Additional Practice Questions

1. Given the graph of $y = \dfrac{x^2 + a}{x + b}$, where $a > 0$ and $b > 0$,

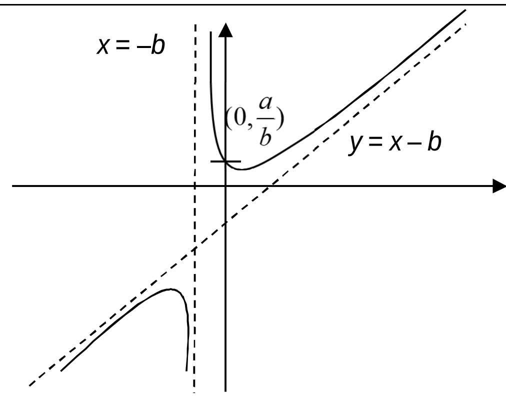
   i) State the coordinates of the intersection(s) of the graph with the axes.

When $x = 0, y = \frac{(0)^2+a}{(0)+b} = \frac{a}{b}$, so $\left(0, \frac{a}{b}\right)$ is the *y*-intercept.
When $y = 0, 0 = \frac{x^2+a}{x+b}, x^2 = -a < 0$ so there is no *x*-intercept.

   ii) Find the equations of the asymptote(s).

$x = -b$; $y = x - b$

   iii) Draw a sketch of the curve, labelling the equations of its asymptotes and coordinates of any intersection with the axes.

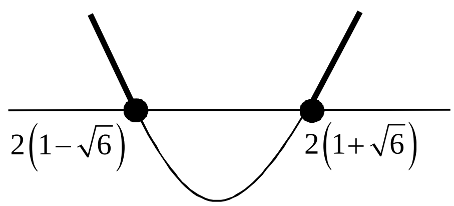
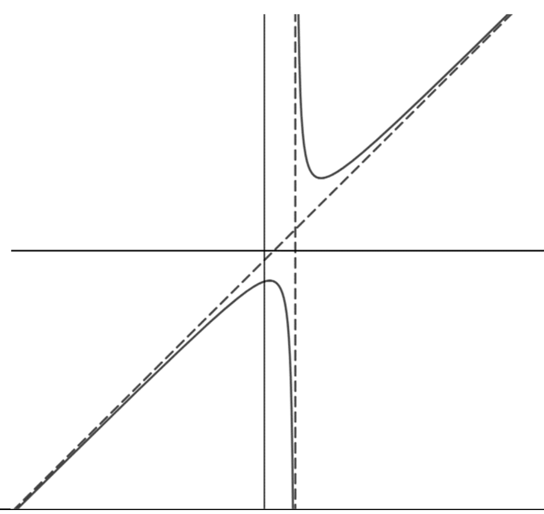


2. **[MI PU2 Promo 9758/2019/02/Q2]**

The curve C has equation $y = \dfrac{x^2 - 4x + 9}{x - 3}$.

(i) Find algebraically the set of values that y can take, leaving your answer in exact form. [4]

(ii) Sketch C, stating the coordinates of the axial intercept, turning points and the equations of any asymptotes. [3]

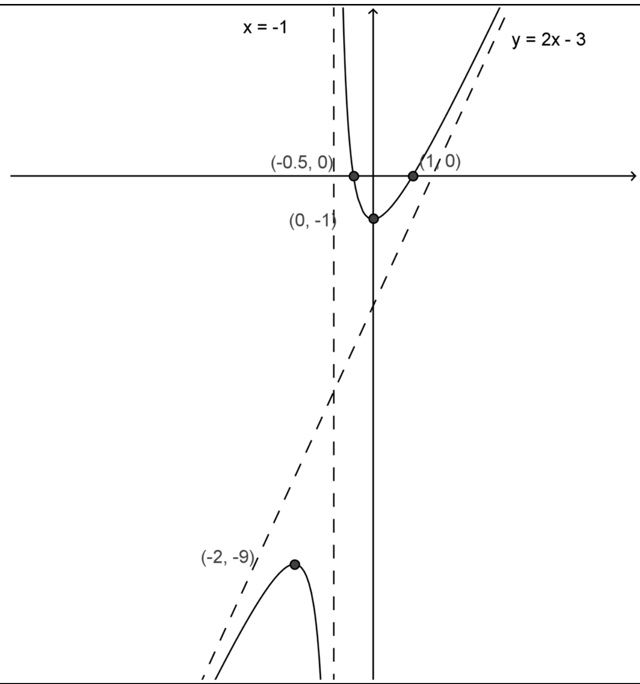
M2 JC1 Promo 9758/2019/02/Q2 (Solutions)	
2(i)	$y=\dfrac{x^2-4x+9}{x-3}$ $y(x-3)=x^2-4x+9$ $x^2-4x-xy+3y+9=0$ $x^2-(4+y)x+3y+9=0$ ..... (*) For values that y can take, the equation (*) has real roots: $b^2-4ac\ge 0$ $\left[-(4+y)\right]^2-4(1)(3y+9)\ge 0$ $(4+y)^2-12y-36\ge 0$ $y^2-4y-20\ge 0$ $y\le 2\left(1-\sqrt{6}\right)$ or $y\ge 2\left(1+\sqrt{6}\right)$ $\left\{y\in\mathbb{R}: y\le 2\left(1-\sqrt{6}\right) \text{ or } y\ge 2\left(1+\sqrt{6}\right)\right\}$ 
2(ii)	$y=\dfrac{x^2-4x+9}{x-3}=x-1+\dfrac{6}{x-3}$ Asymptotes are $y=x-1$ and $x=3$. 

3. The curve C has equation $y = \dfrac{2x^2 - x - 1}{x + 1}$.

   (i) Find the equations of the asymptotes of C.

Vertical Asymptote: $x = -1$ By long division: $y = 2x - 3 + \frac{2}{x+1}$ Therefore, oblique asymptote: $y = 2x - 3$

(ii) Draw a sketch of the curve C, making clear the main relevant features.

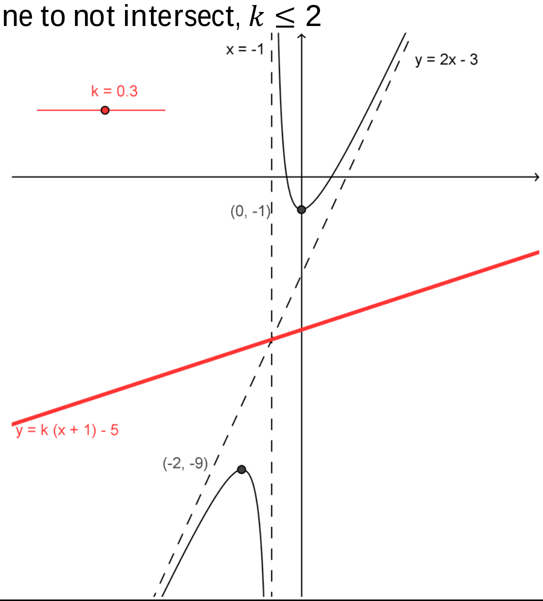


(iii) Find the range of possible values of k such that the line $y = k(x+1) - 5$ and the curve C do not intersect.

