ZHONGHUA SECONDARY SCHOOL
PRELIMINARY EXAMINATION 2024
SECONDARY 4 EXPRESS /
5 NORMAL ACADEMIC

Candidate Name	Class	Register Number
Solution		

ADDITIONAL MATHEMATICS **4049/01**

Paper 1

28 August 2024
2 hours 15 minutes

Candidates answer on the Question Paper.
No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your index number and name on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, glue, or correction fluid.

Answer **all** the questions.
Omission of essential working will result in loss of marks.
The use of an approved scientific calculator is expected, where appropriate.
If the degree of accuracy is not specified in the question, and if the answer is not exact, give the answer to three significant figures.
Give answers in degrees to one decimal place.
For π, use either your calculator value or 3.142, unless the question requires the answer in terms of π.

At the end of the presentation, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is **90**.

90

Setter: Ms Lee Sock Kee
Vetter: Mr Francis Tan and Mr Lionel Ang

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n ,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$\Delta = \frac{1}{2} bc \sin A$$

1. **(a)** Solve the simultaneous equations [3]

$$x - 2y + 4 = 0$$
$$x^2 + y^2 = 2x + 4$$

$x = 2y - 4$ (1)
$x^2 + y^2 = 2x + 4$ (2)

Sub (1) into (2),
$(2y-4)^2 + y^2 = 2x + 4$
$4y^2 - 16y + 16 + y^2 = 4y - 8 + 4$
$5y^2 - 20y + 20 = 0$
$y^2 - 4y + 4 = 0$
$(y-2)^2 = 0$
$y = 2$

Sub $y = 2$ into (1),
$x = 2(2) - 4$
$x = 0$

$\therefore\ x = 0,\ y = 2$

(b) Explain the geometrical meaning of your answer in **(a)**. [1]

The line $x - 2y + 4 = 0$ is a tangent to the circle at (0, 2).

2. **(a)** Express $y = 3 - 8x - 2x^2$ in the form $y = a(x+b)^2 + c$ and hence state the maximum value of y. [3]

$y = -2x^2 - 8x + 3$

$y = -2(x^2 + 4x) + 3$

$y = -2[(x+2)^2 - 2^2] + 3$

$y = -2(x+2)^2 + 8 + 3$

$y = -2(x+2)^2 + 11$

Maximum value of y = 11

(b) Show that there are no values of p for which the curve $y = (p-3)x^2 + 2px + (p+1)$ is always positive. [3]

Always positive,
$p - 3 > 0$ and $b^2 - 4ac < 0$

$p > 3$ and $(2p)^2 - 4(p-3)(p+1) < 0$

$4p^2 - 4(p^2 - 2p - 3) < 0$

$4p^2 - 4p^2 + 8p + 12 < 0$

$8p + 12 < 0$

$p < -\frac{3}{2}$

For the curve y to be always positive, $p > 3$ and $p < -\frac{3}{2}$.

There are no values of p for which y is always positive.

(c) A quadratic equation is given by $hx^2 - 2kx + 6k - 9h = 0$, where h and k are constants and $h \neq 0$.

(i) Show that the equation has real roots for all values of h and k. [3]

$b^2 - 4ac$

$= (-2k)^2 - 4(h)(6k - 9h)$

$= 4k^2 - 24hk + 36h^2$

$= 4(k^2 - 6hk + 9h^2)$

$= 4(k - 3h)^2 \geq 0$

for all values of h and k.

Therefore, the roots are real (shown).

(ii) In the case where the equation has two real and equal roots, express h in terms of k. [2]

$b^2 - 4ac = 0$

$4(k - 3h)^2 = 0$

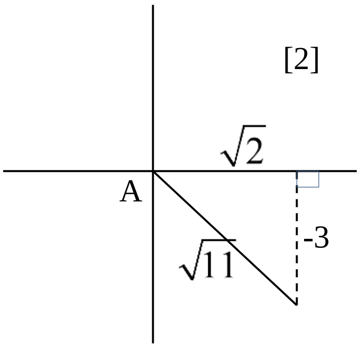
$(k - 3h)^2 = 0$

$k = 3h$

$h = \dfrac{k}{3}$

3. Given that $\cos A = \sqrt{\frac{2}{11}}$ where $180^\circ < A < 360^\circ$, find, without the use of a calculator, the value of

(a) $\tan A$, [2]



A lies in the 4th quadrant.

