JC1 H2 Mathematics (9758)
Term 4 Revision Topical Quick Check
Chapter 7 Differentiation
Chapter 8 Applications of Differentiation

1 **SAJC Promo 9758/2022/Q3b / JPJC Promo 9758/2022/Q2aiii**

Differentiate the following with respect to x.

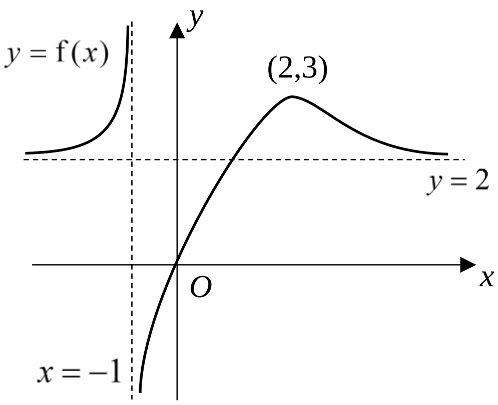
(i) $\ln\left(\dfrac{2x}{\sqrt{x^2+1}}\right)$ [2]

(ii) $\sec 3x \sin^{-1} 2x$ [3]

Solutions	
(i)	$\ln\left(\dfrac{2x}{\sqrt{x^2+1}}\right) = \ln 2 + \ln x - \ln\sqrt{x^2+1}$ $= \ln 2 + \ln x - \dfrac{1}{2}\ln\left(x^2+1\right)$ $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(\dfrac{2x}{\sqrt{x^2+1}}\right)\right) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\ln 2 + \ln x - \dfrac{1}{2}\ln\left(x^2+1\right)\right)$ $= \dfrac{1}{x} - \dfrac{1}{2(x^2+1)}(2x)$ $= \dfrac{1}{x} - \dfrac{x}{x^2+1}$
(ii)	$\dfrac{\mathrm{d}}{\mathrm{d}x}\sec 3x \sin^{-1} 2x$ $= \sec 3x \dfrac{1}{\sqrt{1-(2x)^2}} 2 + \sin^{-1} 2x\left(3\sec 3x \tan 3x\right)$ $= \sec 3x\left(\dfrac{2}{\sqrt{1-4x^2}} + 3\tan 3x \sin^{-1} 2x\right)$

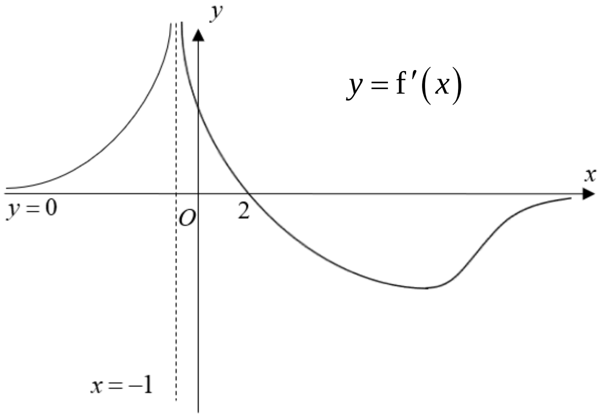
2 **DHS Promo 9758/2022/Q1b**

The graph of $y = \mathrm{f}(x)$ has a maximum turning point at (2,3) and passes through the origin. The lines $x = -1$ and $y = 2$ are asymptotes to the graph, as shown in the diagram below.



Sketch the graph of $y = \mathrm{f}'(x)$, showing clearly the axial intercepts and the asymptotes.

[3]

Solutions	
	

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Tangents and Normals

The equation of the tangent and normal at any point (x_0, y_0) on a curve $y = \mathrm{f}(x)$ is given by:

Tangent	$y - y_0 = m(x - x_0)$	where $m = \frac{\mathrm{d}y}{\mathrm{d}x}$ is the gradient of tangent at the point (x_0, y_0)
Normal	$y - y_0 = -\frac{1}{m}(x - x_0)$	

Note:

- Tangent parallel to x-axis $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 0$
- Tangent parallel to y-axis $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined,

  i.e. DENOMINATOR of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is ZERO.

Let's Try Now.