$y = k(x+1) - 5$ will pass through the point of intersection of the two asymptotes: (-1, 5)

It has gradient k.

For the curve and the line to not intersect, $k \leq 2$



(iv) Find the set of values of x for which $\mathrm{f}(x) = \ln\left(\frac{2x^2 - x - 1}{x+1}\right)$ is defined.

$f(x)$ is defined for $\frac{2x^2-x-1}{x+1} > 0$. Thus $-1 < x < -\frac{1}{2}$ or $x > 1$

4. Consider the curve $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{x^2-1}{x+2}$.

(i) State the coordinates of any points of intersection with the axes.

when $x = 0$, $y = -0.5$
when $y = 0$, $x = \pm 1$
$\therefore$ coordinates are $(0, -0.5)$, $(1, 0)$ and $(-1, 0)$

(ii) Find the equations of the asymptotes.

Vertical asymptote: $x = -2$

$$y = \frac{x^2-1}{x+2} = \frac{x(x+2)-2x-1}{x+2} = x - \left(\frac{2(x+2)-3}{x+2}\right) = x - 2 + \frac{3}{x+2}$$

Oblique asymptote: $y = x - 2$

(iii) Prove, using an algebraic method, that $\mathrm{f}(x) \geq a$ or $\mathrm{f}(x) \leq b$, where a and b are exact values to be determined.

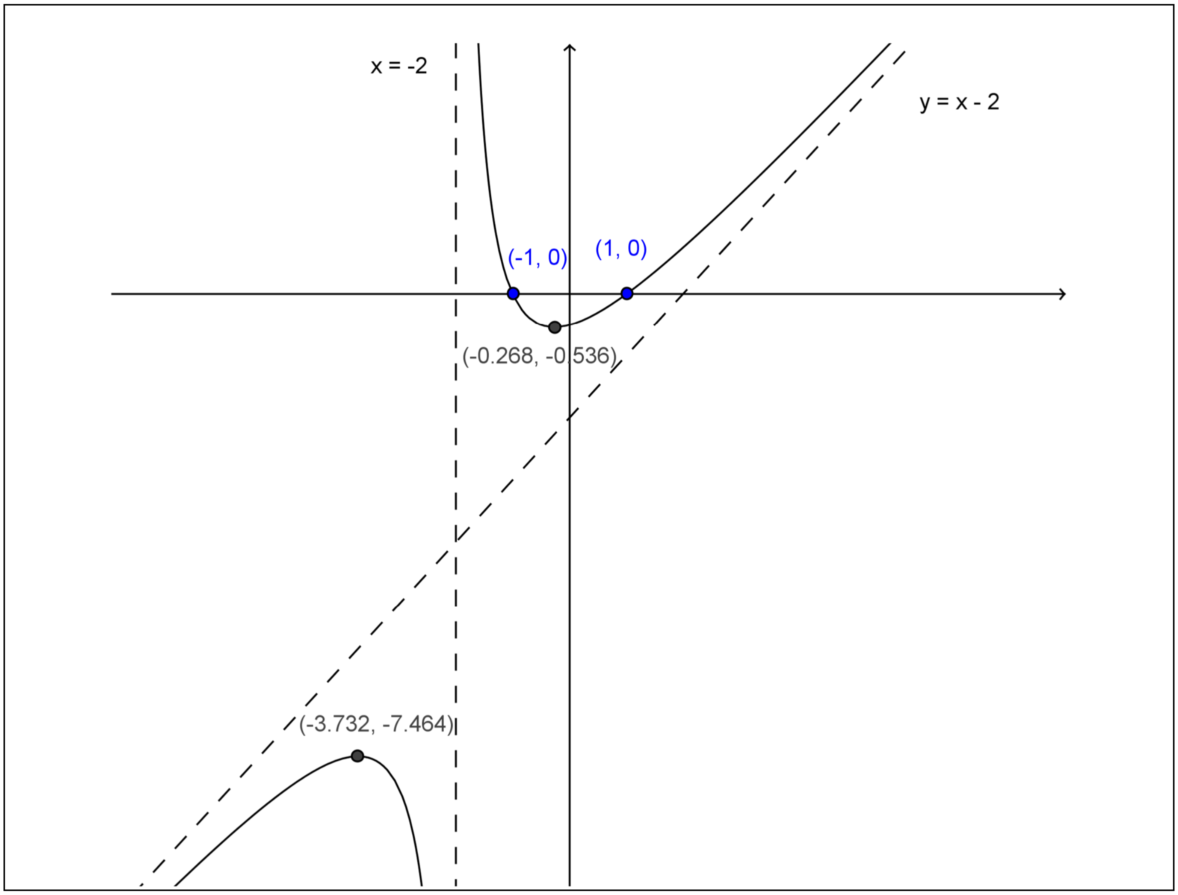
Let $y = \dfrac{x^2-1}{x+2} \Rightarrow x^2 - yx - (1+2y) = 0$

For real x, $y^2 + 4(1+2y) \geq 0 \Rightarrow y^2 + 8y + 4 \geq 0$

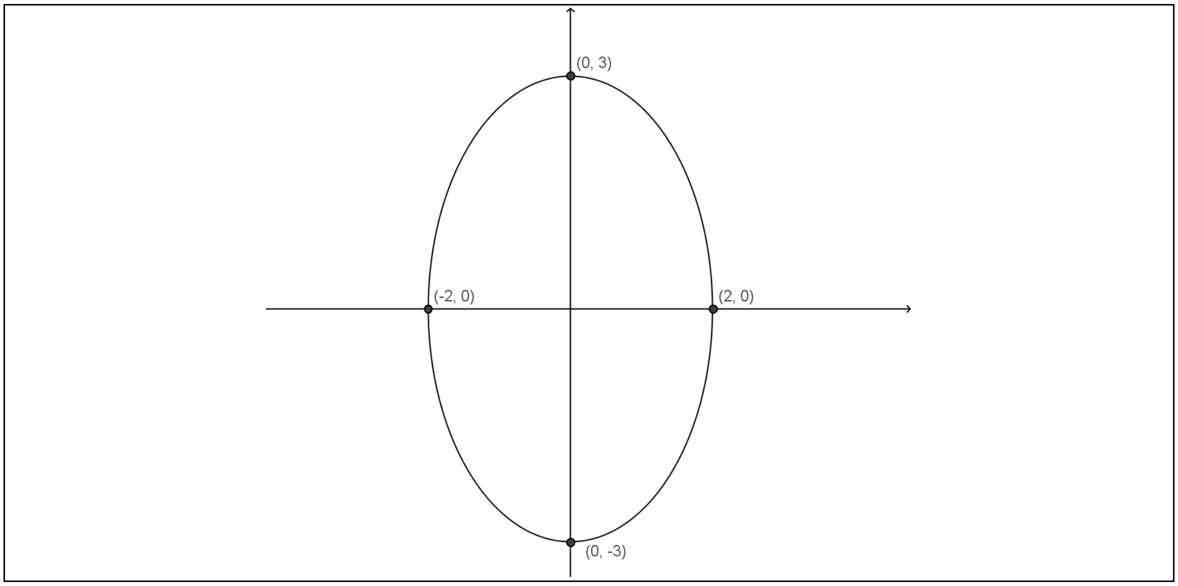
$$\Rightarrow (y+4)^2 - 12 \geq 0 \Rightarrow |y+4| \geq \sqrt{12}$$

$$\Rightarrow y \geq -4 + 2\sqrt{3} \quad \text{or} \quad y \leq -4 - 2\sqrt{3}$$

