H2 MATHEMATICS TUTORIAL SOLUTION
TOPIC TECHNIQUES OF DIFFERENTIATION 2022/JC1

ESSENTIAL PRACTICE

1 **[2014/PJC/Promo/1]**

Differentiate each of the following with respect to *x*.

(i) $e^{\sec x}$ [1]

(ii) $\tan^{-1}\left(x^2\right)$ [2]

(iii) $\cos^4(2x)$ [2]

Solution:

(i) $\frac{d}{dx}\left(e^{\sec x}\right) = (\sec x \tan x)e^{\sec x}$

(ii) $\frac{d}{dx}\left[\tan^{-1}\left(x^2\right)\right] = \frac{2x}{1+x^4}$

(iii) $\frac{d}{dx}\left[\cos^4(2x)\right] = 4\cos^3(2x)(-\sin(2x))2$

$= -8\cos^3(2x)\sin(2x)$

2 **[2013/NYJC/Promo/2]**

Differentiate the following expressions with respect to x, simplifying your answers as far as possible:

(a) $\tan^{-1}\left(\dfrac{2}{x}\right)$, [3]

(b) $\ln\sqrt{\dfrac{1+x}{1-x}}$. [3]

Solution:

(a)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}\left(\frac{2}{x}\right)\right]=\frac{2(-x^{-2})}{1+\left(\frac{2}{x}\right)^2}$$

$$=-\frac{2}{x^2+4}$$

(b) **Method ①:**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\sqrt{\frac{1+x}{1-x}}\right)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(1+x)-\ln(1-x)\right]$$

$$=\frac{1}{2}\left(\frac{1}{1+x}-\frac{-1}{1-x}\right)$$

$$=\frac{1}{1-x^2} \quad \text{or} \quad \frac{1}{(1+x)(1-x)}$$

Method ②: [Not Recommended]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\sqrt{\frac{1+x}{1-x}}\right)=\left(\frac{1}{\sqrt{\frac{1+x}{1-x}}}\right)\left[\frac{1}{2}\cdot\sqrt{\frac{1-x}{1+x}}\left(\frac{1-x+1+x}{(1-x)^2}\right)\right]$$

$$=\frac{1}{2}\left(\frac{1-x}{1+x}\right)\left[\frac{2}{(1-x)^2}\right]$$

$$=\frac{1}{(1-x)(1+x)}$$

3 **[2013/ACJC/Promo/3]**

Differentiate the following with respect to x.

(i) $\sqrt{\cos^{-1}\left(\frac{x}{2}\right)}$, [2]

(ii) $\ln\sqrt{\frac{(x+1)^3}{x^2-1}}$. [2]

Solution:

(i)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{\cos^{-1}\left(\frac{x}{2}\right)}\right)=\frac{1}{2}\left(\cos^{-1}\left(\frac{x}{2}\right)\right)^{-\frac{1}{2}}\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{1}{2}$$
$$=-\frac{1}{2\sqrt{\left(4-x^2\right)\cos^{-1}\left(\frac{x}{2}\right)}}$$

(ii) **Method ①:**
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\sqrt{\frac{(x+1)^3}{x^2-1}}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(\frac{x+1}{\sqrt{x-1}}\right)\right] \qquad \left(\because x>1 \therefore \sqrt{(x+1)^2}=x+1\right)$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(x+1)-\frac{1}{2}\ln(x-1)\right)$$
$$=\frac{1}{x+1}-\frac{1}{2(x-1)}$$

Method ②: [Not Recommended]
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\sqrt{\frac{(x+1)^3}{x^2-1}}\right)=\frac{1}{\sqrt{\frac{(x+1)^3}{x^2-1}}}\times\frac{1}{2}\frac{1}{\sqrt{\frac{(x+1)^3}{x^2-1}}}\times\frac{3(x+1)^2\left(x^2-1\right)-(x+1)^3(2x)}{\left(x^2-1\right)^2}$$
$$=\frac{x-3}{2\left(x^2-1\right)}$$

4 **[2016/PJC/Promo/1]**

Differentiate with respect to x, giving your answers as a single fraction,

(a) $\ln\sqrt{a^2-x^2}$, where a is a constant, [2]

