1	Let P_n denote $\sum_{r=0}^{n} r\left(r!\right)=\left(r+1\right)!-1$ for $n \in \square,\ n \geq 0$. When $n=0$, $\text{LHS}=0\left(0!\right)=0$ $\text{RHS}=\left(0+1\right)!-1=0=\text{LHS}$ Therefore P_0 is true. Assume P_k is true for some $k \in \square,\ k \geq 0$, i.e. $\sum_{r=0}^{k} r\left(r!\right)=\left(k+1\right)!-1$ Want to prove that P_{k+1} is true, i.e. $\sum_{r=0}^{k+1} r\left(r!\right)=\left(k+2\right)!-1$ $\text{LHS}=\sum_{r=0}^{k+1} r\left(r!\right)$ $=\sum_{r=0}^{k} r\left(r!\right)+\left(k+1\right)\left(k+1\right)!$ $=\left(k+1\right)!-1+\left(k+1\right)\left(k+1\right)!$ $=\left(k+1\right)!\left[\left(k+1\right)+1\right]-1$ $=\left(k+1\right)!\left(k+2\right)-1$ $=\left(k+2\right)!-1=\text{RHS}$ Thus P_k is true $\Rightarrow \text{P}_{k+1}$ is true. Since P_0 is true, and P_k is true $\Rightarrow \text{P}_{k+1}$ is true, by mathematical induction, P_n is true for all $n \in \square,\ n \geq 0$.
2	(i) $\int e^{\frac{x}{n}} \cos(nx)\,dx$ $= ne^{\frac{x}{n}} \cos(nx) - \int ne^{\frac{x}{n}} \left(-n\sin(nx)\right) dx$ $= ne^{\frac{x}{n}} \cos(nx) + n^2\left[ne^{\frac{x}{n}} \sin(nx) - \int ne^{\frac{x}{n}} \left(n\cos(nx)\right) dx\right]$ $= ne^{\frac{x}{n}}\left[\cos(nx)+n^2\sin(nx)\right]-n^4\int e^{\frac{x}{n}} \cos(nx)\,dx$ $\left(n^4+1\right)\int e^{\frac{x}{n}} \cos(nx)\,dx = ne^{\frac{x}{n}}\left[\cos(nx)+n^2\sin(nx)\right]$ $\int e^{\frac{x}{n}} \cos(nx)\,dx = \frac{n}{\left(n^4+1\right)} e^{\frac{x}{n}}\left[\cos(nx)+n^2\sin(nx)\right]+C$

	(ii) $\displaystyle\int_{\pi}^{2\pi} \mathrm{e}^{\frac{x}{n}} \cos(nx)\,\mathrm{d}x = \frac{n}{(n^4+1)}\left[\mathrm{e}^{\frac{x}{n}}\left(\cos(nx)+n^2\sin(nx)\right)\right]_{\pi}^{2\pi}$ $\displaystyle = \frac{n}{(n^4+1)}\left[\mathrm{e}^{\frac{2\pi}{n}}\cos(2n\pi)-\mathrm{e}^{\frac{\pi}{n}}\cos(n\pi)\right]$ $\displaystyle = \frac{n}{(n^4+1)}\mathrm{e}^{\frac{\pi}{n}}\left[\mathrm{e}^{\frac{\pi}{n}}-\cos(n\pi)\right]$ $\displaystyle = \begin{cases} \frac{n}{(n^4+1)}\mathrm{e}^{\frac{\pi}{n}}\left(\mathrm{e}^{\frac{\pi}{n}}-1\right) & \text{if } n \text{ is even} \\ \frac{n}{(n^4+1)}\mathrm{e}^{\frac{\pi}{n}}\left(\mathrm{e}^{\frac{\pi}{n}}+1\right) & \text{if } n \text{ is odd} \end{cases}$
3	(i) $(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q}) = 4\mathbf{p}\times\mathbf{p}+10\mathbf{p}\times\mathbf{q}-10\mathbf{q}\times\mathbf{p}-25\mathbf{q}\times\mathbf{q}$ $= 20\mathbf{p}\times\mathbf{q}$ $= 20\begin{pmatrix}2\\1\\a\end{pmatrix}\times\begin{pmatrix}b\\1\\0\end{pmatrix}$ $= 20\begin{pmatrix}-a\\ab\\2-b\end{pmatrix}$ Alternative: $(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q}) = \left(2\begin{pmatrix}2\\1\\a\end{pmatrix}-5\begin{pmatrix}b\\1\\0\end{pmatrix}\right)\times\left(2\begin{pmatrix}2\\1\\a\end{pmatrix}+5\begin{pmatrix}b\\1\\0\end{pmatrix}\right)$ $= \begin{pmatrix}4-5b\\-3\\2a\end{pmatrix}\times\begin{pmatrix}4+5b\\7\\2a\end{pmatrix}$ $= \begin{pmatrix}-6a-14a\\-(8a-10ab-8a-10ab)\\28-35b+12+15b\end{pmatrix}$ $= \begin{pmatrix}-20a\\20ab\\40-20b\end{pmatrix} = 20\begin{pmatrix}-a\\ab\\2-b\end{pmatrix}$

