $k = 3$.

7



The diagram shows a play area for babies that is surrounded by partitions at PQ, QR and PS, where $PQ = 18$ m, $PS = 40$ m, angle SPQ = angle $QRS = 90°$ and the acute angle $PSR = \theta$ can vary.

(i) Show that L m, the length of the partitions can be expressed as $L = 58 + 40\sin\theta - 18\cos\theta$. [3]

$\sin\theta = \frac{PT}{40}$

$PT = 40\sin\theta$

$\angle SPT = 90° - \theta$

$\angle QPU = \theta$

$\cos\theta = \frac{PU}{18}$

$PU = 18\cos\theta$

Need to show
$\sin\theta = ...$
$\cos\theta = ...$

$QR = PT - PU$

$= 40\sin\theta - 18\cos\theta$

$\therefore L = 40 + 18 + 40\sin\theta - 18\cos\theta$

$= 58 + 40\sin\theta - 18\cos\theta$

Need to show 40+18 somewhere

Some students couldn't identify the right-angled Δs to use

(ii) Express L in the form $58 + R\sin(\theta - \alpha)$, where $R > 0$ and $0° \le \alpha \le 90°$. [4]

$$L = 58 + 40\sin\theta - 18\cos\theta$$
$$= 58 + R\sin(\theta - \alpha)$$
$$= 58 + R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

$\therefore$ $R\sin\alpha = 18$ — ①

$R\cos\alpha = 40$ — ②

Some did not write eqⁿ ① & ②.

$\frac{①}{②}$: $\frac{R\sin\alpha}{R\cos\alpha} = \frac{18}{40}$

$\tan\alpha = \frac{9}{20}$

$\alpha \approx 24.227°$

Remind to truncate (24.227°) not round off (24.228°)

①² + ②² : $R^2\sin^2\alpha + R^2\cos^2\alpha = 18^2 + 40^2$

$R = \sqrt{18^2 + 40^2}$

$= 2\sqrt{481}$

Some did not write degree.

$\therefore$ $L = 58 + 2\sqrt{481}\sin(\theta - 24.2°)$ (1 d.p.)

(iii) Given that the exact length of all the partitions is 74 m, find the value of θ. [2]

When $L = 74$

$74 = 58 + 2\sqrt{481}\sin(\theta - 24.227°)$

$16 = 2\sqrt{481}\sin(\theta - 24.227°)$

$\sin(\theta - 24.227°) \approx 0.3647$

Basic $\angle \approx 21.3933°$

$\theta - 24.227° = 21.3933°$

$\theta \approx 45.6203°$

$= 45.6°$ (1 d.p.)

8 The equation of a curve is $y = x\sqrt{x+1}$.

(a) Show that $\frac{d}{dx}(x\sqrt{x+1}) = \frac{3x+2}{2\sqrt{x+1}}$. [4]

$y = x(x+1)^{\frac{1}{2}}$

$\frac{dy}{dx} = (1)(x+1)^{\frac{1}{2}} + \frac{1}{2}(x+1)^{-\frac{1}{2}}(1)(x)$

$= \sqrt{x+1} + \frac{x}{2\sqrt{x+1}}$

$= \frac{2(x+1)+x}{2\sqrt{x+1}}$ Many skipped this step

$= \frac{3x+2}{2\sqrt{x+1}}$ (shown)

Common Mistake

$= \frac{x+1+\frac{1}{2}x}{\sqrt{x+1}}$

How

$= \frac{3x+2}{2\sqrt{x+2}}$

(b) A particle moves along the curve such that the x-coordinate is moving at a constant rate of 0.2 units per second. Find the rate of change of the y-coordinate at the point where $x = 5$. [2]

$\frac{dx}{dt} = 0.2$ units /s

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

When $x = 5$,

$\frac{dy}{dt} = \frac{3(5)+2}{2\sqrt{5+1}}(0.2)$

≈ 0.6940

$= 0.694$ units /s OR $\frac{17}{10\sqrt{6}}$ units /s OR $\frac{17\sqrt{6}}{60}$ units/s