(b) $\tan^{-1}\left(\dfrac{1}{2x}\right)$. [2]

Solution:

(a) **Method ①:**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\sqrt{a^2-x^2}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\ln\left(a^2-x^2\right)\right]$$
$$=\frac{1}{2}\left(\frac{-2x}{a^2-x^2}\right)$$
$$=-\frac{x}{a^2-x^2}$$
$$=\frac{x}{x^2-a^2}$$

Method ②: [Not Recommended]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\sqrt{a^2-x^2}\right)=\frac{1}{\sqrt{a^2-x^2}}.\frac{1}{2}\left(a^2-x^2\right)^{-\frac{1}{2}}.(-2x)$$
$$=-\frac{x}{a^2-x^2}$$
$$=\frac{x}{x^2-a^2}$$

(b)

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}\left(\frac{1}{2x}\right)=\frac{1}{1+\left(\frac{1}{2x}\right)^2}\cdot\frac{(-2)}{(2x)^2}$$
$$=-\left(\frac{1}{1+\frac{1}{4x^2}}\right)\left(\frac{2}{4x^2}\right)$$
$$=-\frac{2}{4x^2+1}$$

5 **[2015/CJC/Promo/1]**

Differentiate the following expressions with respect to x, simplifying your answers whenever possible.

(a) $\tan^{-1}\left(\mathrm{e}^{3x}\right)$, [2]

(b) 5^{2x}, [2]

(c) $\dfrac{ax}{\sqrt{x^3+1}}$, where a is a constant. [3]

Solution:

(a)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}\left(\mathrm{e}^{3x}\right)\right]=\frac{1}{1+\left(\mathrm{e}^{3x}\right)^2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{3x}\right)$$
$$=\frac{3\mathrm{e}^{3x}}{1+\mathrm{e}^{6x}}$$

(b) **Method ①:**

Let $y=5^{2x}$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=5^{2x}\cdot\ln 5\cdot(2)$$
$$=(2\ln 5)\left(5^{2x}\right)$$

Method ②: **Logarithmic Differentiation**

Let $y=5^{2x}$

Take ln on both sides:
$$\ln y=\ln\left(5^{2x}\right)$$
$$\ln y=2x\ln 5$$
$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=2\ln 5$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=y(2\ln 5)$$
$$=(2\ln 5)\left(5^{2x}\right)$$

(c) **<u>Method ①:</u>** **Product Rule**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{ax}{\sqrt{x^3+1}}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(ax\left(x^3+1\right)^{-\frac{1}{2}}\right)$$

$$=a\left(x^3+1\right)^{-\frac{1}{2}}+ax\left(-\frac{1}{2}\right)\left(x^3+1\right)^{-\frac{3}{2}}\cdot\left(3x^2\right)$$

$$=a\left(x^3+1\right)^{-\frac{1}{2}}-\frac{3}{2}ax^3\left(x^3+1\right)^{-\frac{3}{2}}$$

<u>Method ②:</u> **Quotient Rule**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{ax}{\sqrt{x^3+1}}\right)=\frac{a\sqrt{x^3+1}-ax\cdot\left(\frac{1}{2}\right)\left(x^3+1\right)^{-\frac{1}{2}}\left(3x^2\right)}{\left(\sqrt{x^3+1}\right)^2}$$

$$=\frac{a\left(x^3+1\right)^{\frac{1}{2}}-\frac{3}{2}ax^3\left(x^3+1\right)^{-\frac{1}{2}}}{x^3+1}$$

$$=a\left(x^3+1\right)^{-\frac{1}{2}}-\frac{3}{2}ax^3\left(x^3+1\right)^{-\frac{3}{2}}$$

6 **[2010/ACJC/Promo/5]**

Differentiate the following with respect to x, leaving your answers in terms of x.