Given that the **i-** and **j-** components of the vector $20\begin{pmatrix}-a\\ ab\\ 2-b\end{pmatrix}$ are equal,

$-a = ab$

$ab + a = 0$

$a(b+1) = 0$

Since $a \neq 0$, thus $b = -1$

(ii)

$\left|(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q})\right| = 80$

$$\left|20\begin{pmatrix}-a\\ ab\\ 2-b\end{pmatrix}\right| = 80$$

$$\left|\begin{pmatrix}-a\\ -a\\ 2+1\end{pmatrix}\right| = 4$$

$$\sqrt{2a^2+9} = 4$$

$$2a^2+9 = 16$$

$$a^2 = \frac{7}{2}$$

$$a = \pm\sqrt{\frac{7}{2}} \quad or \quad \pm\frac{\sqrt{14}}{2}$$

(iii)

Since $2\mathbf{p}-5\mathbf{q}$ and $2\mathbf{p}+5\mathbf{q}$ are perpendicular,

$$\left(2\mathbf{p}-5\mathbf{q}\right)\cdot\left(2\mathbf{p}+5\mathbf{q}\right) = 0$$

$$4|\mathbf{p}|^2 - 25|\mathbf{q}|^2 = 0$$

$$|\mathbf{p}|^2 = \frac{25}{4}|\mathbf{q}|^2$$

$$= \frac{25}{4}\left((-1)^2+1^2\right)$$

$$= \frac{25}{2}$$

$$|\mathbf{p}| = \frac{5\sqrt{2}}{2}$$

Alternative:

$$(2\mathbf{p}-5\mathbf{q})\cdot(2\mathbf{p}+5\mathbf{q}) = \begin{pmatrix}4+5\\ -3\\ 2a\end{pmatrix}\cdot\begin{pmatrix}4-5\\ 7\\ 2a\end{pmatrix}$$

$$= 16-25-21+4a^2$$

$$= 4a^2-30$$

Since $2\mathbf{p}-5\mathbf{q}$ and $2\mathbf{p}+5\mathbf{q}$ are perpendicular,

$$(2\mathbf{p}-5\mathbf{q})\cdot(2\mathbf{p}+5\mathbf{q})=0$$

$$4a^2-30=0$$

$$a^2=\frac{15}{2}$$

$$|\mathbf{p}|=\sqrt{2^2+1+a^2}=\sqrt{5+\frac{15}{2}}=\sqrt{\frac{25}{2}}=\frac{5\sqrt{2}}{2}$$

4

(a)

Method 1

Since the coefficients are real, $w=2+\mathrm{i}$ is another root of the equation.