$(opp)^2 = (\sqrt{11})^2 - (\sqrt{2})^2$

$opp = -3 \; or \; 3 \; (rej)$

$\tan A = -\frac{3}{\sqrt{2}} \; or \; -\frac{3\sqrt{2}}{2}$

(b) $\sin\left(A - 90^\circ\right)$, [2]

$\sin(A - 90^\circ)$

$= \sin A \cos 90^\circ - \cos A \sin 90^\circ$

$= \sin A(0) - \cos A(1)$

$= -\cos A$

$= -\sqrt{\frac{2}{11}}$

OR

$\sin(A - 90^\circ)$

$= \sin[-(90^\circ - A)]$

$= -\sin(90^\circ - A)$

$= -\cos A$

$= -\sqrt{\frac{2}{11}}$

(c) $\dfrac{1}{\sec 2A}$. [2]

$$\frac{1}{\sec 2A}$$
$$= \cos 2A$$
$$= 2\cos^2 A - 1$$
$$= 2\left(\sqrt{\frac{2}{11}}\right)^2 - 1$$
$$= \frac{4}{11} - 1$$
$$= -\frac{7}{11}$$

4. **(a)** Factorise $27x^3 - \dfrac{y^3}{8}$ completely. [2]

$$27x^3 - \frac{y^3}{8}$$
$$= (3x)^3 - \left(\frac{y}{2}\right)^3$$
$$= \left(3x - \frac{y}{2}\right)\left(9x^2 + \frac{3xy}{2} + \frac{y^2}{4}\right)$$

OR

$$27x^3 - \frac{y^3}{8}$$
$$= \frac{1}{8}\left(216x^3 - y^3\right)$$
$$= \frac{1}{8}\left[(6x)^3 - y^3\right]$$
$$= \frac{1}{8}\left(6x - y\right)\left(36x^2 + 6xy + y^2\right)$$

(b) Express $\dfrac{8x^3-7x^2+4x-3}{(2x^2-x)(2x-1)}$ in partial fractions. [6]

$(2x^2-x)(2x-1)=x(2x-1)^2$

$$\begin{array}{r} 2 \\ 4x^3-4x^2+x\overline{\smash{\big)}8x^3-7x^2+4x-3} \\ \underline{-(8x^3-8x^2+2x)} \\ x^2+2x-3 \end{array}$$

$$\frac{8x^3-7x^2+4x-3}{(2x^2-x)(2x-1)}=2+\frac{x^2+2x-3}{x(2x-1)^2}$$

$$\frac{x^2+2x-3}{x(2x-1)^2}=\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{(2x-1)^2}$$

$$x^2+2x-3=A(2x-1)^2+Bx(2x-1)+Cx$$

When $x=\frac{1}{2}$, $\quad \frac{1}{4}+1-3=\frac{1}{2}c$

$-\frac{7}{4}=\frac{1}{2}C$

$C=-\frac{7}{2}$

When $x=0$, $\quad A=-3$

$1+2-3=-3+B-\frac{7}{2}$

When $x=1$, $\quad 0=-\frac{13}{2}+B$

$B=\frac{13}{2}$

$$2-\frac{3}{x}+\frac{13}{2(2x-1)}-\frac{7}{2(2x-1)^2}$$

5. A curve is such that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x - ax^2}{3}$, where a is a constant.

(a) Given that the curve has a turning point at $(3, 7)$, show that the value of a is $\frac{2}{3}$. [1]

At turning point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$\frac{2x - ax^2}{3} = 0$

$2x - ax^2 = 0$

When $x = 3$,

$2(3) - a(3)^2 = 0$

$6 = 9a$

$a = \frac{2}{3}$ (shown)

(b) Find the range of values of x for which y decreases as x increases. [3]

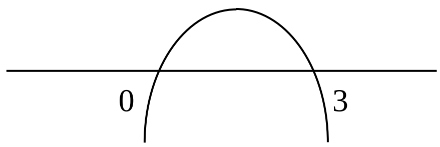
y decreases as x increases, $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$

