Chapter 2 KINEMATICS

Content

- Rectilinear motion
- Non-linear motion

Learning Outcomes

Candidates should be able to:

(a) define displacement, speed, velocity and acceleration.

(b) use graphical methods to represent distance, displacement, speed, velocity and acceleration.

(c) identify and use the physical quantities from the gradients of displacement-time graphs and areas under and gradients of velocity-time graphs, including cases of non-uniform acceleration.

(d) derive, from the definitions of velocity and acceleration, equations which represent uniformly accelerated motion in a straight line.

(e) solve problems using equations which represent uniformly accelerated motion in a straight line, including the motion of bodies falling in a uniform gravitational field without air resistance.

(f) describe qualitatively the motion of bodies falling in a uniform gravitational field with air resistance.

(g) describe and explain motion due to a uniform velocity in one direction and a uniform acceleration in a perpendicular direction.

2 Newtonian Mechanics

Newtonian mechanics is a successful physical theory that explains the relationship between force and motion. Mechanics is the study of how objects move (kinematics) and of the reason why objects move in the way they do (dynamics).

The study of kinematics begins with the introduction of precise terminology and language for describing motion, to reduce ambiguity in expression and confusion in thought. One-dimensional motion is introduced and discussed with the (verbal, mathematical, graphical) language of kinematics before more complex two-dimensional motions such as projectile motion and circular motion are studied. To scope the syllabus, we restrict ourselves to modelling the motion of bodies where effects such as the rotation or even the change in shape of the body are insignificant, and hence such bodies are assumed to be well-described as point objects.

The study of dynamics is grounded on Newton's three laws of motion, which accurately model systems as diverse as the planets of the solar system and helium atoms in a container. Forces play a central role in Newton's laws of motion. However, experiments and observations have proven that the validity of Newtonian mechanics breaks down for objects moving close to the speed of light, or objects at the sub-atomic scale. In these situations, special relativity and quantum mechanics respectively are the more appropriate physical theories that apply.

The concept of energy is one of the most fundamental concepts in science, and is discussed in the context of Newtonian mechanics. Energy is present in various forms, with endless conversion from one form to another. The conservation of energy is an essential principle in Physics. The concept of work links energy and force, as work is a means of energy conversion through the application of a force. In certain situations, the concepts of work and energy can be applied to solve the dynamics of a mechanical system without directly resorting to Newton's laws. Beyond mechanics, this problem-solving approach focusing on energy can be applied to a wide range of phenomena in electromagnetism, and thermal and nuclear physics. The work-energy approach often provides a much simpler analysis than that obtained from the direct application of Newton's laws, since the former deals with scalar rather than vector quantities.

Newtonian mechanics is the foundation of contemporary science and is also the basis for much engineering and applied science. It is of paramount importance to a civil engineer to know the effects of forces acting on a structure such as a bridge. Someone designing a vehicle to break the world speed record had better be conversant in the concepts of force and energy to stand any chance in the competition. While relativistic corrections have become increasingly important for space science, the principles of Newtonian mechanics are still largely used in space missions to the moon and launching satellites into orbit around the Earth.

2.1 Introduction

Kinematics

The study of the motion of objects, with the associated concepts of force and energy, is called *mechanics*.

Mechanics can be further divided into two parts:

1. *kinematics* which describe how objects move and
2. *dynamics* which deals with force and why objects move as they do.

Motion can be categorized into three types:

- translational,
- rotational and
- vibrational.

In this chapter, we are concerned only with translational motion and will treat the moving object as a *particle*, regardless of its size.

Strictly speaking, a particle is a point-like object with mass but no size. However, we can still apply the particle model to objects such as a ball or a car, provided the positions of the objects refer to their centres of mass.

We begin our study with *rectilinear motion*, which is motion in one dimension or motion in a straight line, and then proceed to projectile motion, which is an example of motion in a two-dimensional plane.

Circular motion is another example of motion in a plane and that will be covered in a later chapter.

Frame of Reference and Coordinate System

All measurements of distance or speed are made relative to a frame of reference. Unless otherwise specified, the frame of reference is assumed to be that of a stationary observer on the Earth. Once the frame of reference is specified, it is then represented by a coordinate system.

For motion in a plane, a pair of mutually perpendicular axes is chosen. The most common pair of axes is the horizontal and vertical axes, usually labelled as the *x*- and *y*- axes respectively.

The point of launch or start of motion is usually designated as the origin and the directions for the *x*- and *y*- axes are chosen arbitrarily from the origin.

2.2 Defining the Basics

Important Quantities

Definition

Scalar	**Vector**
Distance, *x*	**Displacement, *s***
The total length of path an object travels. SI unit is *metre* (m).	The distance moved in a specified direction from a reference point. SI unit is *metre* (m).

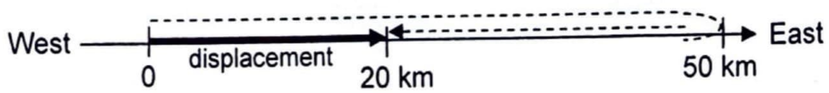
Definition

Speed, *v*	**Velocity, *v***
The **instantaneous speed** of an object is defined as the rate of change of <u>distance</u> travelled with respect to time. $v = \frac{dx}{dt}$ SI unit is *metre per second* (m s^{-1}).	The **instantaneous velocity** of an object is defined as the rate of change of <u>displacement</u> with respect to time. $v = \frac{ds}{dt}$ SI unit is *metre per second* (m s^{-1}).
Average speed refers to the total distance travelled over total time taken. $\langle v \rangle = \frac{\text{total distance travelled}}{\text{total time taken}} = \frac{\Delta x}{\Delta t}$	**Average velocity** refers to the change in displacement over total time taken. $\langle v \rangle = \frac{\text{change in displacement}}{\text{total time taken}} = \frac{\Delta s}{\Delta t}$

Definition

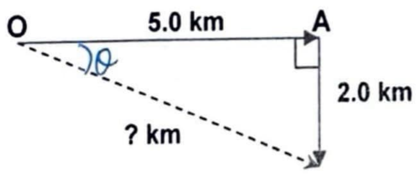
	Acceleration, *a*
No scalar equivalent of acceleration.	The **instantaneous acceleration** of an object is defined as the rate of change of velocity with respect to time. $a = \frac{dv}{dt}$ SI unit is *metre per second squared* (m s^{-2}).
	Average acceleration refers to the change in velocity over time taken. $\langle a \rangle = \frac{\text{change in velocity}}{\text{time taken}} = \frac{\Delta v}{\Delta t}$

Example 1 Consider a car moving 50 km from West to East and then 30 km from East to West.



The total distance it has travelled is 80 km but its displacement is 20 km due East of its starting point.

Example 2 A car travels 5.0 km due East from a point O to a point A and then a further 2.0 km southward. What is the distance covered and the magnitude of its displacement?



Distance covered = 7.0 km
Displacement = $\sqrt{(5^2 + 2^2)}$
= 5.4 km

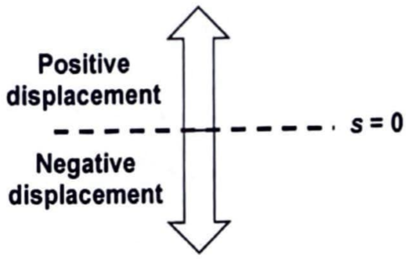
Example 3 An athlete runs at a steady speed clockwise round a running track of perimeter 400 m and completes one round in 1 min 20 s. Calculate
(a) her average speed for one round,
(b) her average velocity for one round.

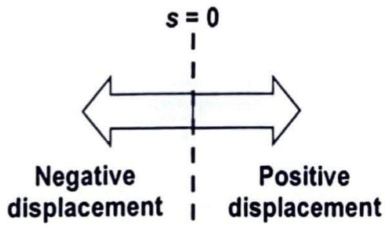
(a) $\text{Average Speed} = \frac{\text{distance travelled}}{\text{time taken}} = \frac{400}{80} = 5.0 \text{ m s}^{-1}$

(b) $\text{Average Velocity} = \frac{\text{change in displacement}}{\text{time taken}} = \frac{0}{80} = 0 \text{ m s}^{-1}$

Sign Conventions

Before solving any problem involving vector quantities (e.g. *s*, *v*, *a*), there is a need to **define the positive direction**. Thereafter, all the quantities will take reference to this defined direction for their sign conventions.