$$\begin{aligned}(w-2+\mathrm{i})(w-2-\mathrm{i})&=(w-2)^2-(\mathrm{i})^2\\&=w^2-4w+4+1\\&=w^2-4w+5\end{aligned}$$

$$w^3+pw^2+qw+30=0$$

$$\left(w^2-4w+5\right)(w+6)=0 \quad \text{(By inspection)}$$

Comparing coefficients of w^2, $p=6-4=2$

Comparing coefficients of w, $q=-24+5=-19$

Method 2

Substitute $w=2-\mathrm{i}$ (or $w=2+\mathrm{i}$) into the given eqn,

$$(2-\mathrm{i})^3+p(2-\mathrm{i})^2+q(2-\mathrm{i})+30=0$$

$$(3-4\mathrm{i})(2-\mathrm{i})+p(3-4\mathrm{i})+q(2-\mathrm{i})+30=0$$

$$(6-3\mathrm{i}-8\mathrm{i}-4)+p(3-4\mathrm{i})+q(2-\mathrm{i})+30=0$$

$$(32+3p+2q)+(-11-4p-q)\mathrm{i}=0$$

Comparing the real parts, $3p+2q=-32$ --- (1)

Comparing the imaginary parts, $4p+q=-11$---- (2)

(1) – (2) × 2: $3p-8p=-32+11\times 2$

$$-5p=-10$$

$$p=2$$

From (2): $q=-11-4\times 2=-19$

$\therefore\ p=2$, $q=-19$

(b)

Substitute $z=3+u\mathrm{i}$ into the given eqn,

$$(3+u\mathrm{i})^2+(-5+2\mathrm{i})(3+u\mathrm{i})+(21-\mathrm{i})=0$$

$$9+6u\mathrm{i}-u^2-15-5u\mathrm{i}+6\mathrm{i}-2u+21-\mathrm{i}=0$$

$$\left(15-2u-u^2\right)+(u+5)\mathrm{i}=0$$

Compare imaginary coefficient: $u+5=0$

$$u=-5$$

$\therefore z=3-5\mathrm{i}$

[Note: if using $15-2u-u^2=0$, need to reject $u=3$]

Method 1

Let the other root be w.

$$z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})=(z-3+5\mathrm{i})(z-w)$$

Comparing coefficients of z,

$$-5+2\mathrm{i}=-w-3+5\mathrm{i}$$

$$w=2+3\mathrm{i}$$

Method 2

Let the other solution be $a+b\mathrm{i}$,

$$z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})$$

$$=(z-(3-5\mathrm{i}))(z-(a+b\mathrm{i}))$$

$$=z^2-(a+b\mathrm{i})z-(3-5\mathrm{i})z+(3-5\mathrm{i})(a+b\mathrm{i})$$

$$=z^2-\left[a+3+(b-5)\mathrm{i}\right]z+(3-5\mathrm{i})(a+b\mathrm{i})$$

Compare the z term: $-(a+3)=-5 \quad \Rightarrow a=2$

$-(b-5)=2 \quad \Rightarrow b=3$

$\therefore z=2+3\mathrm{i}$ is another root.

5

(i)

$$\sum_{n=2}^{N}\frac{2}{n(n-1)^2(n+1)^2}$$

$$=\sum_{n=2}^{N}\left[u_n-u_{n+1}\right]$$

$$=\begin{bmatrix} (u_2-u_3) \\ +(u_3-u_4) \\ +(u_4-u_5) \\ \cdots\cdot \\ \cdots\cdot \\ +(u_{N-1}-u_N) \\ +(u_N-u_{N+1}) \end{bmatrix}$$

$$=u_2-u_{N+1}$$

$$=\frac{1}{2(2^2)(2-1)^2}-\frac{1}{2(N+1)^2((N-1)+1)^2}$$

$$=\frac{1}{8}-\frac{1}{2N^2(N+1)^2}$$

(ii)