$\frac{2x - \frac{2}{3}x^2}{3} < 0$

$2x - \frac{2}{3}x^2 < 0$

$6x - 2x^2 < 0$

$2x(3 - x) < 0$



$x < 0$ *or* $x > 3$

(c) Find the equation of the curve. [4]

$$y = \int \frac{2x - \frac{2}{3}x^2}{3}\,dx$$

$$y = \frac{1}{3}\int \left(2x - \frac{2}{3}x^2\right)dx$$

$$y = \frac{1}{3}\left[\frac{2x^2}{2} - \frac{2x^3}{3(3)}\right] + c$$

$$y = \frac{x^2}{3} - \frac{2x^3}{27} + c$$

When $x = 3$, $y = 7$,

$$7 = \frac{3^2}{3} - \frac{2(3)^3}{27} + c$$

$$7 = 3 - 2 + c$$

$$c = 6$$

$$y = \frac{x^2}{3} - \frac{2x^3}{27} + 6$$

6. **(a)** Prove the identity $(\cot x - \operatorname{cosec} x)^2 = \dfrac{1-\cos x}{1+\cos x}$. [4]

LHS

$= (\cot x - \operatorname{cosec} x)^2$

$= \left(\dfrac{\cos x - 1}{\sin x}\right)^2$

$= \dfrac{(\cos x - 1)^2}{\sin^2 x}$

$= \dfrac{[-(1-\cos x)]^2}{(1-\cos^2 x)}$

$= \dfrac{(1-\cos x)^2}{(1+\cos x)(1-\cos x)}$

$= \dfrac{1-\cos x}{1+\cos x}$ (proven)

(b) Hence, solve the equation $2(\cot x - \operatorname{cosec} x)^2 = 3\cos x$ for $0 \le x \le 2\pi$. [3]

$2\left(\dfrac{1-\cos x}{1+\cos x}\right) = 3\cos x$

$2 - 2\cos x = 3\cos x + 3\cos^2 x$

$3\cos^2 x + 5\cos x - 2 = 0$

$(3\cos x - 1)(\cos x + 2) = 0$

$\cos x = \dfrac{1}{3}$ *or* $\cos x = -2$ (*no solution*)

basic angle, α

$= \cos^{-1}\left(\dfrac{1}{3}\right)$

$= 1.230959$

$x = 1.23$ *or* $2\pi - 1.230959$

$x = 1.23$ *or* 5.05 $(3 s.f.)$

(c) State the number of solutions of the equation $2(\cot 2x - \operatorname{cosec} 2x)^2 = 3\cos 2x$ in the range $-2\pi \le 2x \le 2\pi$. [1]

Angle changes from x to $2x$.

$2x = 1.23$ *or* $2\pi - 1.230959$ *or* -1.23 *or* $-(2\pi - 1.230959)$

$2x = 1.23$ *or* 5.05 *or* -1.23 *or* -5.05

$x = 0.615$ *or* 2.52 *or* -1.23 *or* -2.52 $(3 s.f.)$

There are 4 solutions to the equation.

7 **(a)** The graph of $y = \log_a x$ passes through the points with coordinates (125, 3) and (1, b).

(i) Determine the values of a and b. [2]

Sub $x = 125, y = 3$

$3 = \log_a 125$

$a^3 = 125$

$a = 5$

Sub $x = 1, y = b$

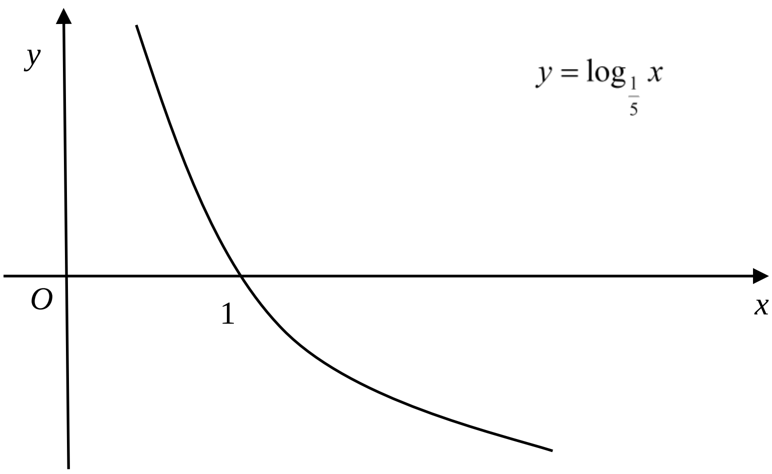
$b = \log_5 1$

$5^b = 1$

$5^b = 5^0$

$b = 0$

(ii) Sketch the graph of $y = \log_{a^{-1}} x$ indicating clearly any intercept on the axes. [2]