Some did not write units /s

(c) Using your result from **part (a)**, evaluate $\int_{-1}^{3} \frac{x}{2\sqrt{x+1}}\,dx$. [4]

$$\int_{-1}^{3} \frac{3x+2}{2\sqrt{x+1}}\,dx = \left[x\sqrt{x+1}\right]_{-1}^{3}$$

$$\int_{-1}^{3} \frac{3x}{2\sqrt{x+1}}\,dx + \int_{-1}^{3} (x+1)^{-\frac{1}{2}}\,dx = \left[x\sqrt{x+1}\right]_{-1}^{3}$$

$$\int_{-1}^{3} \frac{3x}{2\sqrt{x+1}}\,dx + \left[2(x+1)^{\frac{1}{2}}\right]_{-1}^{3} = \left[x\sqrt{x+1}\right]_{-1}^{3}$$

$$\int_{-1}^{3} \frac{3x}{2\sqrt{x+1}}\,dx = \left[x\sqrt{x+1} - 2\sqrt{x+1}\right]_{-1}^{3}$$

$$= \left(3\sqrt{3+1} - 2\sqrt{3+1}\right) - \left(1\sqrt{-1+1} - 2\sqrt{-1+1}\right)$$

$$= (6-4) - 0$$

$$\int_{-1}^{3} \frac{x}{2\sqrt{x+1}}\,dx = \frac{1}{3}(2)$$

$$= \frac{2}{3} \quad \text{or} \quad \approx 0.66667$$

$$= 0.667 \text{ (3s.f.)}$$

Common Mistake

* Many wrote $\int \frac{1}{\sqrt{x+1}} = \ln\sqrt{x+1} + c$

* $\int (x+1)^{-\frac{1}{2}}\,dx \neq \frac{1}{2}(x+1) + c$

Alternative Mtd

$$\int_{-1}^{3} \frac{3x+2}{2\sqrt{x+1}}\,dx = \int_{-1}^{3} \frac{2x+2}{2\sqrt{x+1}}\,dx + \int_{-1}^{3} \frac{x}{2\sqrt{x+1}}\,dx$$

$$\left[x\sqrt{x+1}\right]_{-1}^{3} = \left[\frac{2}{3}\sqrt{(x+1)^3}\right]_{-1}^{3} + \int_{-1}^{3} \frac{x}{2\sqrt{x+1}}\,dx$$

$$\int_{-1}^{3} \frac{x}{2\sqrt{x+1}}\,dx = \left[3\sqrt{3+1} - (-1)\sqrt{-1+1}\right] - \left[\frac{2}{3}\sqrt{(3+1)^3} - \frac{2}{3}\sqrt{(-1+1)^3}\right]$$

$$= 6 - \frac{16}{3}$$

$$= \frac{2}{3}$$

9 The equation of a circle with centre C is $x^2 + y^2 - 8x - 2y - 152 = 0$.

(i) Find the coordinates of C and the radius of the circle. [4]

$x^2 - 8x + y^2 - 2y - 152 = 0$

$(x-4)^2 - 4^2 + (y-1)^2 - 1^2 - 152 = 0$

$(x-4)^2 + (y-1)^2 = 169$

$\therefore$ $C = (4, 1)$

Radius $= \sqrt{169}$

$= 13$ units

Method 2

$2f = -8$, $2g = -2$

$f = -4$, $g = -1$

Centre $= (-f, -g)$

$= (4, 1)$

Radius $= \sqrt{(-g)^2 + (-f)^2 - c}$

$= \sqrt{(-1)^2 + (-4)^2 - (-152)}$

$= \sqrt{169}$

$= 13$ units

Mtd 3 → Sub $x = 9$ and show $y = 13$ w/ conclusion

(ii) Show that the point $P(9, 13)$ lies on the circle. [1]

Method 1

this is a better method!