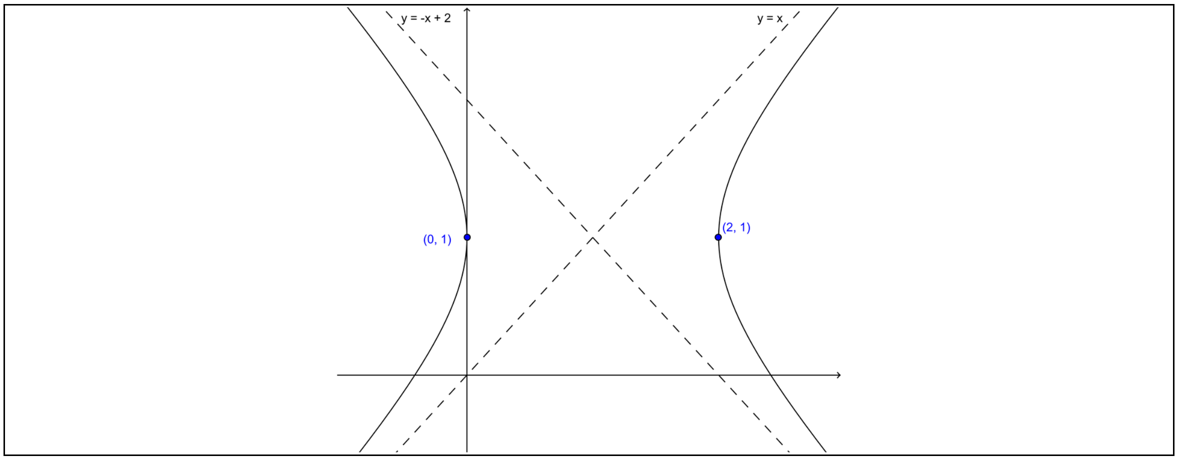
(iv) Sketch $y = \mathrm{f}(x)$, showing the main relevant features of the curve.



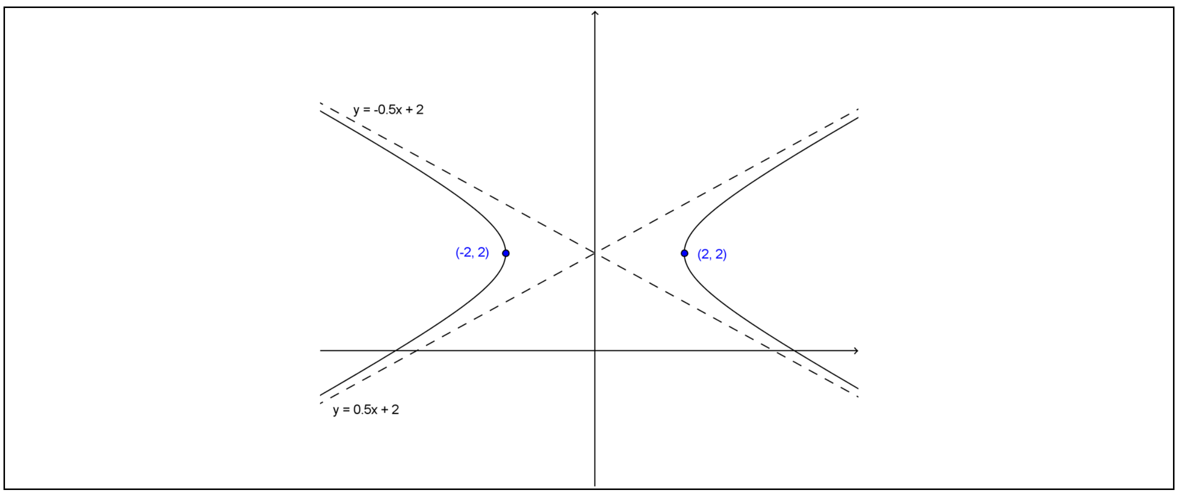
5. (a) $x = 2\sin t, y = 3\cos t \rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$



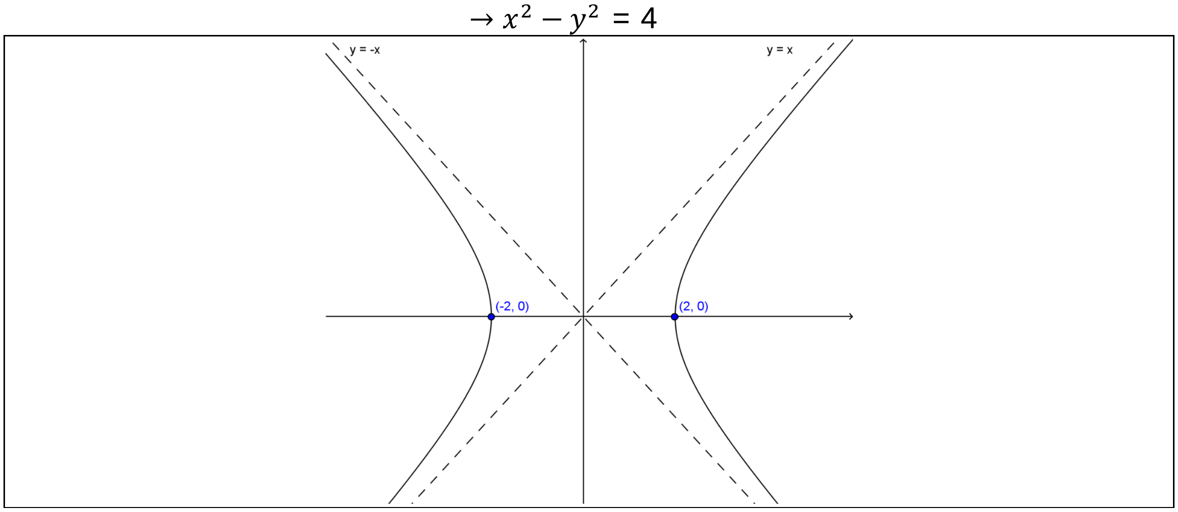
(b) $x = \operatorname{cosec}(t) + 1, y = 1 - \cot(t) \rightarrow 1 + (1-y)^2 = (x-1)^2$