(a) $\mathrm{e}^{\mathrm{e}x}\cot\left(\frac{x}{2}\right)$, [3]

(b) $(2x)^{(1/x)}$. [4]

Solution:

(a)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{\mathrm{e}x}\cot\left(\frac{x}{2}\right)\right]=\mathrm{e}^{\mathrm{e}x}\left[-\frac{1}{2}\operatorname{cosec}^2\left(\frac{x}{2}\right)\right]+\left[\cot\left(\frac{x}{2}\right)\right]\left(\mathrm{e}^{\mathrm{e}x}\right)(\mathrm{e})$$
$$=\mathrm{e}^{\mathrm{e}x}\left[-\frac{1}{2}\operatorname{cosec}^2\left(\frac{x}{2}\right)+\mathrm{e}\cot\left(\frac{x}{2}\right)\right]$$

(b) Let $y=(2x)^{\frac{1}{x}}$

$$\ln y=\frac{1}{x}\ln(2x)$$
$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x}\left(\frac{2}{2x}\right)-\frac{1}{x^2}\ln(2x)$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(2x)^{\frac{1}{x}}}{x^2}\left[1-\ln(2x)\right]$$

7 **[2012/Promo/ACJC/2]**

Differentiate the following with respect to *x*.

(i) $\cos^{-1}(\sin x)$, where $\frac{\pi}{2} < x < \frac{3\pi}{2}$, [2]

(ii) $\ln\sqrt{\frac{e^x+1}{1-e^{-x}}}$. [3]

Solution:

(i)

$$\frac{d}{dx}\left[\cos^{-1}(\sin x)\right] = -\frac{1}{\sqrt{1-(\sin x)^2}}\cdot \cos x$$

$$= \frac{-\cos x}{\sqrt{(\cos x)^2}}$$

$$= \frac{-\cos x}{-\cos x} \quad \left(\because\ \cos x < 0 \text{ when } \frac{\pi}{2} < x < \frac{3\pi}{2} \text{ so } \sqrt{(\cos x)^2} = -\cos x\right)$$

$$= 1$$

(ii)

$$\frac{d}{dx}\left(\ln\sqrt{\frac{e^x+1}{1-e^{-x}}}\right) = \frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{e^x+1}{1-e^{-x}}\right)\right]$$

$$= \frac{1}{2}\frac{d}{dx}\left[\ln\left(e^x+1\right) - \ln\left(1-e^{-x}\right)\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{e^x+1} - \frac{e^{-x}}{1-e^{-x}}\right]$$

$$= \frac{1}{2}\left(\frac{e^x}{e^x+1} - \frac{1}{e^x-1}\right)$$

8 **[2016/CJC/Promo/1]**

Given that $y^3 + \mathrm{e}^{\tan y} = \cos\left(2x + \frac{\pi}{3}\right)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [4]

Solution:

$$y^3 + \mathrm{e}^{\tan y} = \cos\left(2x + \frac{\pi}{3}\right)$$

$$3y^2 \frac{\mathrm{d}y}{\mathrm{d}x} + \sec^2 y \cdot \mathrm{e}^{\tan y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = -2\sin\left(2x + \frac{\pi}{3}\right)$$

$$\left(3y^2 + \sec^2 y \cdot \mathrm{e}^{\tan y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = -2\sin\left(2x + \frac{\pi}{3}\right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2\sin\left(2x + \frac{\pi}{3}\right)}{3y^2 + \sec^2 y \cdot \mathrm{e}^{\tan y}}$$

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9 **[2014/CJC/Promo/2]**

The variables x and y are related by $3^y = xy - \sec x$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [4]

Solution:

$$3^y = xy - \sec x$$

$$3^y \ln 3 \frac{\mathrm{d}y}{\mathrm{d}x} = y + x\frac{\mathrm{d}y}{\mathrm{d}x} - \sec x \tan x$$

$$\left(3^y \ln 3 - x\right)\frac{\mathrm{d}y}{\mathrm{d}x} = y - \sec x \tan x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y - \sec x \tan x}{3^y \ln 3 - x}$$

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10 [2012/Promo/CJC/11(b)]

Given that $y=(2x+1)^x$ for $x>0$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [3]

Solution:

$$y=(2x+1)^x$$
$$\ln y = x\ln(2x+1)$$
$$\left(\frac{1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=x\left(\frac{2}{2x+1}\right)+\ln(2x+1)$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=y\left[\frac{2x}{2x+1}+\ln(2x+1)\right]$$
$$=(2x+1)^x\left[\frac{2x}{2x+1}+\ln(2x+1)\right]$$

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11 [2013/YJC/Promo/4a]

