Check Your Understanding (Hypothesis Testing)

Formulating Null and Alternative Hypothesis (No need to perform the hypothesis testing)

1. For each of the following parts, justify why a z-test (z-test means a test where the test statistic is normally distributed) can be used by writing down the following

   (i) The null and alternative hypotheses;
   (ii) If the sample size is considered large or small;
   (iii) The population variance σ^2, or if unavailable, the unbiased estimate of the population variance s^2 ;
   (iv) If the population is normally distributed or not (if not, also indicate whether an assumption for the population to be normally distributed is needed to complete the test);
   (v) If the Central Limit Theorem is to be used when performing the hypothesis test.

   You do NOT need to actually perform the hypothesis test

   (a) The heights of boys in SAJC are normally distributed with a mean μ cm and standard deviation 5 cm. It is claimed that $\mu = 171$. However, a random sample of 50 SAJC boys has mean height 173 cm. Test, at the 2% significance level, whether the mean height of SAJC boys in general is above 171 cm.

   (b) The heights of boys in SAJC are distributed with a mean μ cm and standard deviation 5 cm. It is claimed that $\mu = 171$. However, a random sample of 50 SAJC boys has mean height 169 cm. Test, at the 2% significance level, whether the mean height of SAJC boys in general is below 171 cm.

   (c) The heights of boys in SAJC are normally distributed with a mean μ cm. It is claimed that $\mu = 171$. However, a random sample of 50 SAJC boys has mean height 173 cm with standard deviation 5 cm. Test, at the 2% significance level, whether the mean height of SAJC boys in general is not equal to 171 cm.

   (d) The heights of boys in SAJC are normally distributed with a mean μ cm and standard deviation 5 cm. It is claimed that $\mu = 171$. However, a random sample of 5 SAJC boys has mean height 173 cm. Test, at the 2% significance level, whether the mean height of SAJC boys in general is above 171 cm.

   (e) The heights of boys in SAJC are distributed with a mean μ cm. It is claimed that $\mu = 171$. However, a random sample of 50 SAJC boys has mean height 173 cm with standard deviation 5 cm. Test, at the 2% significance level, whether the mean height of SAJC boys in general is not equal to 171 cm.

   (f) The heights of boys in SAJC are distributed with a mean μ cm and standard deviation 5 cm. It is claimed that $\mu = 171$. However, a random sample of 5 SAJC boys has mean height 169 cm. Test, at the 2% significance level, whether the mean height of SAJC boys in general is not equal to 171 cm.

Solution:

(a) $H_0 : \mu = 171, H_1 : \mu > 171$, Large sample size, $\sigma^2 = 5^2$
Population is normally distributed, use a z-test, do not need to use CLT

(b) $H_0 : \mu = 171, H_1 : \mu < 171$, Large sample size, $\sigma^2 = 5^2$
Population is not stated to be normally distributed (do not need to assume normality of population), use a z-test, use CLT

(c) $H_0 : \mu = 171, H_1 : \mu \neq 171$, Large sample size, $s^2 = \frac{50}{49}5^2$
Population is normally distributed, use a z-test, do not need to use CLT

(d) $H_0 : \mu = 171, H_1 : \mu > 171$, Small sample size, $\sigma^2 = 5^2$
Population is normally distributed, use a z-test, do not need to use CLT

(e) $H_0 : \mu = 171, H_1 : \mu \neq 171$, Large sample size, $s^2 = \frac{50}{49}5^2$
Population is not stated to be normally distributed (do not need to assume normality of population), use a z-test, use CLT

(f) $H_0 : \mu = 171, H_1 : \mu \neq 171$, Small sample size, $\sigma^2 = 5^2$
Population is not stated to be normally distributed (need to assume normality of population), use a z-test, do not need to use CLT

<u>Sample size large/small, Population normal, Population variance known</u>

2. [N2011/2/Q10 Modified]
In a factory, the time in minutes for an employee to install an electronic component is a normally-distributed continuous random variable T. The standard deviation of T is 5.0 and under ordinary conditions the expected value of T is 38.0. After background music is introduced into the factory, a sample of n components is taken and the mean time taken for randomly chosen employees to install them is found to be $\bar{t}$ minutes. A test is carried out, at the 5% significance level, to determine whether the mean time taken to install a component has been reduced.

(i) State appropriate hypotheses for the test, defining any symbols you use.

(ii) Given that n=50, state the set of values of $\bar{t}$ for which the result of the test would be to reject the null hypothesis.

(iii) It is given instead that $\bar{t}$ =37.1, and the result of the test is that the null hypothesis is not rejected. Obtain an inequality involving *n*, and hence find the set of values that *n* can take.

Solution:

Examiners' Report:

(i) Most candidates correctly define H_0 and H_1 in terms of μ. Although many candidates went on to explain what H_0 and H_1 were (which were not required), very few defined μ and a very small minority defined it correctly as the population mean time taken for the population for employees to install one component.

