Solutions to Tutorial 5A: Differentiation Techniques and Graphical Analysis

Basic Mastery Questions

Q1(a) $\lim_{x\to\infty} \frac{x-1}{x+3} = \lim_{x\to\infty} 1 - \frac{4}{x+3}$

$= 1 - 0$

$= 1$

(b) $\lim_{x\to 1} \frac{x^2+3x+2}{x^2+x+2} = \frac{1+3+2}{1+1+2} = \frac{3}{2}$

(c) $\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)}$

$= \lim_{x\to 3} (x+3)$

$= 3+3 = 6.$

Q2(a) $\frac{d}{dx}\left(\frac{4x^3-2x^2-1}{x}\right) = \frac{d}{dx}(4x^2-2x-x^{-1})$

$= 8x - 2 + \frac{1}{x^2}$

(b) $\frac{d}{dx} 3\left(x+\frac{1}{x}\right)^3 = 9\left(x+\frac{1}{x}\right)^2\left(1-\frac{1}{x^2}\right)$

$= 9\left(1-\frac{1}{x}\right)\left(1+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)^2$

(c) $\frac{d}{dx}\sqrt{2x^2-8} = \frac{d}{dx}(2x^2-8)^{\frac{1}{2}}$

$(x \notin [-2,2]$ $= \frac{1}{2}(2x^2-8)^{-\frac{1}{2}}(4x)$

$= \frac{2x}{\sqrt{2x^2-8}}$

(d) $\frac{d}{dx}\sqrt[3]{x^2+3x} = \frac{d}{dx}(x^2+3x)^{\frac{1}{3}}$

$= \frac{1}{3}(x^2+3x)^{-\frac{2}{3}}(2x+3)$

$= \frac{2x+3}{3(x^2+3x)^{\frac{2}{3}}}$

(e) $\frac{d}{dx}(x^4-1)(x^3-3x+2) = (x^4-1)(3x^2-3)+(x^3-3x+2)(4x^3)$

$= 3x^6-3x^2-3x^4+3+4x^6-12x^4+8x^3$

$= 7x^6-15x^4+8x^3-3x^2+3$

(f) $\frac{d}{dx}\frac{(2x+1)^3}{x-3} = \frac{(x-3)(3)(2x+1)^2(2)-(2x+1)^3(1)}{(x-3)^2} = \frac{(2x+1)^2(6x-18-2x-1)}{(x-3)^2}$