As $N\to\infty$, $\dfrac{1}{2N^2(N+1)^2}\to 0$

$$\sum_{n=2}^{\infty}\frac{2}{n(n-1)^2(n+1)^2}\to\frac{1}{8}$$ which is a constant, hence it is a convergent series.

$$\sum_{n=2}^{\infty}\frac{2}{n(n-1)^2(n+1)^2}=\frac{1}{8}-0$$

$$=\frac{1}{8}$$

(iii)

Method 1

$$\sum_{n=1}^{N}\frac{2N}{(n+1)n^2(n+2)^2}=N\sum_{n=1}^{N}\frac{2}{(n+1)n^2(n+2)^2}$$

$$=N\sum_{n=2}^{N+1}\frac{2}{(n)(n-1)^2(n+1)^2}$$

$$=N\left[\frac{1}{8}-\frac{1}{2(N+1)^2(N+2)^2}\right]$$

$$=\frac{N}{8}\left[1-\frac{4}{(N+1)^2(N+2)^2}\right]$$

Method 2 By listing the terms

$$\sum_{n=2}^{N}\frac{2}{n(n-1)^2(n+1)^2}$$

$$=\frac{2}{2(1)^2(3)^2}+\frac{2}{3(2)^2(4)^2}+\cdots+\frac{2}{N(N-1)^2(N+1)^2}$$

$$\sum_{n=1}^{N}\frac{2N}{(n+1)n^2(n+2)^2}$$

$$=N\left[\frac{2}{2(1)^2(3)^2}+\frac{2}{3(2)^2(4)^2}+\cdots+\frac{2}{(N+1)(N)^2(N+2)^2}\right]$$

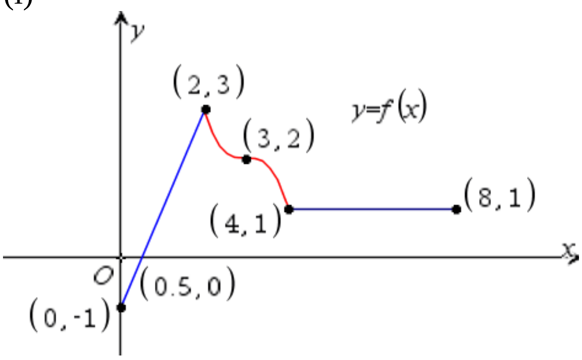
$$=N\sum_{n=2}^{N+1}\frac{2}{n(n-1)^2(n+1)^2}$$

$$=N\left[\frac{1}{8}-\frac{1}{2(N+1)^2(N+2)^2}\right]$$

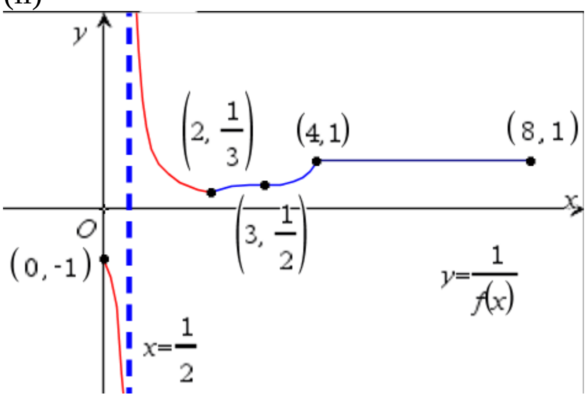
$$=\frac{N}{8}\left[1-\frac{4}{(N+1)^2(N+2)^2}\right]$$