(ii) A few candidates used 0.05 rather than a z-value. Others used +1.645 rather than -1.645. Still others used the wrong inequality or multiplied by a factor $\sqrt{\frac{n}{n-1}}$.

(iii) Candidates found this part rather challenging. Candidates made sign errors manipulating an inequality involving negatives and square root. For full credit, candidates were expected to realise that n was an integer.

(i) Let μ refers to the population mean time taken for the population for employees to install one component.

Test $H_0 : \mu = 38.0$ against $H_1 : \mu < 38.0$ at 5% level of significance

Under H_0, since n is large, $\overline{T} \sim N(38, \frac{5^2}{n})$.

$$Z = \frac{\overline{T} - \mu}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0,1)$$

(ii) Critical region $= \{z : z \le -1.64485\}$

Given that $n = 50$,

$$z_{cal} = \frac{\overline{t} - 38}{\sqrt{\frac{5^2}{50}}},$$

Since H_0 is rejected,

$$z_{cal} \le -1.64485$$

$$\frac{\overline{t} - 38}{\sqrt{\frac{5^2}{50}}} \le -1.64485$$

$$\overline{t} \le 36.8$$

The set of values for $\overline{t} = \{\overline{t} \in \mathbb{R} \mid 0 < \overline{t} \le 36.8\}$

(iii) Critical region $= \{z : z \le -1.64485\}$

Given that $\overline{t} = 37.1$

$$z_{cal} = \frac{37.1 - 38}{\sqrt{\frac{5^2}{n}}}.$$

Since H_0 is not rejected,

$z_{cal} > -1.64485$

$$\frac{37.1-38}{\sqrt{\frac{5^2}{n}}} > -1.64485$$

$$\frac{-0.9\sqrt{n}}{5} > -1.64485$$

$\sqrt{n} < 9.1377$

$n < 83.5$

The set of values for $n = \{n \in \mathbb{Z}^+ \mid 0 < n \le 83\}$

3. [SAJC 2013 Prelim P1/10 (Modified)]
A manufacturer claims that his new dieting pill helps people lose weight. A random sample of 20 people took the pill for a month and the loss in weight (initial weight – final weight) after a month, *x* kg, was summarized as follow

$$\sum x = 20.49$$

Suppose that the population follows a normal distribution and the population standard deviation is known to be 1.6 kg. The manufacturer claims that the average weight loss for people taking the pill is at least μ_0. A test at the 10% level of significance indicates that the manufacturer's claim is valid. Find the largest value of μ_0 .

Solution: The population mean is unknown, hence we need to standardise the normal distribution.

Solution:

$$\bar{x} = \frac{20.49}{20} = 1.0245$$

Let *X* be the r.v. "loss in weight" for a randomly chosen person who took the pill for a month and μ be mean loss weight loss.

Test $H_0 : \mu = \mu_0$ against $H_1 : \mu < \mu_0$ at 10% level of significance

Under H_0, $\bar{X} \sim N\left(\mu_0, \frac{1.6^2}{20}\right)$

$$Z = \frac{\bar{X} - \mu}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0,1)$$

(If the manufacturer's claim is valid, it means that we do not reject H_0)

Critical region $= \{z : z \le -1.2816\}$

$$z_{cal} = \frac{1.0245 - \mu_o}{\sqrt{\frac{1.6^2}{20}}}.$$

Since H_0 is not rejected,

$z_{cal} > -1.2816$

$$\frac{1.0245 - \mu_o}{\sqrt{\frac{1.6^2}{20}}} > -1.64485$$

$1.0245 - \mu_o > -0.45850$

$\mu_o < 1.483$

Largest value of μ_0 is 1.48.

<u>Sample size large, Population non-normal</u>

4. A shopkeeper complains that the average weight of chocolate bars of a certain type that he is buying from a wholesaler is less than the stated value of 8.50g. The shopkeeper weighted 100 bars from a large delivery and found that their weights had a mean of 8.36 g and a standard deviation of 0.72 g.
Using a 5 % significance level, determined whether or not the shopkeeper is justified in his complaint. State clearly the null and alternative hypotheses that you are using, and express your conclusion in words.

Solution:

Unbiased estimate of variance = $\frac{n}{n-1} \times$ sample variance $= \frac{100}{99} \times 0.72^2 = 0.52364$

Let X be the r.v. " weight. of chocolate bar in grams" and μ be mean population weight.

Test $H_0: \mu = 8.50$ against $H_1: \mu < 8.50$ at 5% level of significance

Under H_0,since n = 100 is large, $\overline{X} \sim N\left(8.50, \frac{0.52364}{100}\right)$ approximately by Central Limit Theorem.

Use a one-tailed z-test,

$\bar{x} = 8.36$ gives z_{cal}= - 1.93<-1.64 and p-value = 0.0265 < 0.05

We reject H_0 and conclude at 5% level, there is sufficient evidence at to conclude that the mean weight is less than 8.50g, so the shopkeeper is justified in his complaint.