However, it is possible to define the positive direction opposite to the convention mentioned above whenever it is more convenient to do so.

2.3 Graphs in Kinematics

Graphical Representati-on of Motion

Graphs are very useful in representing the changes that occur during the motion of an object. There are three possible graphs that can provide useful information:

- displacement-time (*s*-*t*) graph
- velocity-time (*v*-*t*) graph
- acceleration-time (*a*-*t*) graph

Information from the graphs is often obtained from
1. direct reading of a point on the line / curve,
2. the gradient of the graph, and
3. the area under the graph.

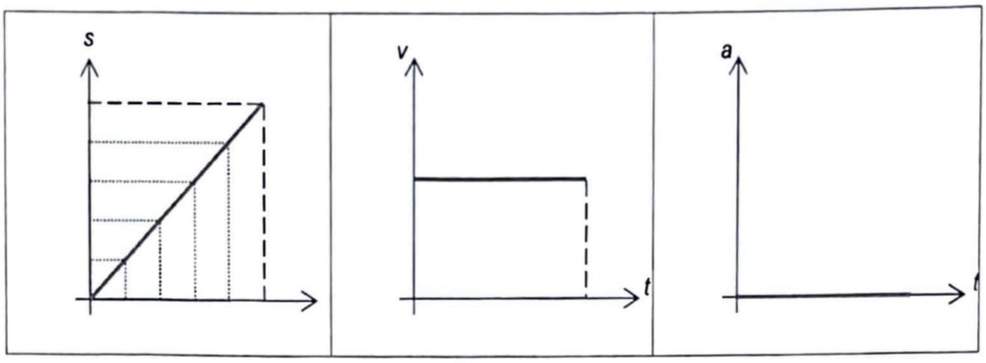
The physical quantities obtained depend on what is being plotted on the graph. **Always look at the axes of a graph very carefully.**

	***s*-*t* graph**	***v*-*t* graph**	***a*-*t* graph**
Gradient at a point	$\frac{ds}{dt}$: instantaneous velocity	$\frac{dv}{dt}$: instantaneous acceleration	no physical significance
Area under graph	no physical significance	$\int v \, dt = \Delta s$ = **change** in displacement	$\int a \, dt = \Delta v$ = **change** in velocity

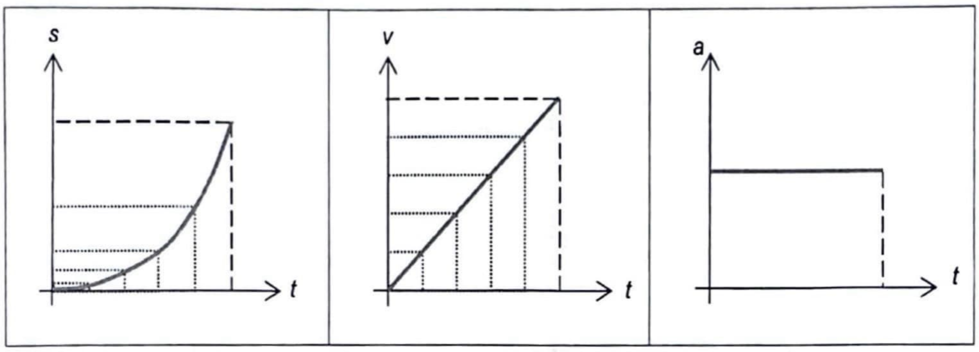
Conversion between graphs

In order to analyze the motion of objects fully, it is important to be able to convert the graphs from one form to another.

Case 1: Constant velocity

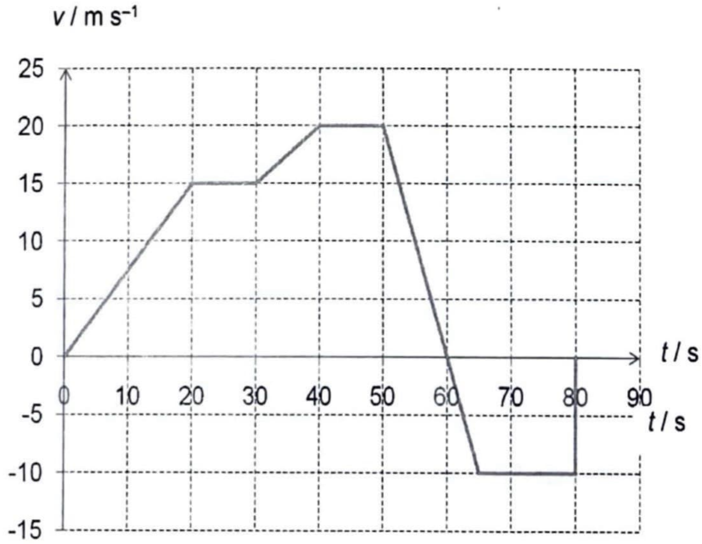


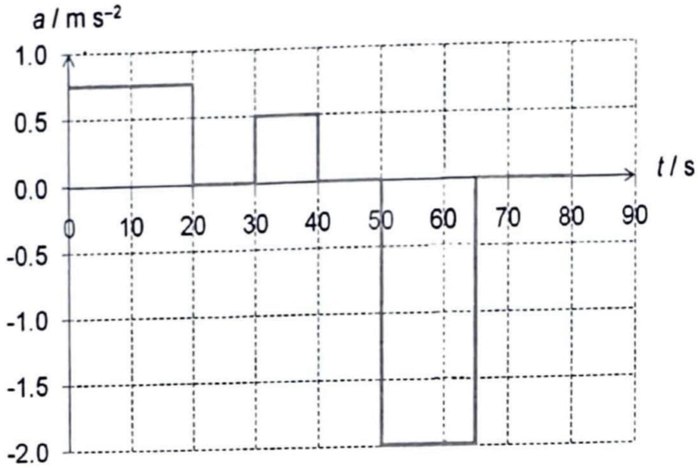
Case 2: Increasing velocity under constant (positive) acceleration

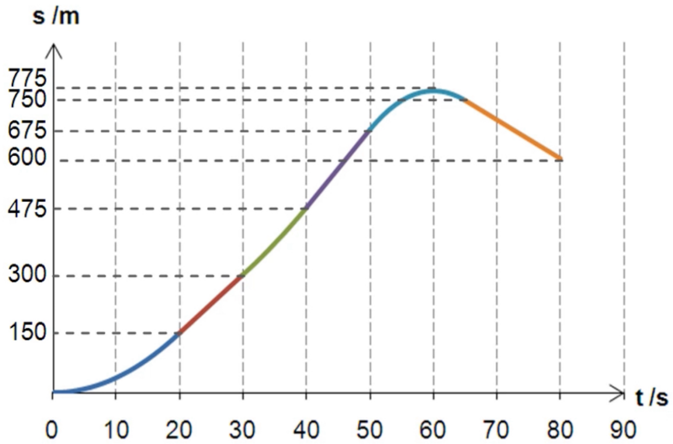


Example 4

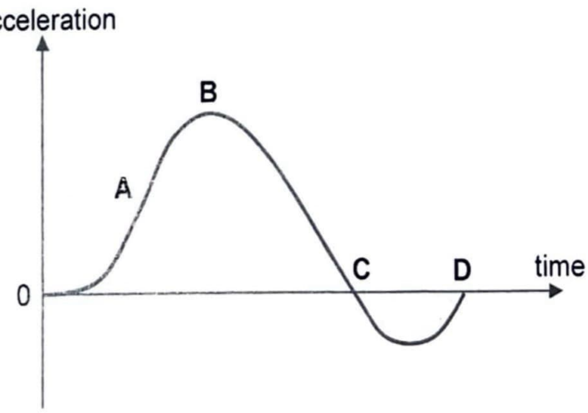
The velocity of an object varies with time as shown on the graph below. Sketch the corresponding acceleration-time and the displacement-time graphs on the axes shown below. Label the vertical axes with appropriate values. Take $s = 0$ at $t = 0$.