3 MI PU2 P1 Promo 9758/2022/Q7

A curve C has parametric equations

$$x = \cos 2\theta, \qquad y = \sin\theta, \qquad \text{for } 0 \le \theta \le \frac{1}{2}\pi .$$

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{4\sin\theta}$. [3]

(ii) Sketch C, showing clearly the features of the curve at the points where $\theta = 0$ and $\theta = \frac{1}{2}\pi$. [2]

(iii) The tangent to the curve C at the point where $\theta = p$ is parallel to the line $2y + x = 0$. Find the equation of this tangent. [4]

(iv) The tangent from part **(iii)** meets the x-axis at P and the y-axis at Q. Find the area of the triangle OPQ. [3]

Solutions

(i)

$x = \cos 2\theta \qquad y = \sin\theta$

$\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -2\sin 2\theta \qquad \dfrac{\mathrm{d}y}{\mathrm{d}\theta} = \cos\theta$

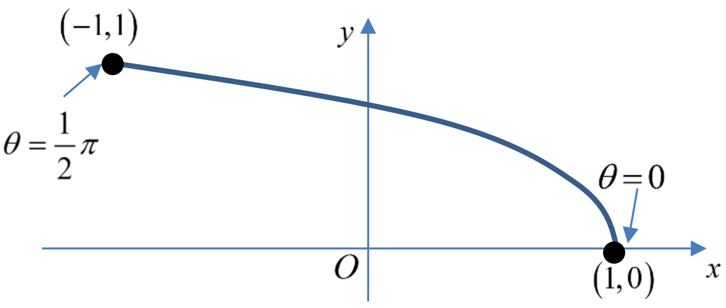
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}}$$

$$\frac{\cos\theta}{-2\sin 2\theta}$$

$$= \frac{\cos\theta}{-4\sin\theta\cos\theta}$$

$$= -\frac{1}{4\sin\theta}$$

(ii)



(iii)

$2y + x = 0 \Rightarrow y = -\frac{1}{2}x$

Tangent parallel to $2y + x = 0$ means $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}$ when $\theta = p$.

$$-\frac{1}{4\sin p} = -\frac{1}{2}$$

$$\sin p = \frac{1}{2}$$

$$\therefore\ p = \frac{\pi}{6}$$

When $p = \frac{\pi}{6}$, $\quad x = \cos\frac{\pi}{3} = \frac{1}{2}$, $\ y = \sin\frac{\pi}{6} = \frac{1}{2}$

Equation of tangent:

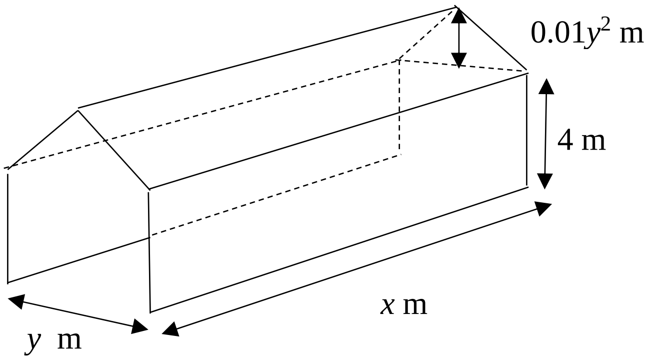
$$y - \frac{1}{2} = -\frac{1}{2}\left(x - \frac{1}{2}\right)$$

$$y = -\frac{1}{2}x + \frac{3}{4}$$

(iv)	When $x = 0, y = \frac{3}{4}$ When $y = 0,\ \frac{1}{2}x = \frac{3}{4} \Rightarrow x = \frac{3}{2}$ $\therefore\ P\left(\frac{3}{2}, 0\right)$ and $Q\left(0, \frac{3}{4}\right)$ Area of Triangle $OPQ = \frac{1}{2} \times \frac{3}{4} \times \frac{3}{2}$ $= \frac{9}{16}$

4 RVHS Promo 9758/2022/Q11(i)-(iii)

To ease food security concerns, the government is building tents at the community garden plots to allow urban farming to take place more efficiently among residents. The organizing committee is building a tent consisting of two rectangular pieces on the roof and two rectangular vertical sides as shown in the diagram below. The base of the tent has sides x m by y m, and covers a total floor area of 40 m^2. The vertical sides of the tent are 4 m tall, and the roof adds another $0.01y^2$ m to the overall height of the tent.



(Diagram not drawn to scale)

(i) The total external surface area of the tent is denoted by A m^2. Show that A is given by

$$A = 8x + 40\sqrt{\frac{16}{25x^2} + 1}\ .$$ [3]

(ii) Suppose that A has a stationary value at some x, show that x satisfies the equation $25x^6 + 16x^4 - 256 = 0$. [3]

(iii) The design team decides that the material for the tent costs \$3.10 per m^2, estimate the minimum total cost of the material for the whole tent. [3]

Solutions	
(i)	Floor area $xy = 40 \Rightarrow y = \frac{40}{x}$ Area $A = 2(4x) + 2x\sqrt{\left(0.01y^2\right)^2 + \left(\frac{y}{2}\right)^2}$ ....(*) $= 8x + 2x\sqrt{\left(\frac{1}{100}\cdot\frac{40^2}{x^2}\right)^2 + \left(\frac{40}{2x}\right)^2}$ $= 8x + 2x\sqrt{\frac{256}{x^4} + \frac{400}{x^2}}$....(**) $= 8x + 2x\sqrt{\left(\frac{400}{x^2}\right)\left(1 + \frac{16}{25x^2}\right)}$ $= 8x + 40\sqrt{\frac{16}{25x^2} + 1}$