(b) Find the values of a and b such that $\lg\left(\dfrac{8}{y}\right) + 4\lg y = a\lg(by)$. [4]

$\lg\left(\dfrac{8}{y}\right) + \lg y^4 = \lg(by)^a$

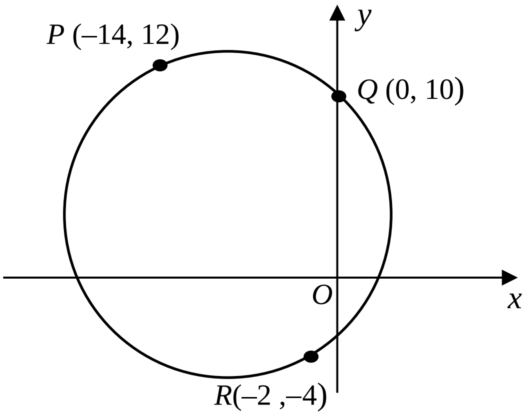
$\lg\left(\dfrac{8}{y} \times y^4\right) = \lg(by)^a$

$\lg\left(8y^3\right) = \lg(by)^a$

$\lg\left(2y\right)^3 = \lg(by)^a$

$\therefore a = 3,\ b = 2$

8 Solutions to this question by accurate drawing will not be accepted.



In the diagram which is not drawn to scale, *P*, *Q* and *R* are points on the circle.

(a) Show that *PR* is the diameter of the circle and hence find the centre of the circle. [5]

Gradient of *PQ*

$= \frac{12-10}{-14-0}$

$= -\frac{1}{7}$

Gradient of *QR*

$= \frac{-4-10}{-2-0}$

$= 7$

Since gradient of *PQ* x gradient of *QR* $= -1$,

line *PQ* is perpendicular to line *QR*.

Angle *PQR* is $90°$ (angle in a semicircle).

$\therefore$ *PR* is the diameter of the circle. (shown)

Centre = $\left(\frac{-14-2}{2}, \frac{12-4}{2}\right)$

$= (-8, 4)$

(b) Find the equation of the circle that passes through the points P, Q and R. [2]

Radius
$=\sqrt{(-8+2)^2+(4+4)^2}$
$=\sqrt{100}$
$=10\,units$

Equation of circle:
$(x+8)^2+(y-4)^2=100$

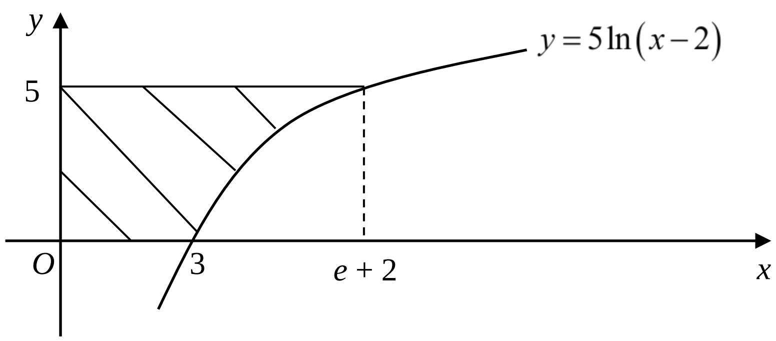
(c) Determine whether the point $S(-14,-2)$ lies inside or outside the circle. [2]

Centre $(-8,4)$

Distance of S from centre
$=\sqrt{(-8+14)^2+(4+2)^2}$ [M1] FT their centre to point S
$=\sqrt{72}$
$=8.4852<10$

Since the distance of point S from centre C is less than the radius, S lies inside the circle. [A1] no FT