$(x\neq 3)$

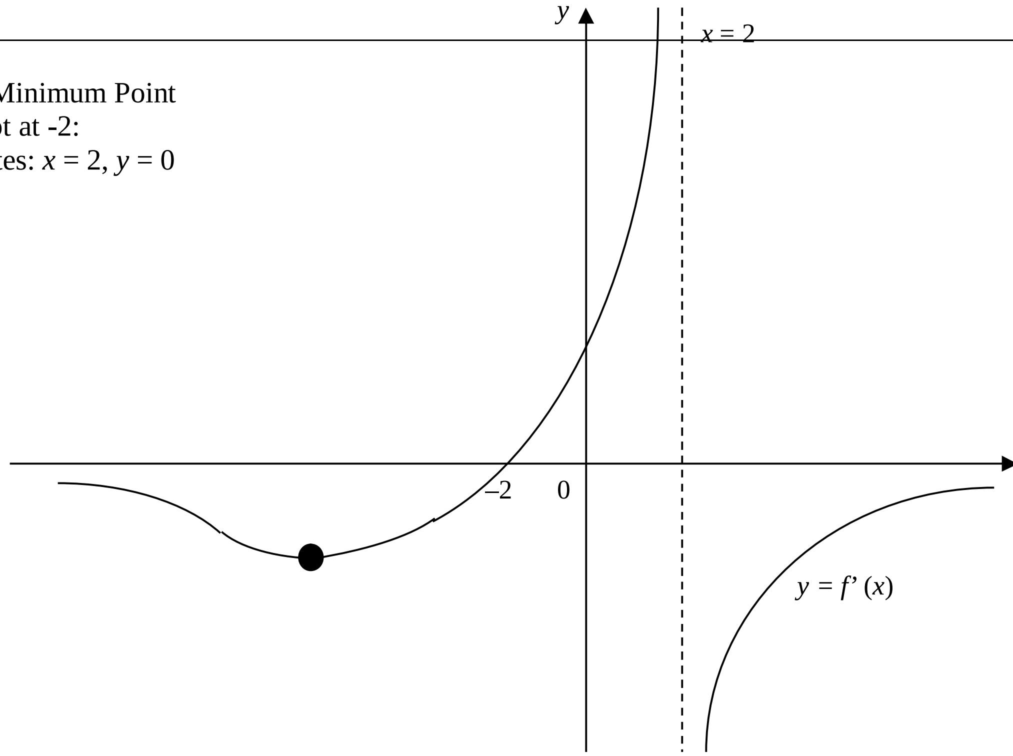
$= \frac{(4x-19)(2x+1)^2}{(x-3)^2}$

Additional Practice Questions

1a.	$\frac{d}{dx}\left(\sin\left(\cos^{-1}(3x)\right)\right)=\cos\left(\cos^{-1}(3x)\right).\frac{-3}{\sqrt{1-9x^2}} \quad =\frac{-9x}{\sqrt{1-9x^2}}$
1b.	$\frac{dx}{d\theta}=1-\cos\theta, \quad \frac{dy}{d\theta}=\sin\theta$ $\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}=\frac{\sin\theta}{1-\cos\theta}$ At point where $\theta=\alpha$, $\frac{dy}{dx}=\frac{1}{2}$ $\Rightarrow \frac{\sin\alpha}{1-\cos\alpha}=\frac{1}{2}$ $\Rightarrow 2\sin\alpha=1-\cos\alpha$ $\Rightarrow 2\sin\alpha+\cos\alpha=1$ (shown)

2(a)	$xy^2+3e^y=4x$ $y^2+2xy\frac{dy}{dx}+3e^y\frac{dy}{dx}=4$ $\frac{dy}{dx}\left(2xy+3e^y\right)=4-y^2 \qquad \Rightarrow \frac{dy}{dx}=\frac{4-y^2}{2xy+3e^y}$
(b)	$\frac{dx}{dt}=2\left(\frac{v\frac{du}{dt}-u\frac{dv}{dt}}{v^2}\right)=2\left(\frac{t+1-t}{(t+1)^2}\right)=\frac{2}{(1+t)^2}$ $\frac{dy}{dt}=\frac{\cos t}{\sin t}=\cot t$ $\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\cot t\left(\frac{(1+t)^2}{2}\right)=\frac{1}{2}(1+t)^2\cot t$
(c)	$\frac{dy}{dx}=1+\sqrt{1-x^2}\left(-\frac{1}{\sqrt{1-x^2}}\right)-\frac{2x\cos^{-1}x}{2\sqrt{1-x^2}}$ $=-\frac{x\cos^{-1}x}{\sqrt{1-x^2}}=-\frac{x\cos^{-1}x}{\sqrt{1-x^2}}\left(\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)=-\frac{x\left(y-x\right)}{1-x^2}$ $=\frac{x\left(x-y\right)}{1-x^2}$ (shown)

3.	(a) $x=\frac{t}{1+t}$ $\quad$ $y=\ln(\cos t)$ $\frac{dx}{dt}=\frac{(1+t)-t}{(1+t)^2}=\frac{1}{(1+t)^2}$ $\quad$ $\frac{dy}{dt}=\frac{-\sin t}{\cos t}=-\tan t$ $\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=-(1+t)^2\tan t$ (b) $\sin^{-1}y+xe^y=3x$ $\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}+xe^y\frac{dy}{dx}+e^y=3$ $\frac{dy}{dx}+\sqrt{1-y^2}\,xe^y\frac{dy}{dx}=\left(3-e^y\right)\sqrt{1-y^2}$ $\frac{dy}{dx}=\frac{(3-e^y)\sqrt{1-y^2}}{1+xe^y\sqrt{1-y^2}}$

	(c) Shape + Minimum Point x-intercept at -2: Asymptotes: $x = 2$, $y = 0$ 

4 $y = x^2 + 2\ln(xy)$

Differentiate implicitly w.r.t x:

$$\frac{dy}{dx} = 2x + 2\left(\frac{1}{x} + \frac{1}{y}\cdot\frac{dy}{dx}\right)$$

$$= 2x + \frac{2}{x} + \frac{2}{y}\cdot\frac{dy}{dx}$$

When $x = 1$, $y = 1$, $\frac{dy}{dx} = 2 + 2 + 2\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -4$

Differentiate implicitly w.r.t. x:

$$\frac{d^2y}{dx^2} = 2 - \frac{2}{x^2} + \frac{2}{y}\cdot\frac{d^2y}{dx^2} - \frac{dy}{dx}\cdot\frac{2}{y^2}\cdot\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = 2 - 2 + 2\frac{d^2y}{dx^2} - (-4)(2)(-4) \Rightarrow \frac{d^2y}{dx^2} = 32$$