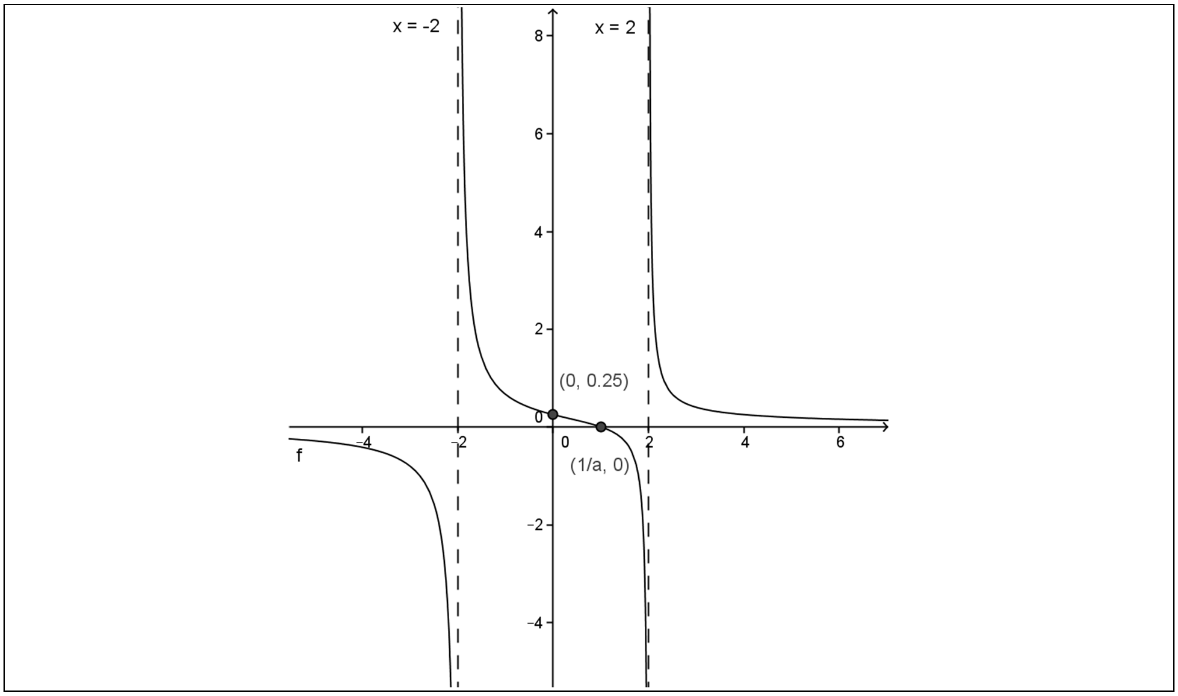
(c) $x = \frac{4\sin t}{\sin(2t)}, y = \tan t + 2 \rightarrow 1 + (y-2)^2 = \left(\frac{x}{2}\right)^2$



(d) $x = t + \frac{1}{t}, y = t - \frac{1}{t} \rightarrow x^2 = t^2 + \left(\frac{1}{t^2}\right) + 2, y^2 = t^2 + \left(\frac{1}{t^2}\right) - 2$
$\rightarrow x^2 - y^2 = 4$



6. Sketch the graph of $y=\dfrac{ax-1}{x^2-4}$, $a>\dfrac{1}{2}$, stating the exact coordinates of any points of intersection with the axes and any stationary points, and the equations of any asymptotes.



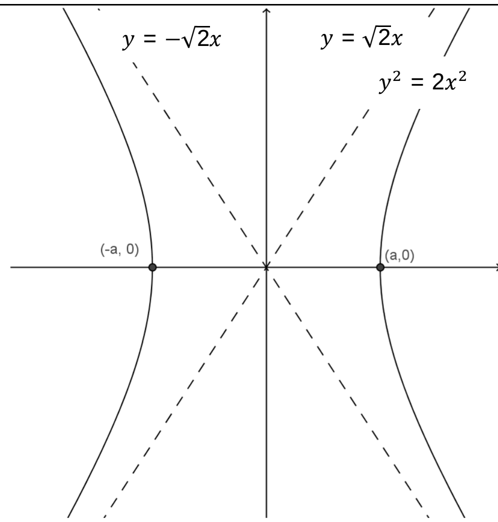
See-Think-Wonder!

Thinking Routine

(See) There is a restriction on *a*
(Think) Why?
To avoid cases where there are repeated roots in both the numerator and denominator. Also, once *a* > 0.5 or *a* < -0.5 the sign of the roots will change (i.e. they will be positive and negative in different regions than before).
(Wonder) What if the restriction were changed or removed?
The graph can look very different, depending on the values of a used.

7. The curve *C* has equation $\frac{x^2}{a^2}-\frac{y^2}{2a^2}=1$, where *a* is a positive constant.

(i) Sketch the graph of *C*, indicating clearly the coordinates of point(s) of intersection with the axes, and the equation(s) of any asymptotes. Leave your answer in terms of *a*.

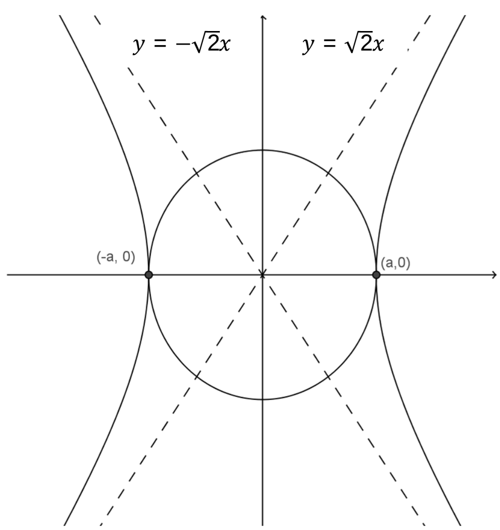


(ii) By sketching a suitable curve on the same diagram, find the solutions to the equation $\frac{x^2}{a^2}-\frac{a^2-x^2}{2a^2}=1$

Replace y^2 with $a^2 - x^2$

$$y^2 = a^2 - x^2 \rightarrow x^2 + y^2 = a^2$$

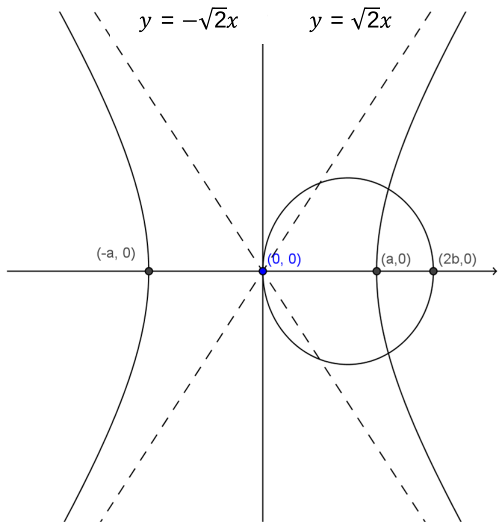
Therefore, draw a circle, centre (0,0) radius *a*.