9 The diagram shows part of the curve $y=5\ln(x-2)$.



(a) Find the exact value of $\int_0^5 x\ \mathrm{d}y$. [3]

$y=5\ln(x-2)$

$\frac{y}{5}=\ln(x-2)$

$e^{\frac{y}{5}}=x-2$

$x=e^{\frac{y}{5}}+2$

$$\int_0^5 (e^{\frac{y}{5}} + 2)\ \mathrm{d}y$$
$$= \left[5e^{\frac{y}{5}} + 2y \right]_0^5$$
$$= (5e + 10) - 5$$
$$= 5e + 5$$

(b) On the diagram above, shade the region whose area is $\int_0^5 x\ \mathrm{d}y$, showing your upper limit clearly. [1]

Must see 5, correct shading and horizontal line on the above diagram.

Need not see $e + 2$

(c) Hence find $\int_3^{e+2} 5\ln(x-2)\ \mathrm{d}x$. [3]

When $x = e + 2$,

$$e + 2 = e^{\frac{y}{5}} + 2$$
$$e = e^{\frac{y}{5}}$$
$$\frac{y}{5} = 1$$
$$y = 5$$

$$\int_3^{e+2} 5\ln(x-2)\ \mathrm{dx}$$
= rectangle $-\ (5e + 5)$
$$= 5(e + 2) -\ (5e + 5)$$
$$= 5e + 10 - 5e - 5$$
$$= 5$$

10 Water is being added at a constant rate of 4 cm^3/s to an inverted right cone.
The height of the cone is twice the radius of the cone.

[The volume of a cone is $\frac{1}{3}\pi r^2 h$.]

(a) Show that the height of the water level in the cone is 6 cm when the volume of water in the cone is 18π cm^3. [2]

$h = 2r$

$r = \frac{h}{2}$

$\frac{1}{3}\pi r^2 h = 18\pi$

$\frac{1}{3}\left(\frac{h}{2}\right)^2 h = 18$

$\frac{h^3}{12} = 18$

$h^3 = 216$

$h = 6cm$ (shown)

OR

$\frac{1}{3}\pi r^2 (2r) = 18\pi$

$\frac{2}{3}\pi r^3 = 18\pi$

$r^3 = 27$

$r = 3$

$h = 3(20$

$h = 6$

(b) Calculate the rate of change of height of the water level when the volume of water is 18π cm^3. Leave your answer in its exact form. [3]

$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$V = \frac{\pi}{12}h^3$

$\frac{dV}{dh}$

$= \frac{\pi}{12}(3)h^2$

$= \frac{\pi}{4}h^2$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$4 = \frac{\pi}{4}h^2 \times \frac{dh}{dt}$

$\frac{dh}{dt} = 4 \div \frac{\pi h^2}{4}$

$\frac{dh}{dt} = \frac{16}{\pi(6)^2}$

$\frac{dh}{dt} = \frac{4}{9\pi} cm/s$

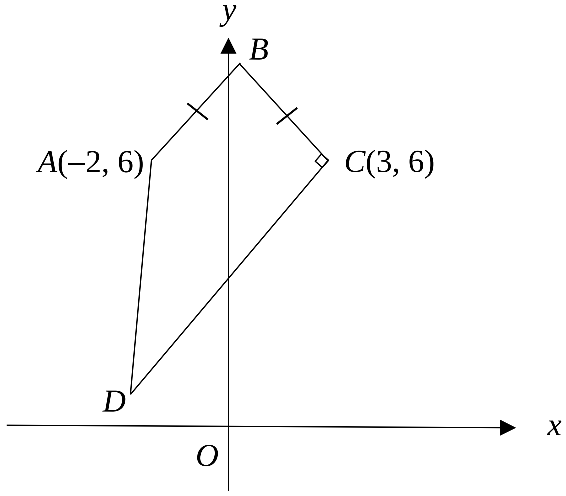
11 (a) Given that $y=(x+5)\sqrt{2x-5}$, show that $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form $\frac{kx}{\sqrt{2x-5}}$, where k is a constant. [2]

$$\frac{dy}{dx}$$
$$=(x+5)\left(\frac{1}{2}\right)(2x-5)^{-\frac{1}{2}}(2)+(2x-5)^{\frac{1}{2}}$$
$$=(x+5)(2x-5)^{-\frac{1}{2}}+(2x-5)^{\frac{1}{2}}$$
$$=(2x-5)^{-\frac{1}{2}}(x+5+2x-5)$$
$$=\frac{3x}{\sqrt{2x-5}}$$