(ii)	$A=8x+40\sqrt{\frac{16}{25x^2}+1}$ $\frac{\mathrm{d}A}{\mathrm{d}x}=8+40\left(\frac{1}{2}\right)\left(\frac{16}{25x^2}+1\right)^{-1/2}\left(-\frac{32}{25x^3}\right)$ $\frac{\mathrm{d}A}{\mathrm{d}x}=8-\frac{128}{5x^3\sqrt{\frac{16}{25x^2}+1}}$ Let $\frac{\mathrm{d}A}{\mathrm{d}x}=0$ $8-\frac{128}{5x^3\sqrt{\frac{16}{25x^2}+1}}=0$ $\sqrt{\frac{16}{25x^2}+1}=\frac{16}{5x^3}$ $\frac{16}{25x^2}+1=\frac{256}{25x^6}$ $25x^6+16x^4-256=0$ (Shown)
(iii)	Solving $25x^6+16x^4-256=0$ using GC Rejecting all negative and complex roots. $x=1.4063$ Using GC, $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=15.67>0$ when x = 1.4063 Hence A is minimum when $x=1.4063$ Minimum Cost = $(3.10)\times\left[8(1.4063)+40\sqrt{\frac{16}{25(1.4063)^2}+1}\right]$ = \$177.53

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Maxima / Minima

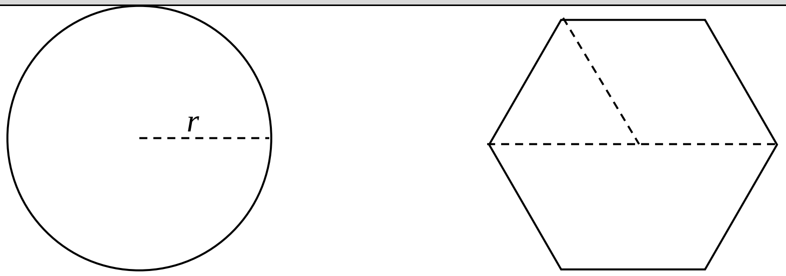
1. Draw a clear diagram with all the given information included.
2. Denote each changing quantity (weight, volume, radius, length etc.) by a variable.
3. Construct the equation(s) relating the variables.
4. Express quantity to be maximised/minimised in terms of a single variable, say x (if there are 2 variables, express 1 in terms of another).
5. Using differentiation, find the **stationary point(s)**.
6. **Use 1st or 2nd derivative test to determine/prove nature of the stationary point**.

Let's Try Now.

5 MJC Prelim 9740/2008/P1/Q13

A piece of wire of length d units is cut into two pieces. One piece is bent to form a circle of radius r units, and the other piece is bent to form a regular hexagon. Prove that, as r varies, the sum of the areas enclosed by the two shapes is a minimum when the radius of the circle is approximately $0.076d$ units. [7]

Solutions



Length of circle = $2\pi r$
Perimeter of hexagon = $d - 2\pi r$

A = total area

$$= \pi r^2 + 6\left(\frac{1}{2}\right)\left(\frac{d-2\pi r}{6}\right)^2 \sin\left(\frac{\pi}{3}\right)$$

$$= \pi r^2 + \frac{\sqrt{3}}{24}(d-2\pi r)^2$$

$$\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r + \frac{\sqrt{3}}{12}(d-2\pi r)(-2\pi)$$

$$= 2\pi r - \frac{\pi\sqrt{3}}{6}(d-2\pi r)$$

At stationary point, $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$

$$\Rightarrow 2\pi r - \frac{\pi\sqrt{3}}{6}\left(d - 2\pi r\right) = 0$$

$$\Rightarrow r\left(2 + \frac{\pi\sqrt{3}}{3}\right) = \frac{d\sqrt{3}}{6}$$

$$\Rightarrow r = \frac{\sqrt{3}d}{2\left(6 + \pi\sqrt{3}\right)}$$

$$\frac{\mathrm{d}^2 A}{\mathrm{d}r^2} = 2\pi + \frac{\pi^2\sqrt{3}}{3} > 0$$

Alternative method (First Derivative)

r	$\left(\frac{\sqrt{3}d}{2\left(6+\pi\sqrt{3}\right)}\right)^-$	$\left(\frac{\sqrt{3}d}{2\left(6+\pi\sqrt{3}\right)}\right)$	$\left(\frac{\sqrt{3}d}{2\left(6+\pi\sqrt{3}\right)}\right)^+$
$\frac{\mathrm{d}A}{\mathrm{d}r}$	$-$	0	$+$
Slope	\	—	/

Thus, the sum of the areas enclosed by the two shapes is a minimum when the radius of the circle is approximately 0.076*d* units.