6	(i) $(x+y)\frac{\mathrm{d}y}{\mathrm{d}x}+ky=2 \qquad \cdots(1)$ Differentiating (1) w.r.t. x: $(x+y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}+k\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $(x+y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+k)\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0 \qquad \cdots(2)$ Differentiating (2) w.r.t. x: $(x+y)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+k)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=0$ $(x+y)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\left(2+3\frac{\mathrm{d}y}{\mathrm{d}x}+k\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ $x=0,\quad y=1:\ \frac{\mathrm{d}y}{\mathrm{d}x}=2-k$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=3k-6$ $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=6k^2-36k+48=6\left(k^2-6k+8\right)$ $\therefore y \ =1+(2-k)x+\left(\frac{3k-6}{2!}\right)x^2+\left(\frac{6\left(k^2-6k+8\right)}{3!}\right)x^3+\ldots$ $=1+(2-k)x+\left(\frac{3k-6}{2}\right)x^2+\left(k^2-6k+8\right)x^3+\ldots$ (ii) $\sin\left(2x+\frac{\pi}{2}\right)=\sin 2x\cos\frac{\pi}{2}+\cos 2x\sin\frac{\pi}{2}=\cos 2x$ $\frac{1}{\sin^2\left(x+\frac{\pi}{2}\right)}=\frac{1}{\cos^2 2x}$ $\approx\left(1-\frac{(2x)^2}{2}\right)^{-2}$ $=\left(1-2x^2\right)^{-2}$ $=1+4x^2+\ldots$

	$4=2\left(\frac{3k-6}{2}\right)$ $k=\frac{10}{3}$
7	(i) $\frac{\mathrm{d}M}{\mathrm{d}t}=k\left(100^2-M^2\right)$, $k>0$ Since $\frac{\mathrm{d}M}{\mathrm{d}t}>0$ and $M>0, \Rightarrow \left(100^2-M^2\right)>0$ and $0<M<100$ $\int \frac{1}{\left(100^2-M^2\right)}\,\mathrm{d}M=\int k\,\mathrm{d}t$ $\frac{1}{200}\ln\left(\frac{100+M}{100-M}\right)=kt+C$ $\ln\left(\frac{100+M}{100-M}\right)=200kt+C'$ $\frac{100+M}{100-M}=A\mathrm{e}^{200kt}$, where $A=\mathrm{e}^{C'}$ When $t=0,\ M=5 \Rightarrow A=\frac{105}{95}=\frac{21}{19}$ When $t=5,\ M=20 \Rightarrow \frac{3}{2}=\frac{21}{19}\mathrm{e}^{1000k}$ $\mathrm{e}^{1000k}=\frac{19}{14}$ *or* $200k=\frac{1}{5}\ln\left(\frac{19}{14}\right)$ Thus $\frac{100+M}{100-M}=\frac{21}{19}\left(\mathrm{e}^{1000k}\right)^{\frac{t}{5}}=\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}$ $100+M=\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}(100-M)$ $M\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1\right]=100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]$ $M=\frac{100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]}{\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1}$ *OR* $\frac{100\left[21\left(\frac{19}{14}\right)^{\frac{t}{5}}-19\right]}{21\left(\frac{19}{14}\right)^{\frac{t}{5}}+19}$ *OR* $\frac{100\left[\left(\frac{19}{14}\right)^{\frac{t}{5}}-\frac{19}{21}\right]}{\left(\frac{19}{14}\right)^{\frac{t}{5}}+\frac{19}{21}}$ (ii) When t = 15, $M=\frac{100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{3}-1\right]}{\frac{21}{19}\left(\frac{19}{14}\right)^{3}+1}=46.847$ $M\approx 47$ (nearest whole number)

	(iii) **<u>Method 1:</u> Graphical Method** Sketch the graphs of M=f(t) and M=80 From the graph, when $t > 34.336397$, $M > 80$ Least number of days required is 35. **<u>Method 2:</u> Use GC table** When $t = 34$, $M = 79.627 < 80$ When $t = 35$, $M = 80.718 > 80$ When $t = 36$, $M = 81.756 > 80$ $\Rightarrow t \geq 35$ Thus least number of days required is 35. **<u>Method 3:</u>** $\dfrac{100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]}{\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1} > 80$ $\frac{5}{4}\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right] > \frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1$ $\frac{1}{4}\cdot\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}} > \frac{9}{4}$ $\left(\frac{19}{14}\right)^{\frac{t}{5}} > \frac{57}{7}$ $t > \dfrac{5\ln\left(\frac{57}{7}\right)}{\ln\left(\frac{19}{14}\right)} = 34.336397$ Least number of days required is 35.
8	(i)  Range of f is $[-1, 3]$ or $R_f = [-1, 3]$ or $R_f = \{y : -1 \leq y \leq 3\}$