It is given that $y^x=\mathrm{e}^y x^2$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y, simplifying your answer. [4]

Solution:

$$y^x=\mathrm{e}^y x^2$$
$$x\ln y=\ln\left(\mathrm{e}^y x^2\right)$$
$$x\ln y = y+2\ln x$$
$$\ln y + x\left(\frac{1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2}{x}$$
$$\left(\frac{x}{y}-1\right)\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{x}-\ln y$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{2}{x}-\ln y}{\frac{x}{y}-1}$$
$$=\frac{\frac{2-x\ln y}{x}}{\frac{x-y}{y}}$$
$$=\frac{y(2-x\ln y)}{x(x-y)}$$

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12 [2014/MJC/Promo/5]

(a) Differentiate $\dfrac{x-2x^3}{\ln x}$ with respect to x. [2]

(b) Given that $0<x<\dfrac{\pi}{2}$, show that $\dfrac{d}{dx}\left[\sin^{-1}(\cos x)\right]=k$, where k is a real constant to be determined. [3]

(c) Given that $e^{xy}=\left(1+y^2\right)^2$, find $\dfrac{dy}{dx}$ in terms of x and y, simplifying your answer. [4]

Solution:

(a) **Method ①: Quotient Rule**

$$\frac{d}{dx}\left(\frac{x-2x^3}{\ln x}\right)=\frac{\left(1-6x^2\right)\ln x-\left(\frac{1}{x}\right)\left(x-2x^3\right)}{(\ln x)^2}$$
$$=\frac{\left(1-6x^2\right)\ln x-\left(1-2x^2\right)}{(\ln x)^2}$$
$$=\frac{\left(1-6x^2\right)\ln x+2x^2-1}{(\ln x)^2}$$

Method ②: Product Rule

$$\frac{d}{dx}\left(\frac{x-2x^3}{\ln x}\right)=\frac{d}{dx}\left[\left(x-2x^3\right)(\ln x)^{-1}\right]$$
$$=\left(1-6x^2\right)(\ln x)^{-1}+\left(x-2x^3\right)\left[-(\ln x)^{-2}\left(\frac{1}{x}\right)\right]$$
$$=\frac{1-6x^2}{\ln x}-\frac{1-2x^2}{(\ln x)^2}$$
$$=\frac{\left(1-6x^2\right)\ln x-\left(1-2x^2\right)}{(\ln x)^2}$$
$$=\frac{\left(1-6x^2\right)\ln x+2x^2-1}{(\ln x)^2}$$

(b)
$$\frac{d}{dx}\left[\sin^{-1}(\cos x)\right]=\frac{-\sin x}{\sqrt{1-\cos^2 x}}$$
$$=\frac{-\sin x}{\sqrt{\sin^2 x}}$$
$$=\frac{-\sin x}{\sin x}\quad(\text{since } x \text{ is acute})$$
$$=-1$$

$\therefore\ k=-1$

(c) **Method ①:**

$$e^{xy} = (1+y^2)^2$$

$$\left(y + x\frac{dy}{dx}\right)e^{xy} = 2\left(1+y^2\right)\left(2y\frac{dy}{dx}\right)$$

$$ye^{xy} + xe^{xy}\frac{dy}{dx} = 4y\left(1+y^2\right)\frac{dy}{dx}$$

$$\left[4y\left(1+y^2\right) - xe^{xy}\right]\frac{dy}{dx} = ye^{xy}$$

$$\frac{dy}{dx} = \frac{ye^{xy}}{4y\left(1+y^2\right) - xe^{xy}}$$

Method ②:

$$e^{xy} = (1+y^2)^2$$

$$xy = 2\ln(1+y^2)$$

$$y + x\frac{dy}{dx} = \frac{2\left(2y\frac{dy}{dx}\right)}{1+y^2}$$

$$y\left(1+y^2\right) + x\left(1+y^2\right)\frac{dy}{dx} = 4y\frac{dy}{dx}$$

$$\left[4y - x\left(1+y^2\right)\right]\frac{dy}{dx} = y\left(1+y^2\right)$$

$$\frac{dy}{dx} = \frac{y\left(1+y^2\right)}{4y - x\left(1+y^2\right)}$$