(b) Hence, find $\int\frac{x-4}{\sqrt{2x-5}}\,\mathrm{d}x$. [4]

$$\int\frac{x-4}{\sqrt{2x-5}}\,\mathrm{d}x$$
$$=\int\frac{x}{\sqrt{2x-5}}\,dx-4\int\frac{1}{\sqrt{2x-5}}\,dx$$
$$=\frac{1}{3}\int\frac{3x}{\sqrt{2x-5}}\,dx-4\int(2x-5)^{-\frac{1}{2}}\,dx$$
$$=\frac{1}{3}(x+5)\sqrt{2x-5}-\frac{4(2x-5)^{\frac{1}{2}}}{\frac{1}{2}(2)}+c$$
$$=\frac{1}{3}(x+5)\sqrt{2x-5}-4\sqrt{2x-5}+c$$

12 Solutions to this question by accurate drawing will not be accepted.



The diagram (not drawn to scale) shows a quadrilateral *ABCD* such that *AB* = *BC* and angle $BCD = 90°$. Point *A* is $(-2, 6)$ and point *C* is $(3, 6)$. Given that the area of triangle *ABC* is 7.5 square units and point *D* lies on the line $y + x + 2 = 0$,

(a) show that the coordinates of *B* is $\left(\frac{1}{2}, 9\right)$. [2]

Given that *AB* = *BC*, triangle *ABC* is an isosceles triangle.

$$x_B = \frac{-2+3}{2}$$
$$x_B = \frac{1}{2}$$

Area of triangle *ABC* = 7.5

$$\frac{1}{2}(5)(h) = 7.5$$
$$h = 3$$

$$y_B = 6 + 3$$
$$y_B = 9$$
$$B\left(\frac{1}{2}, 9\right) \quad \text{(shown)}$$

(b) Find the coordinates of D. [4]

Gradient of BC

$$=\frac{9-6}{\frac{1}{2}-3}$$

$$=-\frac{6}{5}$$

Gradient of CD

$$=-1\div\left(-\frac{6}{5}\right)$$

$$=\frac{5}{6}$$

Equation of CD:

$$y-6=\frac{5}{6}(x-3)$$

$$y=\frac{5}{6}x-\frac{15}{6}+6$$

$$y=\frac{5}{6}x+\frac{7}{2} \qquad (1)$$

$$y+x+2=0 \qquad (2)$$

Sub (1) into (2),

$$\frac{5}{6}x+\frac{7}{2}+x+2=0$$

$$\frac{11}{6}x=-\frac{11}{2}$$

$$x=-3$$

$$y=\frac{5}{6}(-3)+\frac{7}{2}$$

$$y=1$$

$$D(-3,1)$$

(c) Find the area of *ABCD*. [2]

Area of *ABCD*

$$=\frac{1}{2}\begin{vmatrix} 3 & \frac{1}{2} & -2 & -3 & 3 \\ 6 & 9 & 6 & 1 & 6 \end{vmatrix}$$

$$=\frac{1}{2}\left[(27+3-2-18)-(3-18-18+3)\right]$$

$$=\frac{1}{2}(40)$$

$$=20\,units^2$$

(d) If *ABCT* is a parallelogram, find the coordinates of *T*. [2]

Midpoint of *AC* = midpoint of *BD*

$$\left(\frac{-2+3}{2},\frac{6+6}{2}\right)=\left(\frac{\frac{1}{2}+x}{2},\frac{9+y}{2}\right)$$

$$\left(\frac{1}{2},6\right)=\left(\frac{1}{4}+\frac{x}{2},\frac{9+y}{2}\right)$$

$$\frac{1}{4}+\frac{x}{2}=\frac{1}{2}$$

$$1+2x=2$$

$$2x=1$$

$$x=\frac{1}{2}$$

$$\frac{9+y}{2}=6$$

$$9+y=12$$

$$y=3$$

$$T\left(\frac{1}{2},3\right)$$

-End of paper-