The two curves intersect at $x = \pm a$,
therefore the solutions are $x = \pm a$

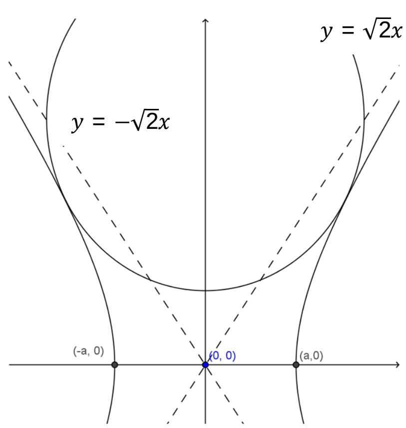
(iii) Find the range of values of b for which the graph of $\left(x-b\right)^2+y^2=b^2$ has two points of intersection with C.

$(x-b)^2+y^2=b^2$ is a circle, centre $(b,0)$ radius b.
It won't have any intersection if $|2b|<a$
It will have only one intersection if $2b=\pm a$
Therefore, $b>\frac{a}{2}$ or $b<-\frac{a}{2}$



(iv) The graph of $x^2+\left(y-b\right)^2=c^2$ has two points of intersection with C for any constant b. Find c in terms of a and b.

$x^2+(y-b)^2=c^2$ has two points of intersection with $\frac{x^2}{a^2}-\frac{y^2}{2a^2}=1$
This is a circle, with radius c, centred on $(0, b)$.
Therefore, the two points of intersection should have the same y value.



Therefore, for fixed constants a, b:

$$\frac{c^2-(y-b)^2}{a^2}-\frac{y^2}{2a^2}=1$$

Should have only one root.

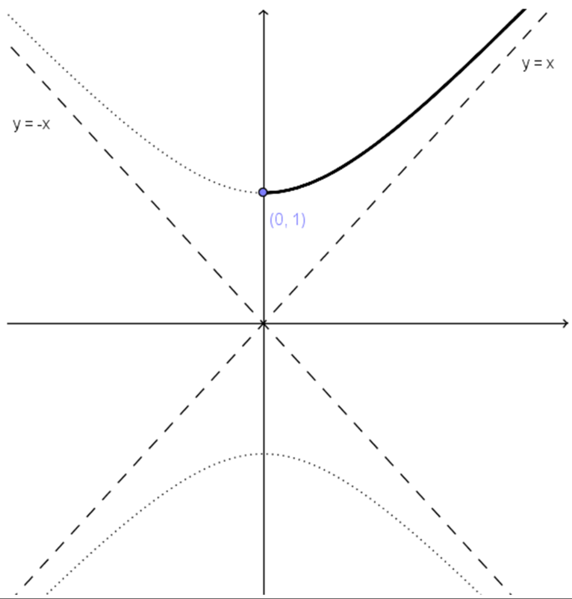
$$2c^2-2(y-b)^2-y^2=2a^2$$
$$2c^2-2(y^2-2by+b^2)-y^2-2a^2=0$$
$$-3y^2+4by+2c^2-2a^2-2b^2=0$$
$$\text{Discriminant}=0$$
$$16b^2-4(-3)(2c^2-2a^2-2b^2)=0$$
$$24\left(c^2-(a^2+b^2)\right)=-16b^2$$
$$c^2=-\frac{2}{3}b^2+(a^2+b^2)$$
$$c^2=a^2+\frac{1}{3}b^2$$
$$c=\pm\sqrt{a^2+\frac{1}{3}b^2}$$

8. Find the cartesian equations and coordinates of the intersections of the following curves with the x and y–axes (if any):

(i) $x = t^2$, $y = \sqrt{t^4 + 1}$ [2]

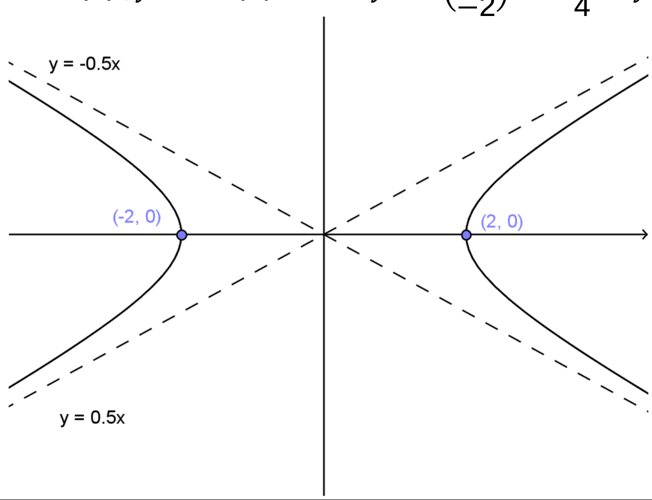
$x = t^2, y = \sqrt{t^4 + 1} \rightarrow y^2 = x^2 + 1$

Notice the restriction, $x, y \geq 0$, $y \geq 1$



(ii) $x = -2\sec\theta$, $y = \tan\theta$ [2]

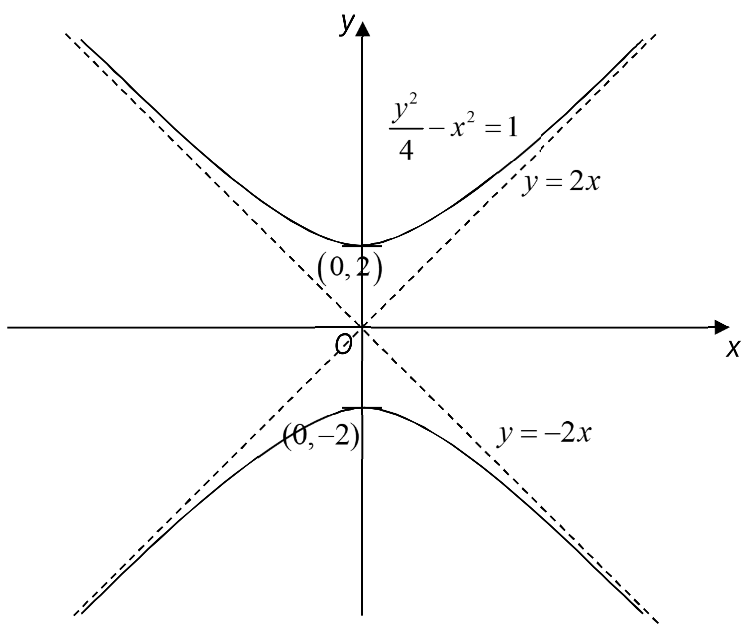
$$x = -2\sec(\theta), y = \tan(\theta) \rightarrow 1 + y^2 = \left(\frac{x}{-2}\right)^2 \rightarrow \frac{x^2}{4} - y^2 = 1$$



9. 2016/DHS/Promo/10

The curve C_1 has equation $\dfrac{y^2}{4} - x^2 = 1$ and the curve C_2 has equation $y = k - \dfrac{1}{x}$ where k is a positive constant. The horizontal asymptote of C_2 is a tangent to C_1.