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Rate of Change

$\dfrac{\mathrm{d}y}{\mathrm{d}t}$ denotes the rate of change of variable y with respect to time t.

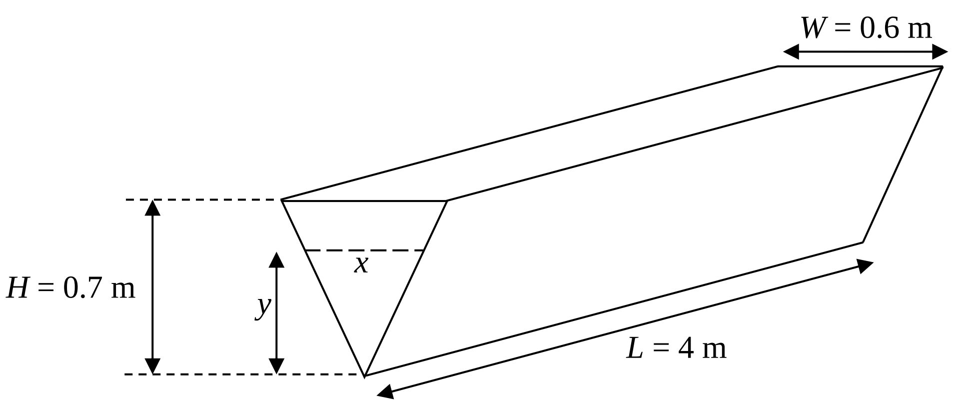
$\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{\mathrm{d}y}{\mathrm{d}x}\times\dfrac{\mathrm{d}x}{\mathrm{d}t}$ (Chain Rule) is useful for rates of change problems.

1. Draw a clear diagram with all the given information included.
2. Denote each changing quantity (weight, volume, radius, length etc.) by a variable.
3. Construct the equation(s) relating the variables.
4. Use Chain Rule to link up the derivatives and rate given.

Let's Try Now.

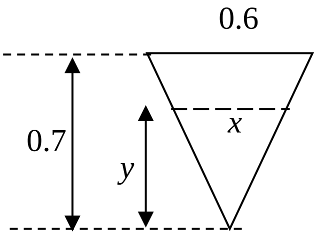
6 JPJC Promo 9758/2022/Q5

The diagram shows a V-shaped tank with dimensions $L = 4$ m, $W = 0.6$ m and $H = 0.7$ m. The tank is initially empty. Water is pumped into the tank at a rate of 0.0025 m^3/s. At any instant from the start of water flowing into the tank, the water in the tank has a depth of y m and a surface width of x m.



(i) Find the rate of change of the water depth when $y = 0.4$ m, leaving your answer to 4 decimal places. [4]

(ii) Find, to the nearest second, the time taken to completely fill up the tank from the instant when $y = 0.4$ m. [2]

Solutions	
(i)	Using similar triangles, $\frac{x}{y}=\frac{0.6}{0.7}$ $\therefore x=\frac{6}{7}y$ Let the volume of water in the tank be V. $V=\frac{1}{2}xy(4)=2y\left(\frac{6}{7}y\right)=\frac{12}{7}y^2$ $\frac{\mathrm{d}V}{\mathrm{d}y}=\frac{24}{7}y$ Since $\frac{\mathrm{d}V}{\mathrm{d}t}=\left(\frac{\mathrm{d}V}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)$, $\therefore 0.0025=\frac{24}{7}(0.4)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)$ $\frac{\mathrm{d}y}{\mathrm{d}t}=0.0018$ m/s (4 d.p) 
(ii)	Time taken = $\left[\frac{1}{2}(0.6)(0.7)(4)-\frac{12}{7}(0.4)^2\right]\div 0.0025\approx 226\text{s}$ (nearest second) Or Time taken = $\left[\frac{12}{7}(0.7)^2-\frac{12}{7}(0.4)^2\right]\div 0.0025\approx 226\text{s}$ (nearest second)

Answer Key

No.	**Year**	**JC**	**Answers**
1	2022	SAJC	**(i)** $\frac{1}{x}-\frac{x}{x^2+1}$ **(ii)** $\sec 3x\left(\frac{2}{\sqrt{1-4x^2}}+3\tan 3x\sin^{-1}2x\right)$
2	2022	DHS	
3	2022	MI	**(iii)** $y=-\frac{1}{2}x+\frac{3}{4}$ **(iv)** $\frac{9}{16}$
4	2022	RVHS	**(iii)** $177.53
6	2022	JPJC	**(i)** $\frac{\mathrm{d}y}{\mathrm{d}t}=0.0018$ m/s (4 d.p) **(ii)** 226 seconds