Example 5 The graph below shows the acceleration-time graph of an object moving from rest. At which points on the graph does the object have its (i) greatest velocity and (ii) greatest displacement?



(i) **C**

- *Change in velocity* can be found by taking the *area* under the *a-t* graph. At C, the *a-t* graph gives the greatest area under the graph.
- From **C to D,** the *a-t* graph gives a negative area which means that velocity is **decreasing**.
- Therefore, the velocity at **D** is smaller than the velocity at **C**.

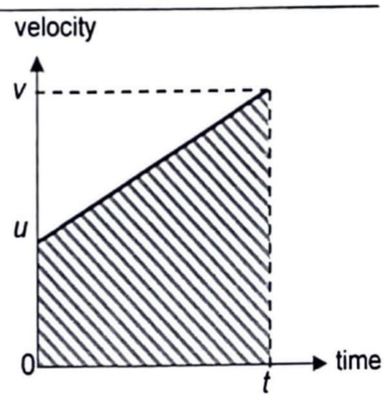
(ii) **D**

- As negative area is smaller than positive area, the decrease in velocity is not sufficient to reverse the direction of velocity.
- Hence displacement keeps increasing till point D.

2.4 Kinematics Equations of Uniformly-Accelerated Motion

1-D Motion with Constant Acceleration

The motion of an object whose velocity is increasing at a steady rate is called **uniformly accelerated motion**. The graph to the right shows the velocity-time graph of an object moving with constant acceleration. Its initial velocity is u and its velocity at time t later is v.



Derivation

Acceleration a = gradient of v-t graph

$$a = \frac{v-u}{t}$$

Hence,

Formula

$$\boxed{v = u + at} \quad \ldots(1)$$

Displacement s = Area under v-t graph.

Formula

$$\boxed{s = \frac{1}{2}(u+v)t} \quad \ldots(2)$$

Substituting (1) into (2),

$$s = \frac{1}{2}(u + (u + at))t$$

Formula

$$\boxed{s = ut + \frac{1}{2}at^2} \quad \ldots(3)$$

From (1), $t = \dfrac{v-u}{a}$ and substituting into (2),

$$s = \frac{1}{2}(u+v)\left(\frac{v-u}{a}\right) = \frac{v^2-u^2}{2a}$$

Formula

$$\boxed{v^2 = u^2 + 2as} \quad \ldots(4)$$

Note!

These 4 equations apply only to motion in a straight line with constant acceleration.

Example 6 A light plane must reach a speed of 35 m s^{-1} for take-off. Calculate,

(a) the time needed for the plane to reach its take-off speed if it accelerates uniformly from rest at 2.8 m s^{-2},

(b) the length of the runway that is needed.

Soln

(a) Using $v = u + at$,

$35 = 0 + (2.8)\,t$

$t = 12.5$ s (3 sf)

or = 13 s (2 sf)

(b) Using $s = ut + \frac{1}{2}at^2$

$s = 0 + \frac{1}{2}(2.8)(12.5)^2$

= 219 m (3 sf)

or = 220 m (2 sf)

Example 7 An object starts from rest and experiences a uniform acceleration of 0.50 m s^{-2} while moving down an inclined plane 9.0 m long. When it reaches the bottom, the object moves up another plane with uniform acceleration, where, after moving 15 m, it comes to rest.

Determine,

(a) the speed of the object at the bottom of the first plane,

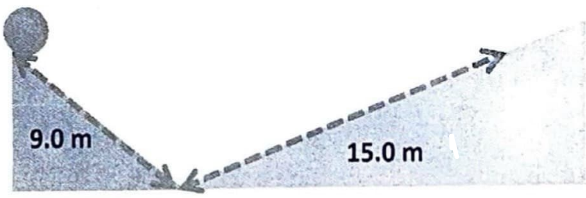
(b) the time taken for it to move down the first plane,

(c) its acceleration along the second plane, and

(d) its speed 8.00 m along the second plane.

(a) For the first plane:

$u_1 = 0$; $a_1 = 0.50$ m s^{-2} ; $s_1 = 9.0$ m



Soln

(a) Using $v^2 = u^2 + 2as$,

$v_1^2 = 0 + 2(0.50)(9.0)$

$v_1 = 3.0$ m s^{-1}

(b) $v_1 = u_1 + a_1 t_1$ or $s_1 = u_1 t_1 + \frac{1}{2}a_1 t_1^2$

$9.0 = 0 + \frac{1}{2}(0.50)t_1^2$

$t_1 = 6.0$ s

(b) Using $v = u + at$

$3.0 = 0 + (0.50)t_1$

$t_1 = 6.0$ s

(c) For the second plane:

$u_2 = 3.0$ m s^{-1} ; $v_2 = 0$; $s_2 = 15$ m

Using $v^2 = u^2 + 2as$,

$0 = 3.0^2 + 2a_2(15.0)$

$a_2 = -0.30$ m s^{-2}

= 0.30 m s^{-2} down the 2nd slope

(d)

(d) $v^2 = u_2^2 + 2a_2 s$

$v = 3.0^2 + 2(-0.30)(8.0)$

$v = \pm 2.05$ m s^{-1}

speed is 2.05 m s^{-1}

2.5 Free Falling Objects

Objects Moving Vertically in a Uniform Gravitational Field

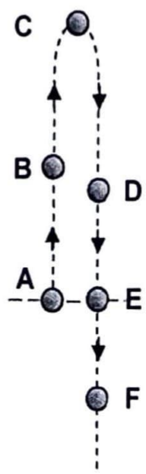
(without air-resistance)

A very important example of uniformly accelerated motion is the vertical motion of an object in a uniform gravitational field. In the absence of air resistance, we can say this object is undergoing **free fall**.

In such case, whether the object is thrown upwards or downwards or released from rest, the object experiences a constant **acceleration directed downwards (towards centre of the Earth)** with magnitude of $g = 9.81$ m s^{-2}. This is known as the acceleration due to free fall, and it is assumed to be a constant value near the Earth's surface.

For illustration, let us consider the motion of a ball that is projected vertically upwards and falling back to its original position and beyond. (upwards is defined as the positive direction and the reference point is set at point A)

	displacement	velocity	acceleration
A	zero	+	–
B	+	+	–
C	+	0	–
D	+	–	–
E	0	–	–
F	–	–	–



Example 8

A ball was released from rest at a height of 10 m. A second ball was released from the same point 1.0 s later. Taking the acceleration of free-fall to be 9.81 m s^{-2}, calculate
(a) the time taken for the first ball to reach the ground, and
(b) the height of the second ball when the first ball hits the ground.

Taking downwards as positive,

(a)

$u = 0$, $s = 10$ m, $a = 9.81$ m s^{-2}, $t = ??$

Applying $s = ut + \frac{1}{2}at^2$,

$10 = 0 + \frac{1}{2}(9.81)\,t^2$

$t = 1.428 = 1.43$ s

(b)

From (a), Ball 1 takes 1.428 s to hit the ground

$\therefore t_2 = (1.428 - 1) = 0.428$ s

$u_2 = 0$, $a = 9.81$ m s^{-2}, $t_2 = 0.428$ s, $s_2 = ??$

Using $s_2 = u_2t_2 + \frac{1}{2}at_2^2$,

$s_2 = 0 + \frac{1}{2}(9.81)(0.428)^2$

$= 0.899$ m

Hence, height of 2nd ball from ground

$= 10 - 0.899$

$= 9.10$ m

Example 9 A stone is thrown vertically upward with a speed of 12 m s^{-1} from the edge of a cliff 100 m high.

(a) Calculate
  (i) the time it takes to reach the bottom of the cliff,
  (ii) its speed just before hitting the ground,
  (iii) the total distance it travels.

(b) Sketch a graph showing how the velocity of the stone varies with time and label the axes with the appropriate values. Take t = 0 to be the time at which the stone is thrown.

(a)(i) Taking upward as positive and the level of the cliff as the reference axis,

u = 12 m s^{-1}, s = -100 m, a = -9.81 m s^{-2}, t = ??

$s = ut + \frac{1}{2}at^2$
$-100 = 12t + \frac{1}{2}(-9.81)t^2$
$4.905t^2 - 12t - 100 = 0$

t = 5.90 s   or   –3.54 s (n.a.)