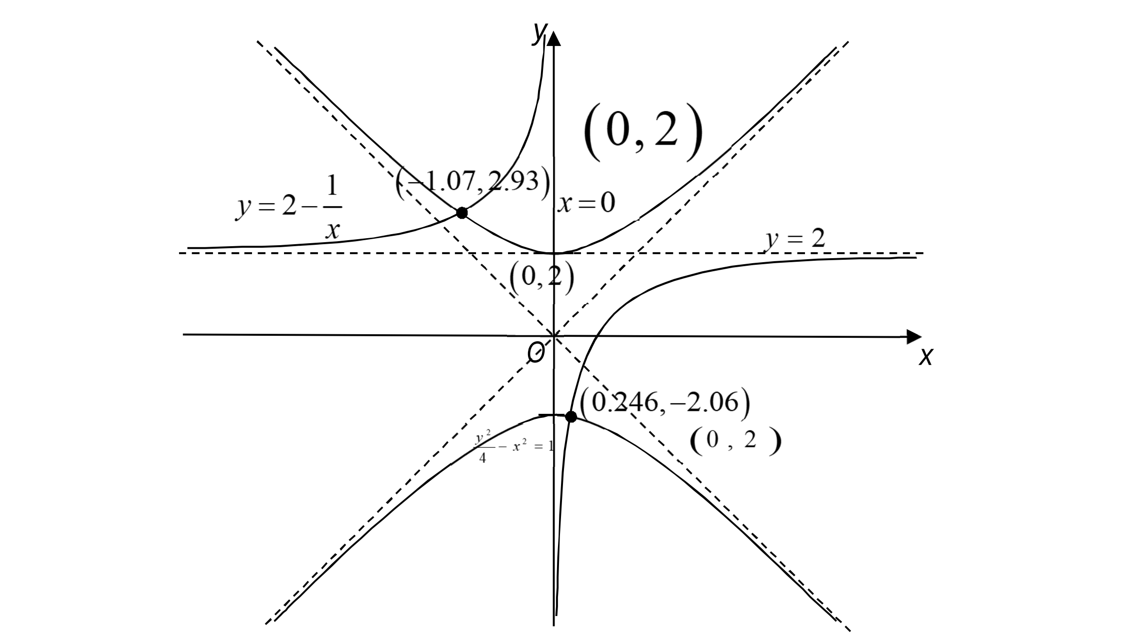
(i) Sketch C_1, giving the equations of any asymptotes and the coordinates of any turning points. [3]



(ii) Explain why $k = 2$. [1]

$y = a$ is a line with gradient 0. Since it is tangent to C_1, it must be tangent to the points at $x = 0$, which has tangent $y = 2$ at $(0, 2)$ and tangent $y = -2$ at $(0, -2)$. Since a is positive, hence the tangent line involved must be $y = 2$, i.e. $a = 2$.

(iii) On the same diagram as C_1, sketch C_2 showing the coordinates of the points of intersection of C_1 and C_2. [3]



(iv) It is given that if $[f(x)]^2 \le g(x)$, then $-\sqrt{g(x)} \le f(x) \le \sqrt{g(x)}$. Using this result and the graphs in part **(iii)**, solve the inequality $\left(2-\frac{1}{x}\right)^2 \le 4(x^2+1)$. [2]

$$\left(2-\frac{1}{x}\right)^2 \le 4\left(x^2+1\right)$$

Using the given result with $f(x) = 2-\frac{1}{x}$ and $g(x) = 4\left(x^2+1\right)$,

$$-2\sqrt{x^2+1} \le 2-\frac{1}{x} \le 2\sqrt{x^2+1}$$

From graph in (iii), $x \le -1.07$ or $x \ge 0.246$.

10. 2019/NJC/J1 MYE/8

The curve C_1 has equation $y = \frac{x^2-3ax+9}{x+a}$, $a > 0$.

(i) Find, in terms of a, the equations of the asymptotes of C_1. [2]

(ii) Given that C_1 has exactly one x-intercept, show that $a = 2$. [3]

The curve C_2 has parametric equations $x = \tan\theta$, $y = \frac{9}{2}\sin^2\theta$, $0 \le \theta < \frac{\pi}{2}$. It is given that the gradient of C_2 is zero at the point where $x = 0$.

(iii) What can be said about the values of x and y for C_2 as $\theta \to \frac{\pi}{2}$? [1]

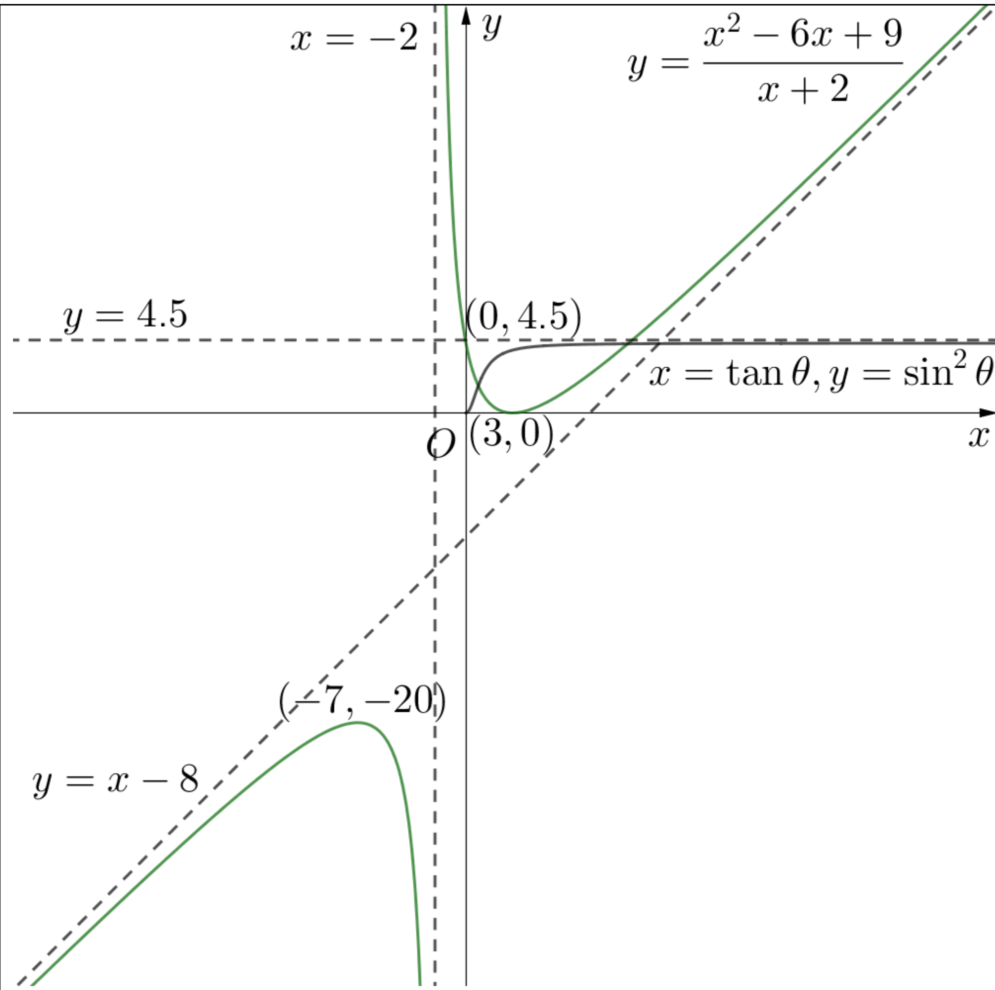
(iv) Sketch C_1 and C_2 on the same diagram, stating the equations of any asymptotes, and the coordinates of turning points and of points where the curves cross the axes. [4]