(ii)



(iii)

$$\int_{-6}^{-4} \mathrm{f}(-x)\,\mathrm{d}x = \int_{4}^{6} \mathrm{f}(x)\,\mathrm{d}x$$

$$= \text{area of rectangle}$$

$$= 2$$

9

$$\mathrm{f}(x) = \sin 2x + \cos 2x$$

$$R = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\tan\alpha = 1 \quad \Rightarrow \quad \alpha = \frac{\pi}{4}$$

$$\mathrm{f}(x) = \sin 2x + \cos 2x = \sqrt{2}\sin\left(2x + \frac{\pi}{4}\right)$$

(i)

Transforming $y = \sin x$ to $y = \sqrt{2}\sin\left(2x + \frac{\pi}{4}\right)$

Sequence of Transformation:

Either

A: A translation of $\frac{\pi}{4}$ units in the negative *x*-direction

B: A scaling/stretch with scale factor $\frac{1}{2}$ parallel to the *x*-axis.

C: A scaling/stretch with scale factor $\sqrt{2}$ parallel to the *y*-axis.

Acceptable sequence: ABC, ACB, CAB.

OR $y = \sqrt{2}\sin\left[2\left(x + \frac{\pi}{8}\right)\right]$

D: A scaling/stretch with scale factor $\frac{1}{2}$ parallel to the *x*-axis.

E: A translation of $\frac{\pi}{8}$ units in the negative *x*-direction.

F: A scaling/stretch with scale factor $\sqrt{2}$ parallel to the *y*-axis.

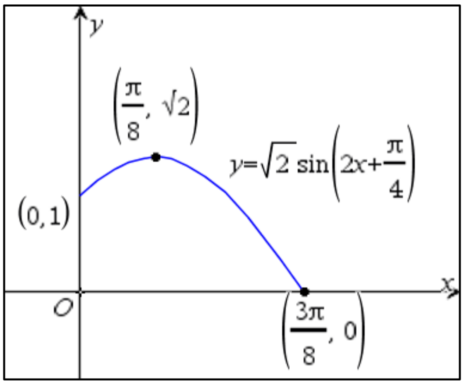
Acceptable sequence: DEF, DFE, FDE

(ii)

Max point occurs when $\sin\left(2x+\frac{\pi}{4}\right)=1$

$\Rightarrow \quad \left(2x+\frac{\pi}{4}\right)=\frac{\pi}{2}$

$\Rightarrow \quad x=\frac{\pi}{8}, y=\sqrt{2}$

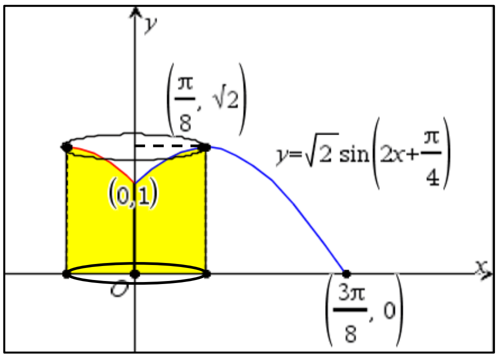


(iii)

$y=\sqrt{2}\sin\left(2x+\frac{\pi}{4}\right)$

The curve is one-one for $0\le x\le\frac{\pi}{8}$,

thus inverse function exists.