(ii) $v = u + at$
  = 12 + (–9.81)(5.90)
  = –45.9 m s^{-1} (-ve sign means v is downwards),
∴ speed is 45.9 m s^{-1}

Or:

$v^2 = u^2 + 2as$
  = 12^2 + 2(–9.81)(–100)
v = - 45.9 m s^{-1} (reject +45.9 m s^{-1} as stone is falling downwards)

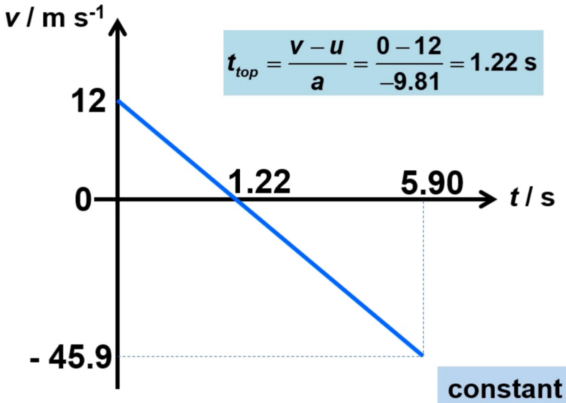
(iii) From cliff-top to maximum height:

u = 12 m s^{-1}, a = –9.81 m s^{-2}, v = 0 m s^{-1}

$v^2 = u^2 + 2as$
$0 = 12^2 + 2(-9.81)\,s$
s = 7.339 m

Total distance travelled = (2 x 7.339) + 100
  = 115 m

(b)



$$t_{top} = \frac{v-u}{a} = \frac{0-12}{-9.81} = 1.22\text{ s}$$

constant gradient of g = –9.81 m s^{-2}

Objects Released from Rest in a Uniform Gravitational Field

(with air-resistance)

In the absence of air resistance, all falling objects have the same acceleration of free-fall, independent of their mass. This is **not true** when there is air resistance.

When an object moves through a fluid (liquid or gas), it experiences a drag force, or air resistance, if the fluid is air. This force acts opposite in direction to the velocity. Under non-turbulent conditions, the magnitude of this drag force F_D is proportional to the instantaneous velocity v:

$$F_D = k\,v \qquad \text{where } k \text{ is a constant.}$$

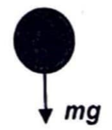
At higher velocities, the magnitude of the drag force F_D' is proportional to the square of the velocity v^2:

$$F_D' = k'\,v^2 \qquad \text{where } k' \text{ is a constant.}$$

Let us now consider the motion of an object released from rest in a uniform gravitational field where there is air resistance.

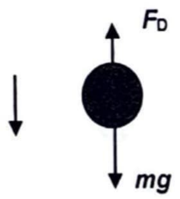
Forces acting on the falling body | **Description of motion of the object**

Initial $v = 0$



- At the point of release, velocity v is zero.
- The only force acting on object at this instan gravitational force.
- Body falls with an acceleration of g.

v increases

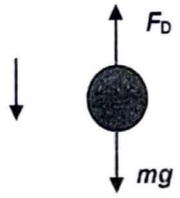


$$mg - F_D = ma$$

$$a = g - \frac{F_D}{m}$$

- As body gains velocity, air resistance, which is dependent on velocity, starts to increase.
- Resultant force on body = $mg - F_D$, where F_D is the drag force.
- Body is still accelerating, acceleration < g.
- Acceleration a is affected by the drag force and the mass of object.

constant v
$v = v_T$



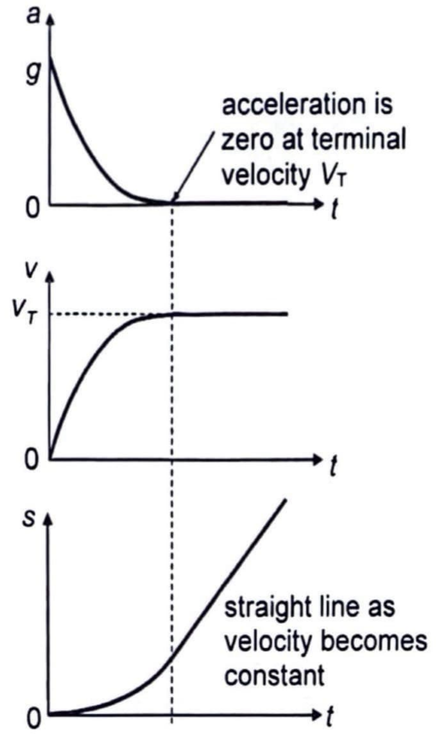
$$mg - F_D = 0$$

$$F_D = mg$$

- Eventually, drag force increases to a point where it is equal in magnitude to the weight of the body.
- Resultant force acting on body = 0.
- ∴ acceleration of body = 0.
- Body attains a constant velocity, called the **terminal velocity,** v_T.

Kinematics Graphs (with air-resistance)

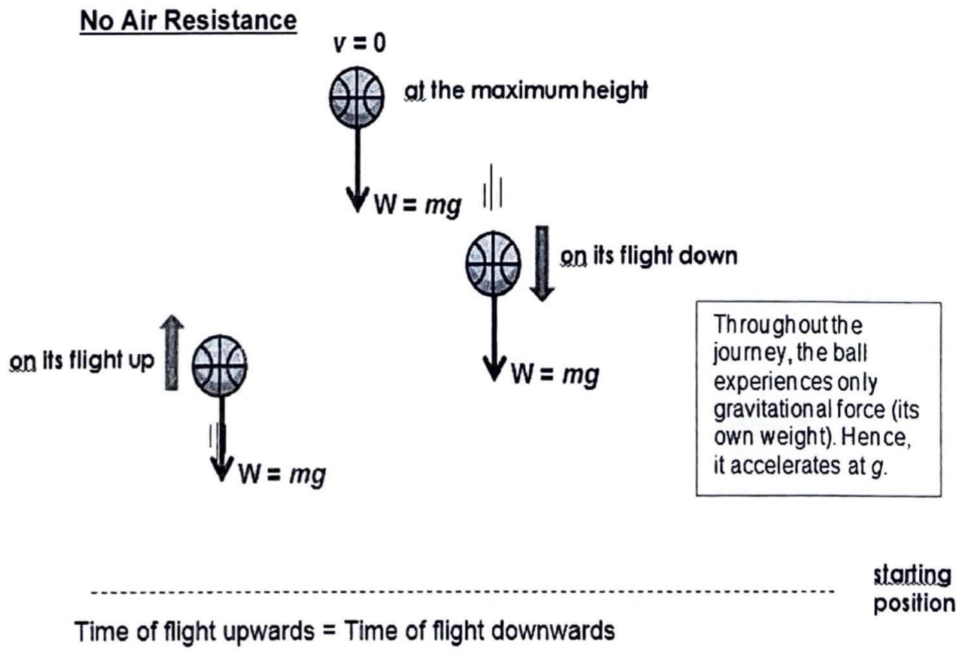
The following graphs show the acceleration, velocity and displacement of the object described above.



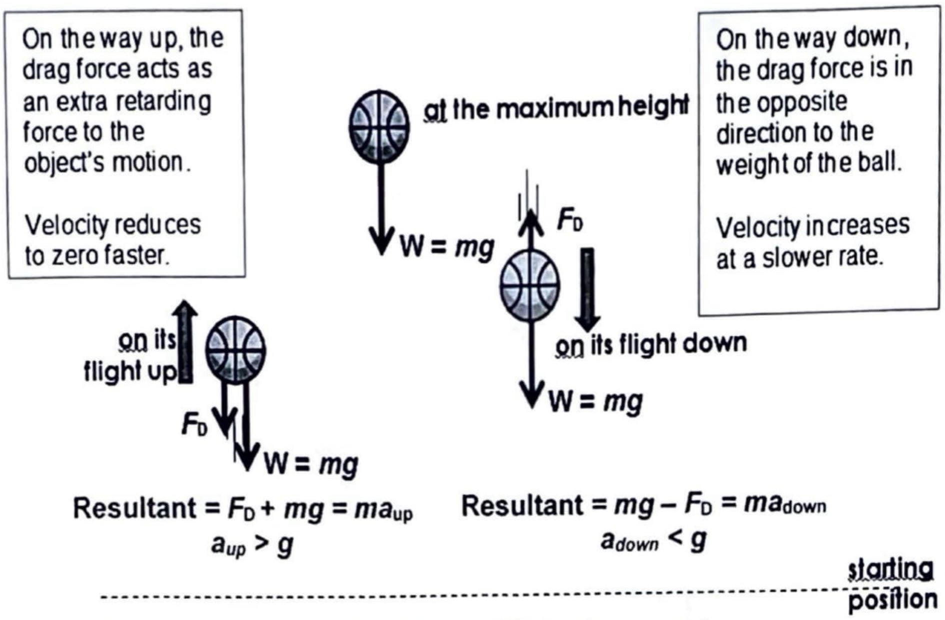
Effect of Air Resistance on Vertical Motion in a Uniform Gravitational Field

With air resistance, an object **thrown** upwards with a certain speed experiences a downward drag force in addition to its own weight. It slows down until it comes to rest at the highest point of its motion. It then reverses its direction of motion and accelerates downwards (refer to previous page).