5. Tensile testing is a fundamental materials science test in which a sample is subjected to a controlled tension until failure. A test on a sample of 6 plastic strips manufactured by a company showed a mean ultimate tensile strength of 75 MPa(force per unit area) and a standard deviation of 5 MPa, whereas the manufacturer claimed a mean ultimate tensile strength of 80 MPa. Assuming that the ultimate tensile strengths are normally distributed with standard deviation 5 MPa, can we support the manufacturer's claim at a level of significance of 5%? Explain what is meant by "5% level of significance" in the context of this question.

Solution:

Let X be the r.v. " breaking strength in kg" and μ be mean population tensile strength.

Test $H_0 : \mu = 80$ against $H_1 : \mu \neq 80$ at 5% level of significance

Under $H_0, \overline{X} \sim N\left(80, \frac{5^2}{6}\right)$.

Use a one-tailed z-test,

$\overline{x} = 75$ gives z_{cal} = - 2.45<-1.96 and p-value = 0.0143 < 0.05

We reject H_0 and conclude at 5% level, there is is sufficient evidence at 5% level of significance that the manufacturer's claim is not true.

5% level of significance means that there is a probability of 0.05 of concluding that the mean tensile strength is not 80 MPa when in fact it is 80 MPa.

Sample size large, Population non-normal, Population variance unknown

6. At an early stage in analysing the marks scored by the large number of candidates in an examination paper, the Examining Board takes a random sample of 250 candidates and finds that the marks, x, of these candidates give $\Sigma x = 11872$ and $\Sigma x^2 = 646\ 193$. Using the figures obtained in this sample, the null hypothesis $\mu = 49.5$ is tested against the alternative hypothesis $\mu < 49.5$ at the α % significance level.

Determine the set of values of α for which the null hypothesis is rejected in favour of the alternative hypothesis.

Solution:

$$\bar{x} = \frac{\sum x}{n} = \frac{11872}{250} = 47.488$$

Unbiased estimate of variance

$$= \frac{1}{n-1}\left[\sum x^2 - \frac{\left[\sum x\right]^2}{n}\right] = \frac{1}{249}\left[646193 - \frac{11872^2}{250}\right] = 330.9858$$

Let X be the r.v. " marks scored" and μ be mean marks.

Test $H_0 : \mu = 49.5$ against $H_1 : \mu < 49.5$ at α% level of significance

Under H$_0$, since n is large, $\bar{X} \sim N(49.5, \frac{330.9858}{250})$ approximately by Central Limit Theorem,

$$Z = \frac{\overline{X} - \mu}{\sqrt{\frac{s^2}{n}}} \sim N(0,1)$$

$\overline{x} = 47.488$ gives z_{cal} = -1.75 and p – value = 0.0402

Since H$_0$ is rejected, p-value $\leq \frac{\alpha}{100}$.

$\frac{\alpha}{100} \geq 0.0402$

$\alpha \geq 4.02.$

Hence the set of values of $\alpha = \{\alpha \in \mathbb{R}: 4.02 \leq \alpha < 100\}$

7. A company supplies sugar in small packets. The mass of sugar in one packet is denoted by X grams. The masses of a random sample of 9 packets are summarised by

$$\sum x = 86.4, \sum x^2 = 835.92$$

(i) Calculate unbiased estimates of the mean and variance of X.

The mean mass of sugar in a packet is claimed to be 10 grams. The company directors want to know whether the sample indicates that this claim is incorrect.

(ii) Explain why the Central Limit Theorem does not apply in this context.

(iii) Explain what is meant by “5% level of significance” in the context of this question.

(iv) Suppose now that the population distribution is normal and population variance of X is known and it is equal to the unbiased estimator of population variance found in (i) , carry out a test at 5% level of significance.

Solution:

(i) Unbiased estimate of the population mean, $\bar{x} = \frac{1}{n}\sum x = \frac{86.4}{9} = 9.6$

Unbiased estimate of the population variance,

$s^2 = \frac{1}{8}\left[835.92 - \frac{86.4^2}{9}\right] = 0.81$

(ii) Sample size is small, we cannot use CLT.

(iii) 5% level of significance means that:
There is a probability of 0.05 of concluding that the mean mass is not 10 g when in fact it is 10 g.
Or There is a probability of 0.05 of concluding that the claim is incorrect when in fact it is correct.

(iv) Let μ be the mean population mass.

$H_0 : \mu = 10$ against $H_1 : \mu \neq 10$ at 5 % significance level.

Under H_0, $\bar{X} \sim N(10, \frac{0.81^2}{9})$.

$Z = \frac{\bar{X} - \mu}{\sqrt{\frac{s^2}{n}}} \sim N(0,1)$

Using a 2-tailed z-test,

$\bar{x} = 9.6$ gives z_{calc}= - 1.33 and p-value = 0.182 > 0.05

We do not reject H_0 and conclude that at 5% significance level that the mean mass of sugar is not 10g.