$\sin\left(2x+\frac{\pi}{4}\right)=\frac{y}{\sqrt{2}}$

$2x+\frac{\pi}{4}=\sin^{-1}\frac{y}{\sqrt{2}}$

$x=\frac{1}{2}\left[\sin^{-1}\left(\frac{y}{\sqrt{2}}\right)-\frac{\pi}{4}\right]$

Volume = Volume of cylinder - $\pi\int_1^{\sqrt{2}} x^2 \ \mathrm{d}y$

$=\pi\left(\frac{\pi}{8}\right)^2\sqrt{2}-\pi\int_1^{\sqrt{2}}\frac{1}{4}\left[\sin^{-1}\left(\frac{y}{\sqrt{2}}\right)-\frac{\pi}{4}\right]^2 \ \mathrm{d}y$

$=0.6506458$

≈ 0.6506 (4 d.p.)

10 (i)

Let the foot of perpendicular be N.

<u>Method 1</u>

Equation of the line that passes through A and perpendicular to p_1 is

$l_A : \mathbf{r}=\begin{pmatrix}6c\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix}, \lambda\in\mathbb{R}$.

Since N lies on l_A, $\overrightarrow{ON}=\begin{pmatrix}6c\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix}$, for some $\lambda\in\mathbb{R}$.

$$\left[\begin{pmatrix}6c\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix}\right]\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}=8$$

$$6c+2+6\lambda=8$$

$$\lambda=1-c$$

$$\overrightarrow{ON}=\begin{pmatrix}6c\\0\\2\end{pmatrix}+(1-c)\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}1+5c\\2-2c\\3-c\end{pmatrix}$$

Hence, N is the point $\left(1+5c,\,2-2c,\,3-c\right)$.

<u>Method 2</u>

Let C denote the point (0, 4, 0). Then C lies on p_1 since

LHS of eqn. of p_1 = 0 + 8+0 = 8 = RHS of eqn. of p_1

$$\overrightarrow{AC}=\begin{pmatrix}0\\4\\0\end{pmatrix}-\begin{pmatrix}6c\\0\\2\end{pmatrix}=\begin{pmatrix}-6c\\4\\-2\end{pmatrix}$$

$$\overrightarrow{AN}=\frac{\begin{pmatrix}-6c\\4\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}}{\sqrt{1+4+1}}\left(\frac{\begin{pmatrix}1\\2\\1\end{pmatrix}}{\sqrt{1+4+1}}\right)$$

$$=(1-c)\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$\therefore\overrightarrow{ON}=\overrightarrow{OA}+\overrightarrow{AN}=\begin{pmatrix}6c\\0\\2\end{pmatrix}+(1-c)\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}1+5c\\2-2c\\3-c\end{pmatrix}$$

Hence, N is the point $\left(1+5c,\,2-2c,\,3-c\right)$.

(ii)

Let **b** be the position vector of point B.

By Ratio Theorem, $\begin{pmatrix}6c\\0\\2\end{pmatrix}+\mathbf{b}=2\begin{pmatrix}1+5c\\2-2c\\3-c\end{pmatrix}$