(v) Find the coordinates of the points of intersection of C_1 and C_2. [3]

Solution:

(i)	$y=\dfrac{x^2-3ax+9}{x+a}$ $y=x-4a+\dfrac{9+4a^2}{x+a}$ The equations of asymptotes are $y=x-4a$ and $x=-a$
(ii)	For C_1 to have exactly one x-intercept, there is only one solution to the equation $\dfrac{x^2-3ax+9}{x+a}=0$. $x^2-3ax+9=0$ Discriminant = 0 $(-3a)^2-4(a)(9)=0$ $9a^2=36$ $a^2=4$ since $a>0$ $a=2$ (shown)
(iii)	As $\theta\to\dfrac{\pi}{2}$, $x\to+\infty$ and $y\to4.5$.

(iv)



(v)

$$\sin^2\theta+\cos^2\theta=1$$

$$\sin^2\theta\left(1+\frac{1}{\tan^2\theta}\right)=1$$

Substitute $x=\tan\theta,\ y=4.5\sin^2\theta$ to find the Cartesian equation of C_2 .

$$\frac{y}{4.5}\left(1+\frac{1}{x^2}\right)=1$$

$$y=4.5\left(1+\frac{1}{x^2}\right)^{-1}=\frac{9x^2}{2\left(1+x^2\right)}$$

By G.C., the coordinates of points of intersection are
(0.794, 1.74) and (10.5, 4.46)

Alternatively,

Substitute $x=\tan\theta,\ y=4.5\sin^2\theta$ into C_1

$$\frac{9}{2}\sin^2\theta=\frac{\tan^2\theta-6\tan\theta+9}{\tan\theta+2}$$

By G.C., $\theta=0.67129$ or 1.47541.

	Hence, substituting into C_2, we have (0.794, 1.74) and (10.5, 4.46).

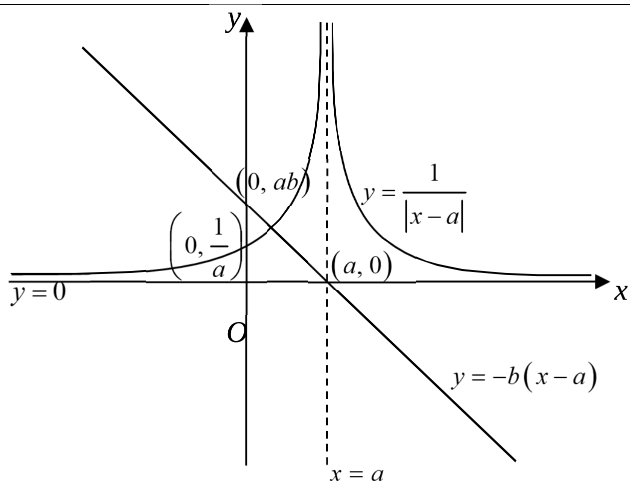
11. 2019/MI/J2 Prelim/5

(i) On the same axes, sketch the graphs of $y=\dfrac{1}{|x-a|}$ and $y=-b(x-a)$, where a and b are positive constants and $ab>\dfrac{1}{a}$, stating clearly any axial intercepts and equations of any asymptotes. [3]

(ii) Given that the solution to the inequality $\dfrac{1}{|x-a|}>-b(x-a)$ is $\dfrac{1}{2}<x<1$ or $x>1$, find the values of a and b. [4]

(iii) Using the values of a and b found in part **(ii)**, write down the solution to the inequality $\dfrac{1}{x-a}>-b(x-a)$.

(i)



(ii) **Method 1**:

From graph in **(i)**, one of the solution to $\dfrac{1}{|x-a|}>-b(x-a)$ is $x>a$.

By observation, since $x>1$, $a=1$

When $x=\dfrac{1}{2}$, $y=\dfrac{1}{\left|\frac{1}{2}-1\right|}=2$

Subst $x=\dfrac{1}{2}$ and $y=2$ into $y=-b(x-1)$

$2=-b\left(\dfrac{1}{2}-1\right)$

$b=4$

	----- **Method 2**: From graph in **(i)**, one of the solution to $\frac{1}{\|x-a\|} > -b(x-a)$ is $x > a$. By observation, since $x > 1$, $a = 1$ To find the *x*-coordinate of the point of intersection: $\frac{1}{-(x-1)} = -b(x-1)$ $(x-1)^2 = \frac{1}{b}$ $x-1 = \pm\frac{1}{\sqrt{b}}$ When $x = \frac{1}{2}$, $\frac{1}{2} - 1 = \pm\frac{1}{\sqrt{b}}$ $-\frac{1}{2} = -\frac{1}{\sqrt{b}}$ or $-\frac{1}{2} = \frac{1}{\sqrt{b}}$ (rej. since $\sqrt{b} > 0$) $\sqrt{b} = 2$ $b = 4$
(iii)	$x > 1$

12. **CJC 2020 Promo 9758/2020/Q8**

The curve C has equation

$$y = \frac{3x^2 - 2x - 1}{x+1} \text{ for } x \in \mathbb{R}, x \neq -1.$$

(i) Find, using an algebraic method, the range of values that y cannot take. [3]

(ii) Sketch C, stating clearly the equations of asymptotes, the coordinates of the axial intercepts and the coordinates of the turning points. [4]

(iii) Verify that $(-1, -8)$ lies on the graph $y = m(x+1) - 8$, where $m \in \mathbb{R}$. Hence find the range of values of m for which the equation $\frac{3x^2 - 2x - 1}{x+1} = m(x+1) - 8$ has 2 distinct real roots. [2]