The following illustrates how air resistance affects the maximum height reached by the object, and its asymmertical effect on the time taken for the upward and downward motion.



With Air Resistance



Time of flight upwards < Time of flight downwards

The smaller acceleration on the way down means that the average speed on the way down is also smaller compared to on the way up. Since the distance travelled on the way up is same as on the way down, it takes a longer time to fall from the maximum height to the orginal position.

Think! Note: Kinematics equations cannot be used to calculate time of flight when there is air resistance effect. Do you know why?

Exercise How will the graph in Example 9(b) be drawn differently if there is air resistance?

Bouncing Ball

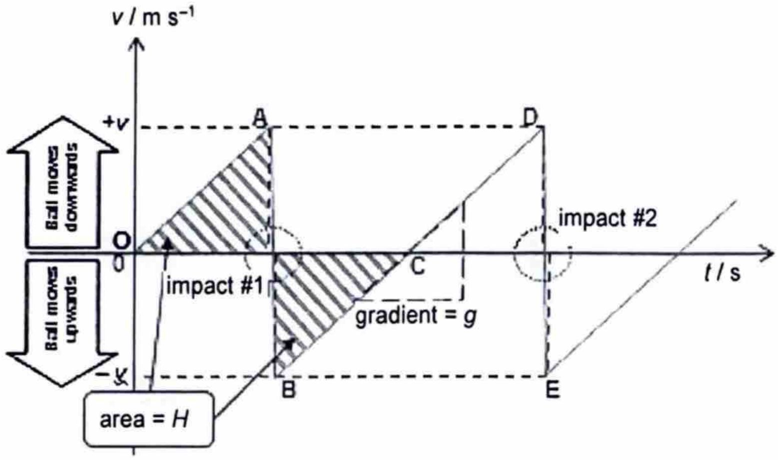
(elastic collision)

Consider a ball that is released from rest at a height H above the floor. Taking downwards as positive and assuming that

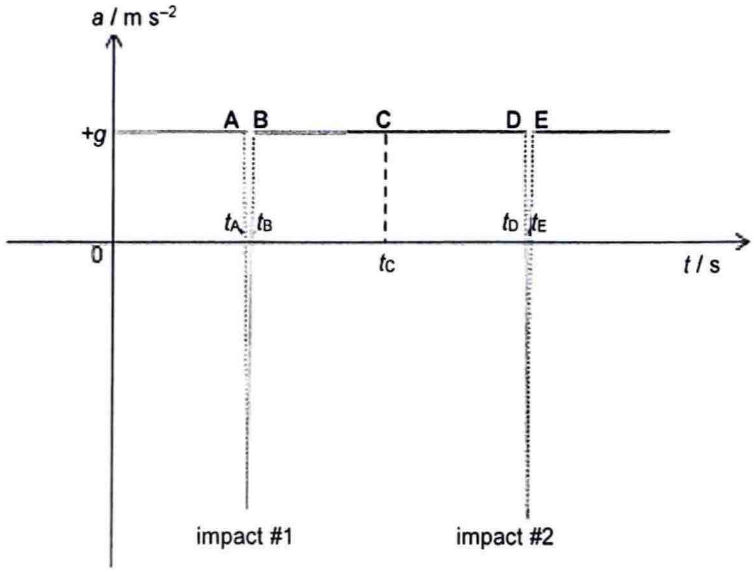
- there is no air resistance, and
- it rebounds without loss of energy on impact with the floor (i.e. the collision with the floor is *elastic*),

its motion can be represented in the graphs below:

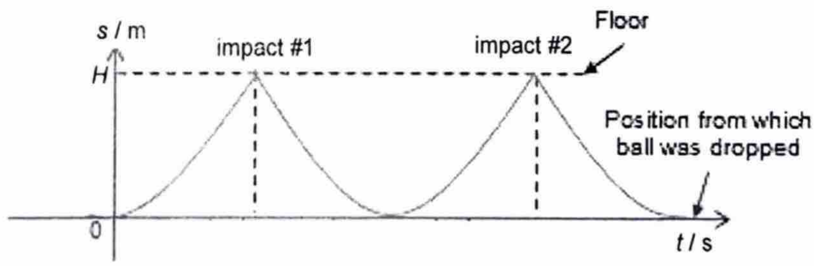
velocity-time graph



acceleration-time graph



displacement-time graph



O: Ball is released from rest at a height *H* from the ground.

OA: Ball accelerates at *g* on its way down.

A: Ball hits the ground with max speed *v*.

AB: Ball compresses before it finally stops moving. It then uncompresses before leaving the floor with max speed *v* in the opposite direction. Its acceleration is upward and is much larger than *g*.

B: Ball leaves the ground with the same speed *v* when it hits the ground.

BC: Ball decelerates at *g* on its way up.

C: Ball reaches the height of release and comes to instantaneous rest. The cycle repeats.

Note:

- Throughout its journey in air, the ball experiences acceleration due to free fall *g* (neglecting effects of air resistance).
- The duration of the impact is very short, so points A and B, as well as points D and E, happen almost at the same instant, ie the line AB and DE are almost vertical.
- The shape of the *s-t* graph between successive impacts is parabolic.

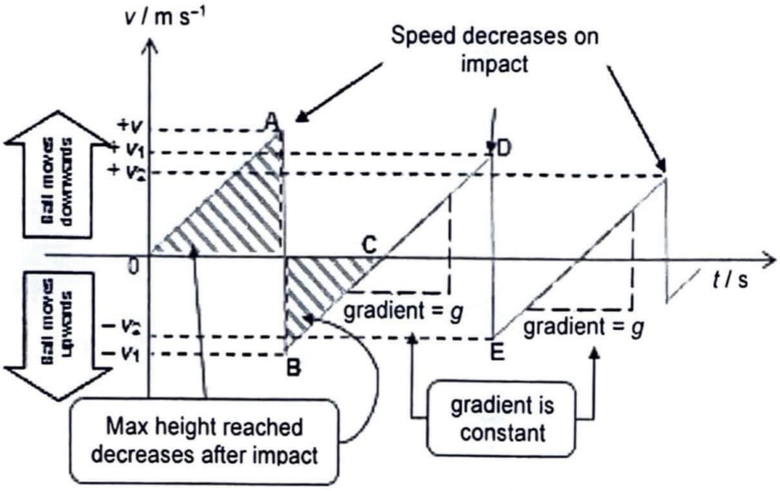
Bouncing Ball

(inelastic collision)

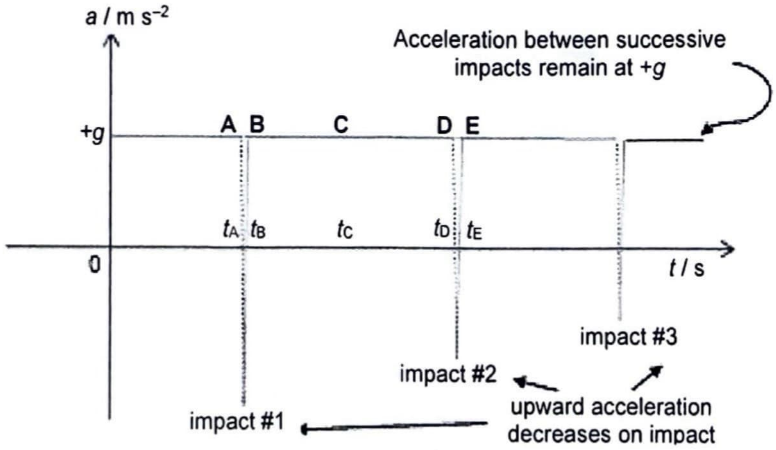
If the ball loses energy on impact with the floor (collision is inelastic), it rebounds with a smaller kinetic energy than just before impact, and hence maximum rebound velocity is not constant. The time taken between each subsequent bounce also decreases.