Length of $CP = \sqrt{(9-4)^2 + (13-1)^2}$

$= \sqrt{25 + 144}$

$= 13$ units

Since length of CP = radius of circle, P lies on the circle.

Method 2

$x^2 + y^2 - 8x - 2y - 152 = 0$ — ①

Sub $(9, 13)$ into ①,

$9^2 + 13^2 - 8(9) - 2(13) - 152 =$

$0 = 0$

Need this explanation in words to be correct to get mark

Since $(9, 13)$ satisfies the equation of the circle, the point lies on the circle.

Common Mistake

1) Not because eqn $= 0$, but $(9,13)$ satisfies the eqn

(iii) A line, l is parallel to the tangent to the circle at P. Q is a point on line l such that P is the midpoint of the line QC.

(a) Find the equation of the line l. [5]

Gradient of CP $= \frac{13-1}{9-4}$

$= \frac{12}{5}$

Common Mistake

① Students find eq^n of normal/tangent at P instead of eq^n of line, l

Gradient of line $l = -\frac{5}{12}$

Let coordinates of Q $= (x, y)$

$\left(\frac{x+4}{2}, \frac{y+1}{2}\right) = (9, 13)$

$\frac{x+4}{2} = 9$ $\quad\quad$ $\frac{y+1}{2} = 13$

$x = 14$ $\quad\quad$ $y = 25$

$\therefore$ Eq^n of l : $y - 25 = -\frac{5}{12}(x-14)$

$y = -\frac{5}{12}x + \frac{185}{6}$

OR

$12y = -5x + 370$

Common Mistake

① line l, // to tangent does not mean line l will not intersect circle

② line $l \neq$ tang to circl

(b) Explain if this line will intersect the circle. [1]

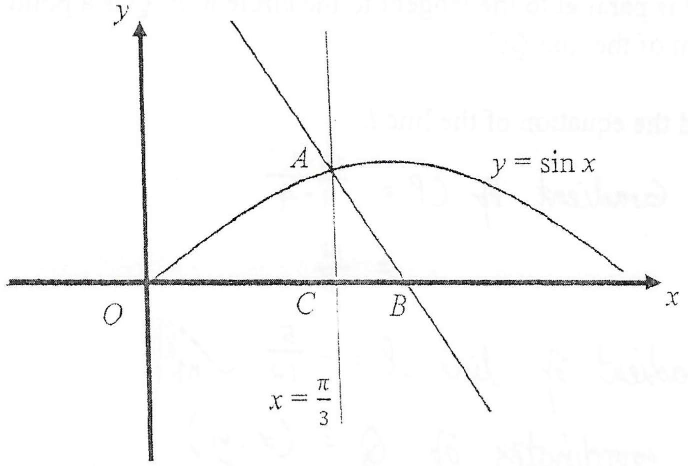
Mtd ① : Since line l is parallel to tangent at P and P is midpoint of QC,

$\therefore$ QC > radius of circle

Hence line l will not intersect the circle.

Mtd ② : Some showed $D < 0$, $\therefore$ l will not intersect the circle.

10



The diagram shows part of the curve of $y = \sin x$.

A is a point on the curve that lies on the line $x = \dfrac{\pi}{3}$. The line AC is perpendicular to the x-axis.

(i) If the normal of the curve at point A meets the x-axis at B, find the exact coordinates of A and B. [6]

$y = \sin x$

When $x = \frac{\pi}{3}$

$y = \sin \frac{\pi}{3}$

$= \frac{\sqrt{3}}{2}$

$\therefore$ Coordinates of $A = \left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$

$\frac{dy}{dx} = \cos x$

When $x = \frac{\pi}{3}$,

$\frac{dy}{dx} = \cos \frac{\pi}{3}$

$= \frac{1}{2}$

$\therefore$ Eq'n of AB: $y - \frac{\sqrt{3}}{2} = -2\left(x - \frac{\pi}{3}\right)$