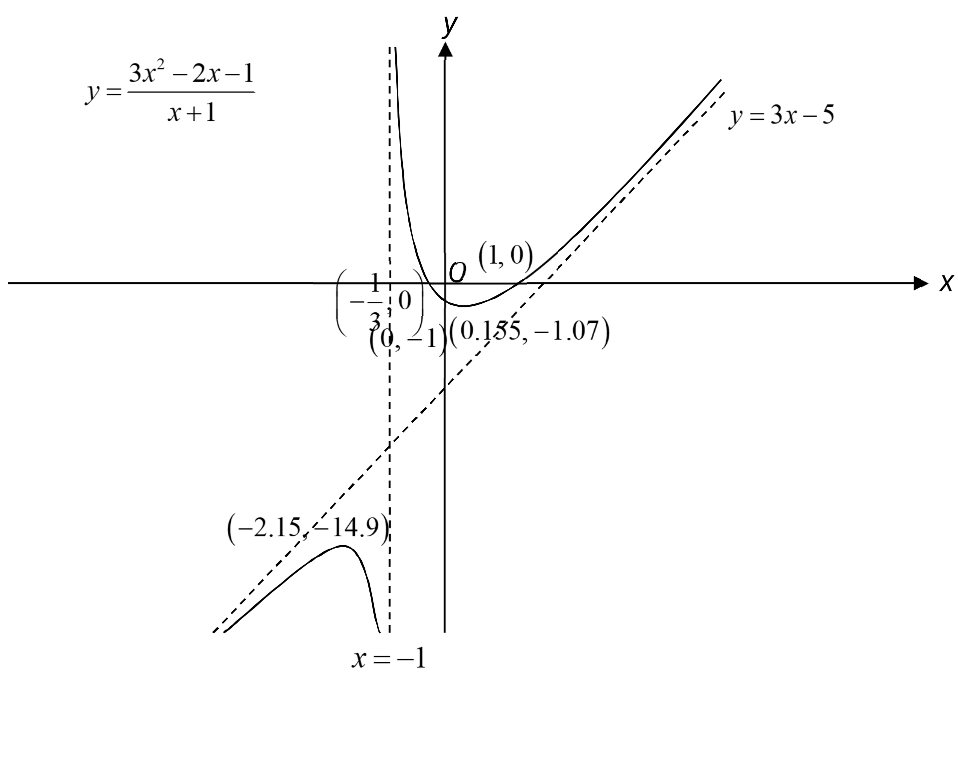
(i)	$y=\dfrac{3x^2-2x-1}{x+1}$ $yx+y=3x^2-2x-1$ $3x^2+(-2-y)x+(-1-y)=0$ Consider $b^2-4ac<0$, $(-2-y)^2-4(3)(-1-y)<0$ $4+4y+y^2+12+12y<0$ $y^2+16y+16<0$ Let $y^2+16y+16=0$

Method 1	Method 2	Method 3
From GC, $y=-14.9$ or -1.07	$y=\dfrac{-16\pm\sqrt{16^2-4(1)(16)}}{2}$ $=\dfrac{-16\pm\sqrt{192}}{2}$ $=-8\pm4\sqrt{3}$	$y^2+16y+16=0$ $(y+8)^2-64+16=0$ $(y+8)^2=48$ $y+8=\pm4\sqrt{3}$ $y=-8\pm4\sqrt{3}$

$\therefore -14.9<y<-1.07$ OR

$-8-4\sqrt{3}<y<-8+4\sqrt{3}$

(ii)



(iii)	When $x=-1$, $y=m(-1+1)-8$ $=-8$ $\therefore\ (-1,\ -8)$ lies on the graph. $m>3$

13.	**NYJC Promo 9758/2020/Q10 (part)**

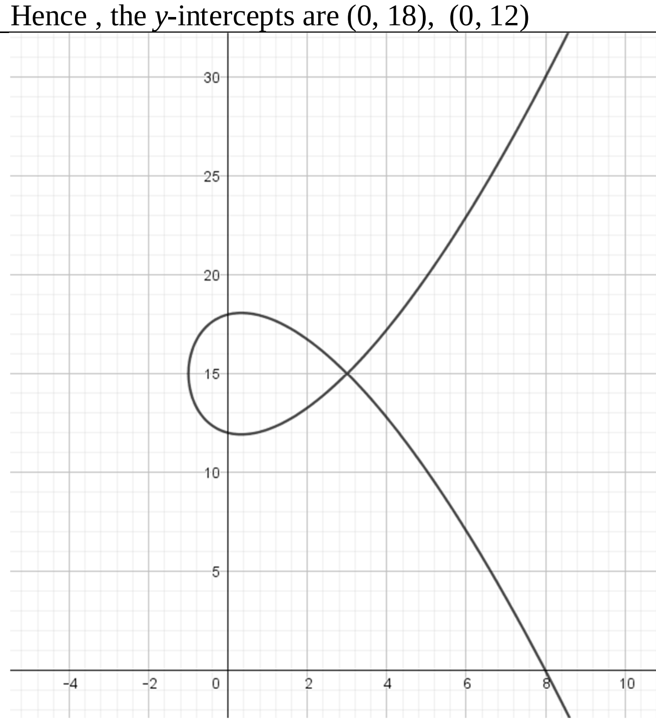
A curve C has parametric equation

$$x=t^2-1, \qquad y=4t-t^3+15, \quad \text{where } t\in\mathbb{R}.$$

(i) Show that the curve C cuts the x-axis at the point (8, 0) and the y-axis at (0, 12) and (0, 18). [3]

(ii) Sketch C, giving the equation of the line of symmetry. [2]

(iii) Find the exact coordinates of the point where C crosses itself. [2]

(i)	Show that the curve C cuts the x-axis at (8, 0). Let $y=0$. $y=4t-t^3+15=0 \quad \Rightarrow \quad t=3$. Thus $x=3^2-1=8$. Hence, the curve C cuts the x-axis at (8, 0). To find the y-intercepts, we let x = 0, then $x=t^2-1=0 \quad \Rightarrow \quad t=1, \quad -1$. When t = 1, $y=4t-t^3+15=4-1+15=18$ When $t=-1$, $y=4t-t^3+15=-4+1+15=12$ Hence , the y-intercepts are (0, 18), (0, 12)
(ii)	

	When $t = 0$, $y = 4t - t^3 + 15 = 15$. Equation of the symmetric axis is: $y = 15$.
(iii)	Observe that the point where C crosses itself lies on the symmetric axis with equation $y = 15$, thus its y-ordinate is 15. Substitute $y = 15$ into $y = 4t - t^3 + 15$, $15 = 4t - t^3 + 15 \Rightarrow t(4 - t^2) = 0 \Rightarrow t = 0, \quad 2, \quad -2$ When $t = 0$, $x = t^2 - 1 = -1$, $y = 15$, $(-1, 15)$ is not the required point. When $t_1 = \pm 2$, $x = t^2 - 1 \Rightarrow x = (\pm 2)^2 - 1 = 3$ Thus, the curve crosses itself at $(3, 15)$ **Alternately**, Let $x = t_1^2 - 1 = t_2^2 - 1$ -----------------------(1) and $y = 4t_1 - t_1^3 + 15 = 4t_2 - t_2^3 + 15$ -------------(2) where $t_1 \neq t_2$. From (1), $t_1 = t_2$, $-t_2$ Obviously, we reject $t_1 = t_2$. Substitute $t_1 = -t_2$ into (2), $4(-t_2) - (-t_2)^3 = 4t_2 - t_2^3 \Rightarrow 8t_2 - 2t_2^3 = 0 \Rightarrow 2t_2(4 - t_2^2) = 0$ As $t_2 \neq 0$, $t_2^2 = 4 \Rightarrow t_2 = \pm 2$. Hence $t_1 = \mp 2$. From (1), $x = (\pm 2)^2 - 1 = 3$ From (2), $y = 4(2) - (2)^3 + 15 = 15$ or $y = 4(-2) - (-2)^3 + 15 = 15$ Thus, the curve crosses itself at $(3, 15)$.