Taking downwards as positive and assuming that there is no air resistance, its motion can be represented in the graphs below:

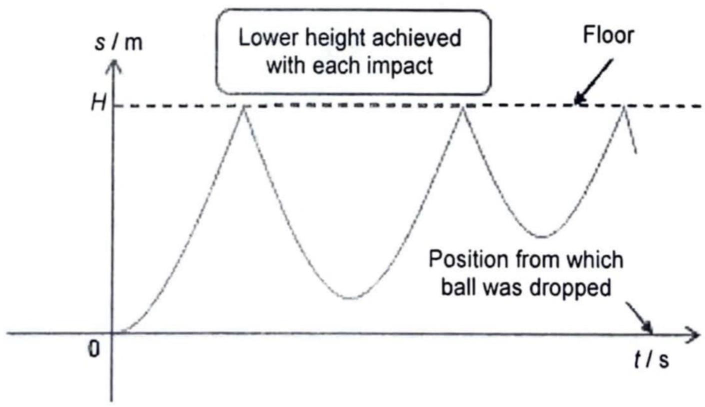
velocity-time graph



acceleration-time graph



displacement-time graph



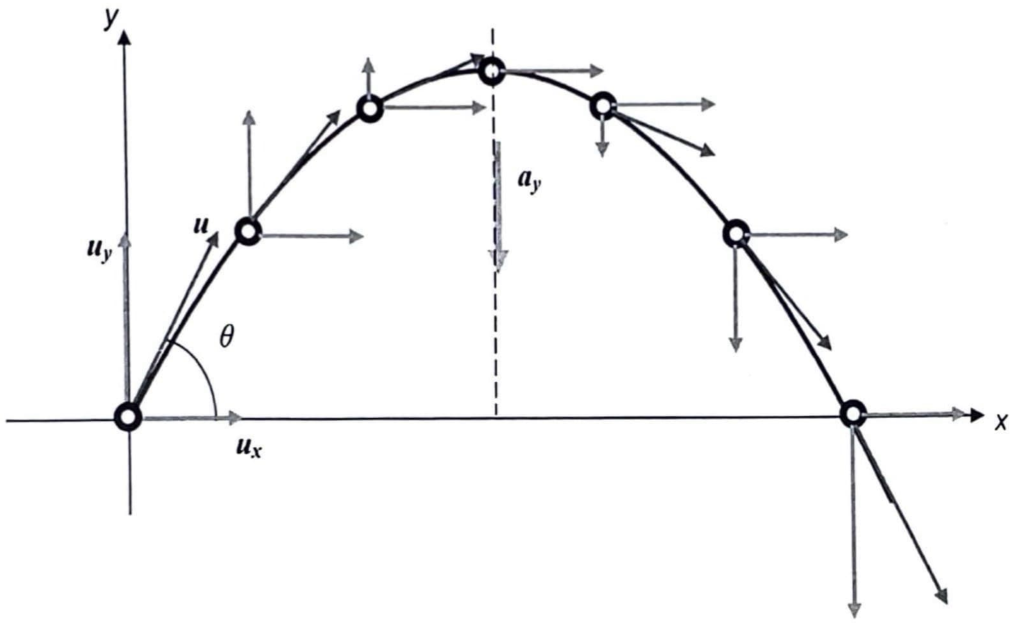
2.6 Non-Linear Motion

Projectile Motion

An object that is projected near the surface of the Earth, such as a kicked football or a batted base-ball, describes a curved path in a vertical plane. This kind of motion is called **projectile motion.** Projectile motion involves motion in the x and y-directions simultaneously and the vertical and horizontal motions of a projectile are independent of each other.

Hence, projectile motion can be analysed in the following steps:

1) Resolving the displacement, velocity and acceleration vectors into their ***horizontal*** and ***vertical*** **components.**
2) Applying the kinematic equations in each direction.



Key Characteristics of a Projectile Motion

The path of the projectile is symmetrical about the vertical through the highest point. The time the object takes to go from $y = 0$ to the highest point equals to the time the object takes to go from the highest point to $y = 0$.

Assumptions

There is no air resistance and the projected object experiences:

- zero horizontal acceleration
- a vertical acceleration $a_y = g$ (constant value of 9.81 m s^{-2}) directed downwards.

With this assumption, we find that the path of a projectile is always a parabola. (Refer to Appendix for a mathematical proof.)

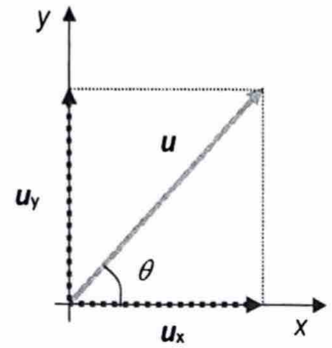
Important Terms:

- **Trajectory** – the path described by an object, which for this chapter is parabolic.
- **Range** – the horizontal displacement between the point of projection and the point of impact.
- **Angle of projection** – the angle between the direction of projection and the horizontal plane through the point of projection.
- **Time of flight** – time taken from the point of projection to the point of impact

Resolution of Vectors

As mentioned above, in solving problems of projectile motion, you will always need to resolve velocity into its components. The following is a quick recap of the concepts you have learnt.

From the figure below, the initial velocity of the projectile is represented by a vector $\boldsymbol{u}$ which makes an angle θ with the ground. $\boldsymbol{u}$ can be resolved into 2 components, $\boldsymbol{u}_x$ (the horizontal component) and $\boldsymbol{u}_y$ (the vertical component)



The magnitudes of the components are given by:

$$u_x = u \cos\theta$$
$$u_y = u \sin\theta$$

If both the components are known, the magnitude and direction of the vector can be found.

Magnitude of vector:

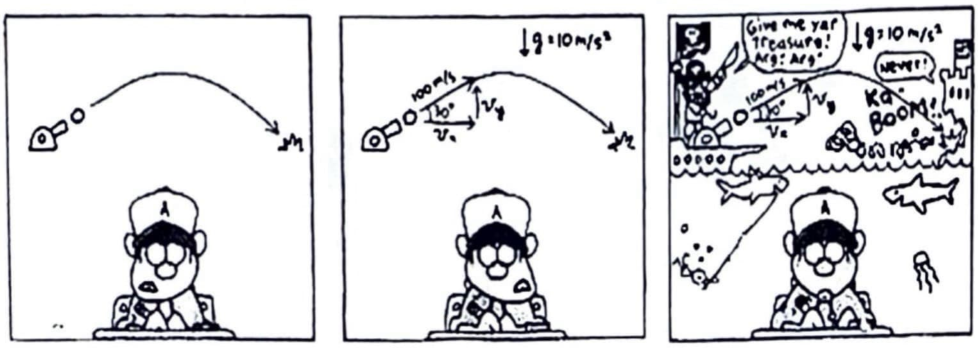
$$u = \sqrt{{u_x}^2 + {u_y}^2}$$

Direction of vector:

$$\tan\theta = \frac{u_y}{u_x}$$

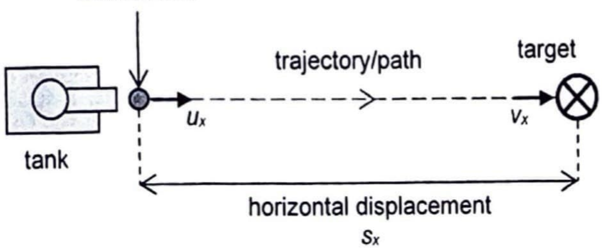
Trajectory of a Projectile

Let's consider Peter's problem: cannon ball shot at an angle to the horizon.