$$\mathbf{b}=2\begin{pmatrix}1+2c\\2-2c\\2-c\end{pmatrix}$$

	Since B lies in p_2, $$2\begin{pmatrix}1+2c\\2-2c\\2-c\end{pmatrix}\cdot\begin{pmatrix}3\\3\\-2\end{pmatrix}=4$$ $$(3+6-4)+c(6-6+2)=2$$ $$5+2c=2$$ $$c=-\frac{3}{2}$$ (iii) Using GC, $$l:\mathbf{r}=\begin{pmatrix}-\frac{16}{3}\\\frac{20}{3}\\0\end{pmatrix}+\lambda\begin{pmatrix}7\\-5\\1\end{pmatrix},\lambda\in\mathbb{R}.$$ (iv) If all the three planes meet in *l*, and *l* lies in p_3. I.e The direction vector of *l* is perpendicular to the normal vector of p_3. $$\begin{pmatrix}m\\0\\1\end{pmatrix}\cdot\begin{pmatrix}7\\-5\\1\end{pmatrix}=0$$ $$7m+1=0$$ $$m=-\frac{1}{7}$$ $$\begin{pmatrix}-\frac{16}{3}\\\frac{20}{3}\\0\end{pmatrix}\cdot\begin{pmatrix}-\frac{1}{7}\\0\\1\end{pmatrix}=n$$ $$n=\frac{16}{21}$$ (v) Since the 3 planes have no common point, *l* must be parallel to p_3 but *l* does not lie on p_3. Thus $m=-\frac{1}{7}$ and

	$\begin{pmatrix} -\frac{16}{3} \\ \frac{20}{3} \\ 0 \end{pmatrix} \square \begin{pmatrix} -\frac{1}{7} \\ 0 \\ 1 \end{pmatrix} \neq n$ $n \neq \frac{16}{21}$
11	(i) Let l be the slant height of the cone. $l^2 = h^2 + r^2 \quad ----(1)$ Using similar triangles, $\frac{h-3}{l} = \frac{3}{r}$ $l = \frac{rh-3r}{3} \quad ----(2)$ Equating (1) and (2), $\left(\frac{rh-3r}{3}\right)^2 = h^2 + r^2 \quad ----(*)$ $r^2h^2 - 6r^2h + 9r^2 = 9h^2 + 9r^2$ $r^2(h^2 - 6h) = 9h^2$ $\therefore r = \frac{3h}{\sqrt{h^2-6h}}$ (Since $r > 0$) (ii) Volume of cone, $V = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3}\pi\left(\frac{3h}{\sqrt{h^2-6h}}\right)^2 h$ $= \frac{3\pi h^3}{h^2-6h}$ $= \frac{3\pi h^2}{h-6}$ $\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{6\pi h(h-6) - 3\pi h^2}{(h-6)^2}$ $= \frac{3\pi h^2 - 36\pi h}{(h-6)^2}$ $\frac{\mathrm{d}V}{\mathrm{d}h} = 0 \quad \Rightarrow \quad 3\pi h^2 - 36\pi h = 0$ $h(h-12) = 0$ $h = 12$ or $h = 0$ (reject $\because h > 0$)

h	12^-	12	12^+
Sign of $\frac{\mathrm{d}V}{\mathrm{d}h}$	– ve	0	+ ve
Tangent	╲	—	╱

Thus, V is a minimum when $h = 12$

When $h = 12$,

$$r = \frac{3(12)}{\sqrt{(12)^2 - 6(12)}} = \frac{6}{\sqrt{2}} \qquad (\approx 4.2426)$$

$$V = \frac{3\pi(12)^2}{12-6} = 72\pi \qquad (\approx 226.195)$$

(iii)

Let R be the radius of the snowball

$$S = 4\pi R^2 \qquad \Rightarrow \qquad \frac{\mathrm{d}S}{\mathrm{d}t} = 8\pi R\frac{\mathrm{d}R}{\mathrm{d}t}$$

$$V = \frac{4}{3}\pi R^3 \qquad \Rightarrow \qquad \frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi R^2\frac{\mathrm{d}R}{\mathrm{d}t}$$

When $R = 2.5$, $\frac{\mathrm{d}S}{\mathrm{d}t} = -0.75 \quad \Rightarrow \quad 8\pi(2.5)\frac{\mathrm{d}R}{\mathrm{d}t} = -0.75$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{3}{80\pi} \quad or \quad -\frac{0.0375}{\pi} \quad or \quad -0.0119366$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi(2.5)^2\left(-\frac{3}{80\pi}\right) = -\frac{15}{16} \quad or \quad -0.9375$$

At the instant when $R = 2.5$ m, the rate of decrease of volume is 0.9375 m^3 per minute.