When $y = 0$,

$0 - \frac{\sqrt{3}}{2} = -2\left(x - \frac{\pi}{3}\right)$

$\frac{\sqrt{3}}{4} = x - \frac{\pi}{3}$

$x = \frac{\sqrt{3}}{4} + \frac{\pi}{3}$

$\therefore$ Coordinates of $B = \left(\frac{\sqrt{3}}{4} + \frac{\pi}{3}, 0\right)$

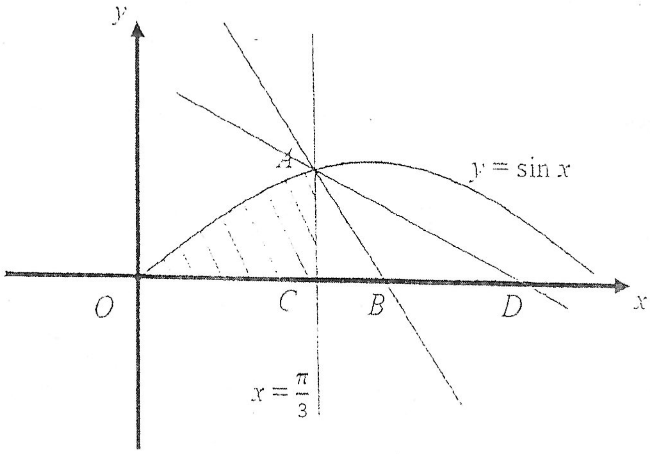
$\left(\frac{4\pi + 3\sqrt{3}}{12}, 0\right)$

Common Mistake

① $\frac{dy}{dx} = -\cos x$

② Give answers in 3 s.f. instead of exact

(ii)



D is a point on the x-axis such that the shaded area OAC and triangle ACD are equal.

Find the distance BD in the form $\frac{m\sqrt{3}}{n}$. [6]

Let coordinates of $D = (x, 0)$

$$\int_0^{\frac{\pi}{3}} \sin x \, dx = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\left(x - \frac{\pi}{3}\right)$$

$$\left[-\cos x\right]_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right)$$

$$\left(-\cos\frac{\pi}{3}\right) - \left(-\cos 0\right) = \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right)$$

$$-\frac{1}{2} - (-1) = \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right)$$

$$\frac{2}{\sqrt{3}} = x - \frac{\pi}{3}$$

$$x = \frac{2}{\sqrt{3}} + \frac{\pi}{3}$$

$$\therefore BD = \left(\frac{2}{\sqrt{3}} + \frac{\pi}{3}\right) - \left(\frac{\sqrt{3}}{4} + \frac{\pi}{3}\right)$$

$$= \frac{2\sqrt{3}}{3} - \frac{\sqrt{3}}{4}$$

$$= \frac{8\sqrt{3} - 3\sqrt{3}}{12}$$

$$= \frac{5\sqrt{3}}{12} \text{ units}$$

OR $\int_0^{\frac{\pi}{3}} \sin x \, dx$

$$= \left[-\cos x\right]_0^{\frac{\pi}{3}}$$

$$= \left(-\cos\frac{\pi}{3}\right) - \left(-\cos 0\right)$$

$$= -\frac{1}{2} - (-1)$$

$$= \frac{1}{2} \text{ units}^2$$

$$\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)(CD) = \frac{1}{2}$$

$$CD = \frac{2}{\sqrt{3}} \text{ units}$$

$$BC = \frac{\sqrt{3}}{4} + \frac{\pi}{3} - \frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{4} \text{ units}$$

$$BD = \frac{2}{\sqrt{3}} - \frac{\sqrt{3}}{4}$$

$$= \frac{2\sqrt{3}}{3} - \frac{\sqrt{3}}{4}$$

$$= \frac{8\sqrt{3} - 3\sqrt{3}}{12}$$

$$= \frac{5\sqrt{3}}{12}$$

End of Paper