Horizontal Component

If we're looking down from above, we would observe the cannon ball to be moving in a straight line. Neglecting air resistance, the cannon ball moves in a straight line with constant velocity.



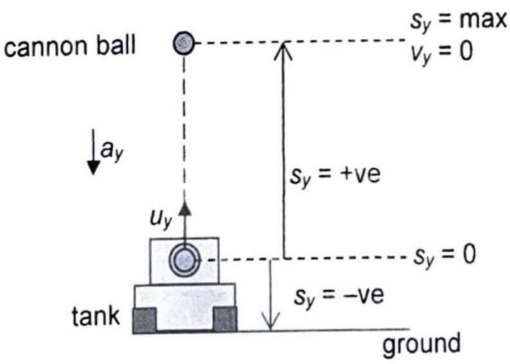
Hence,

$a_x = 0$

$u_x = v_x$

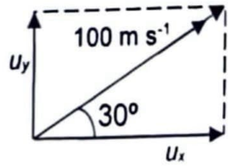
Vertical Component

If we're looking from the front view of the tank, we would observe the cannon ball decelerate while moving upwards until it stops, then accelerate downwards. The magnitude of its deceleration and acceleration is the acceleration of free fall g or 9.81 m s^{-2}.



Note!
Remember to define your directions and assign the correct sign to the different vectors!

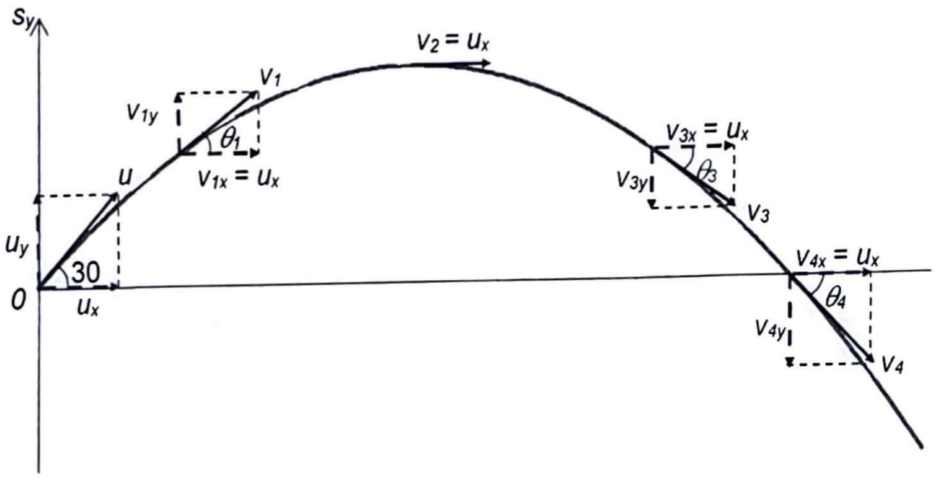
According to Peter's diagram,



$$u_x = 100\cos 30° = 86.6 \text{ m s}^{-1}$$
$$u_y = 100\sin 30° = 50.0 \text{ m s}^{-1}$$

The velocity v of the cannon ball at any time in the motion can be described as:

$v = \sqrt{v_x^2 + v_y^2}$ where $v_x = v\cos\theta$ & $v_y = v\sin\theta$ and at an angle θ to the horizontal $= \tan^{-1}\left(\frac{v_y}{v_x}\right)$.



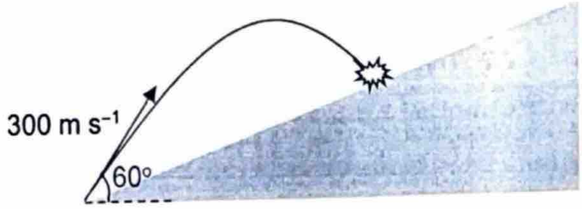
Note!s

1. When attempting to solve problems on projectile motion, identify all the data given as s_x, s_y, u_x, u_y, v_x, v_y and a_y.

2. Remember that the time taken t is the same for both motion descriptions because they are describing the same motion path – the key to solving many projectile motion problems!

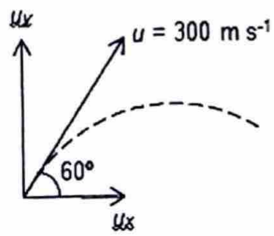
Example 10
Resolving Vertical & Horizontal Components

An artillery shell at the base of a mountain is fired with an intial velocity of 300 m s^{-1} at an angle of 60° from the horizontal, as shown in the diagram. The shell explodes on the side of the mountain 40 s after firing.



(a) Calculate the magnitude of the horizontal and vertical components of the initial velocity.

(b) Find the horizontal and vertical distance of the point where the shell explodes relative to the firing point.



a)

$$u_x = u\cos 60 = 300\cos 60 = 150 \text{ ms}^{-1}$$

$$u_y = u\sin 60 = 300\sin 60 = 260 \text{ ms}^{-1}$$

b)

Horizontally,

$u_x = u\cos 60$

$a_x = 0$

$t = 40$

$s_x = ?$

$$s_x = u_x t = (150)(40) = 6000 \text{ m}$$

Vertically,

$u_y = u\sin 60$

$a_y = -9.81 \text{ ms}^{-2}$

$t = 40$

$s_y = ?$

$$s_y = u_y t + \frac{1}{2}a_y t^2 = (300\sin 60)(40) + \frac{1}{2}(-9.81)(40^2) = 2540 \text{ m}$$

Example 11
Projected Horizontally
($u_y = 0$)

A stone is projected horizontally with a speed of 15 m s^{-1} from the top of a cliff 100 m high. Determine
(a) the velocity of the stone just before it hits the ground,
(b) the horizontal distance from the base of the cliff to where the stone lands.

Taking downwards as positive,

Vertically	Horizontally
$u_y = 0$	$u_x = 15$
$a_y = 9.81$	$a_x = 0$
$s_y = 100$	$s_x = ?$

(a) $v_x = 15$

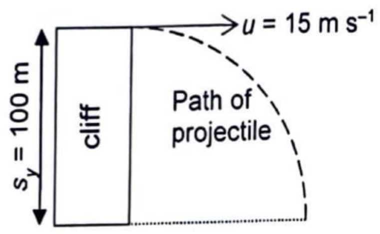
using $v^2 = u^2 + 2as$

$$v_y^2 = u_y^2 + 2a_y s_y = 0 + 2(9.81)(100) = 1962$$

$$v = \sqrt{v_y^2 + v_x^2} = \sqrt{1962 + (15)^2} = 46.765 \text{ m s}^{-1} = 46.8 \text{ m s}^{-1}$$

$$\tan\theta = \frac{v_y}{v_x}$$

$$\theta = \tan^{-1}\left(\frac{44.294}{15}\right) = 72.2^\circ$$



(b)

$$s = ut + \frac{1}{2}at^2$$
$$s_y = u_y t + \frac{1}{2}a_y t^2$$
$$100 = 0 + \frac{1}{2}(9.81)t^2$$
$$t = 4.515 \text{ s}$$

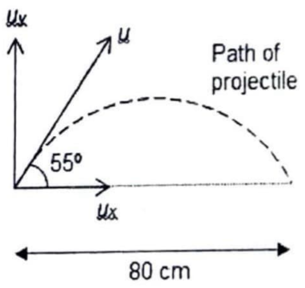
OR

$$v = u + at$$
$$v_y = u_y + a_y t$$
$$t = \frac{v_y - u_y}{a_y} = \frac{\sqrt{1962} - 0}{9.81} = 4.515$$

$$s_x = u_x t = (15)(4.515) = 67.7 \text{ m}$$

Example 12
Horizontal Range

Locusts have been observed to jump distances up to 80 cm on a level floor. Photographs of their jump show that they usually take off at an angle of about 55° from the horizontal. Calculate the initial speed of locusts jumping 80 cm horizontally with a takeoff angle of 55°.



Taking upwards as positive.

Vertically	Horizontally
$u_y = u\sin 55^\circ$	$u_x = u\cos 55^\circ$
$a_y = -9.81$	$a_x = 0$
$s_y = 0$	$s_x = 0.8$

Vertically

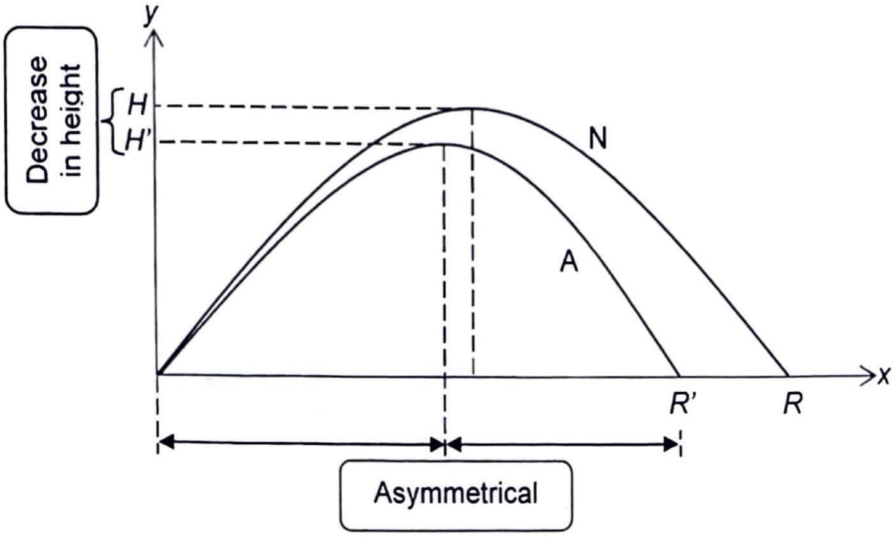
$$s = ut + \frac{1}{2}at^2$$
$$s_y = u_y t + \frac{1}{2}a_y t^2$$
$$0 = (u\sin 55^\circ)t + \frac{1}{2}(-9.81)t^2$$
$$t = 0$$
or
$$t = \frac{u\sin 55^\circ}{4.905} \Rightarrow \text{eqn 1}$$

Horizontally

$$s_x = u_x t$$
$$0.8 = (u\cos 55^\circ)t$$
$$t = \frac{0.8}{u\cos 55^\circ} \Rightarrow \text{eqn 2}$$

$$\frac{u\sin 55^\circ}{4.905} = \frac{0.8}{u\cos 55^\circ}$$
$$u^2 = \frac{(0.80)(4.905)}{(\sin 55^\circ)(\cos 55^\circ)}$$
$$u = 2.89$$

Effect of Air Resistance on Projectile Motion



Curve N shows the trajectory of a projectile in the absence of air resistance. It is a parabola. The maximum height reached is H and the horizontal range is R.

Curve A shows the trajectory of the same projectile, thrown at the same speed and at the same angle to the horizontal, when there is air resistance. Note that

1. the trajectory is no longer symmetrical,
2. the maximum height reached is lower and
3. the horizontal range is shorter.

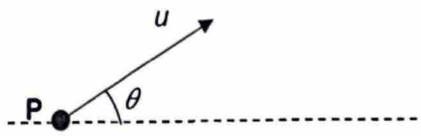
Why?
The projectile experiences air resistance in the direction opposite to its velocity. Hence, the air resistance has both a horizontal component and a vertical component.

When the projectile is on its way up, the vertical component of the air resistance acts downward. This means that v_y reduces to zero after moving through a shorter vertical displacement. Hence, the maximum height reached is lower.

In addition, the horizontal component of the velocity decreases throughout the motion due to the horizontal component of the air resistance. Hence the range is shorter, and the horizontal displacement of the projectile on its way down is also shorter than that on the way up.

Appendix PROOF OF PARABOLIC PATH OF A PROJECTILE

Consider a particle **P** with velocity *u* projected at an angle θ from the ground. Assume air resistance is negligible.



Applying the equations of motion in both directions,

$$s_X = (u\cos\theta)t \qquad \text{..................(1)}$$

$$s_Y = (u\sin\theta)t - \frac{1}{2}gt^2 \qquad \text{..................(2)}$$

Rearranging (1), we get :

$$t = \frac{s_X}{u\cos\theta} \qquad \text{................(3)}$$

Substitute (3) in to (2):

$$s_Y = (u\sin\theta)\left(\frac{s_X}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{s_X}{u\cos\theta}\right)^2$$

$$s_Y = (u\tan\theta)s_X - \left(\frac{g}{2u^2\cos^2\theta}\right)s_X{}^2$$

The above expression shows that the relationship between the vertical displacement S_Y and the horizontal displacement S_X is a quadratic one.

Hence, the trajectory of a projectile travelling through negligible air resistance is a parabola.

RAFFLES INSTITUTION
YEAR 5-6 PHYSICS DEPARTMENT

Tutorial 2 KINEMATICS

Self-Check Questions

S1 Define the terms *displacement*, *speed, velocity* and *acceleration*, and state whether they are scalar or vector quantities.

S2 State the significance of the slope and/or the area under the graphs of
(a) displacement-time
(b) velocity – time, and
(c) acceleration – time.

S3 Derive the equations of motion and state the conditions necessary for these equations to be valid.

S4 Describe qualitatively the motion of an object falling in a uniform gravitational field with air resistance. Sketch the *displacement – time, velocity – time* and *acceleration – time* graphs for such an object.

S5 Describe what is meant by projectile motion and state the assumption that is made when analysing projectile motion.

S6 State and explain the meaning of the terms trajectory and range.

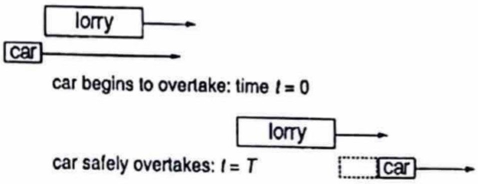
S7 For projectile motion without resistance, state what happens to
(a) the vertical component of velocity at the maximum height, and
(b) the horizontal component of velocity at all points throughout the flight.

S8 Describe the effects of air resistance on the trajectory of projectile motion.

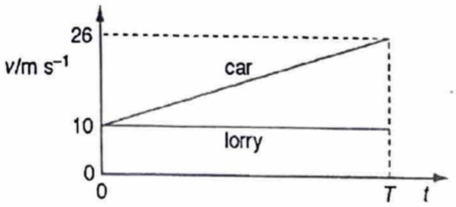
Self-Practice Questions

SP1

The minimum time T required for a car safely to overtake a lorry on the motorway is measured from the time the front of the car is level with the rear of the lorry, until the rear of the passing car is a full car-length ahead of the lorry.



The car is 3.5 m long and the lorry is 17.0 m long. The graph shows the variation with time t of the speeds v of the car and the lorry.



What is the value of T?

A 0.86 s

B 1.2 s

C 2.6 s

D 3.0 s

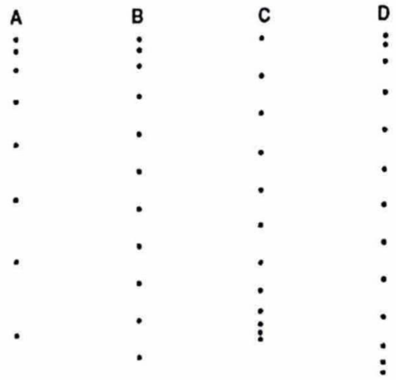
N02/I/4

SP2

A steel sphere is released from rest at the surface of a deep tank of viscous oil. A multiple exposure photograph is taken of the sphere as it falls.

The time interval between exposures is always the same.

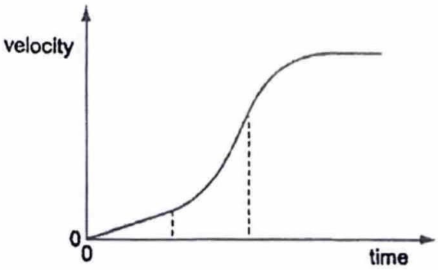
Which of the following could represent this photograph?



N04/I/4

SP3

The graph shows how the velocity of a racing car changes with time.



Which statement describes the acceleration?

A A constant positive acceleration is followed by an acceleration increase and then a negative acceleration.

B The acceleration increases positively in the first two sections and then decreases to zero.

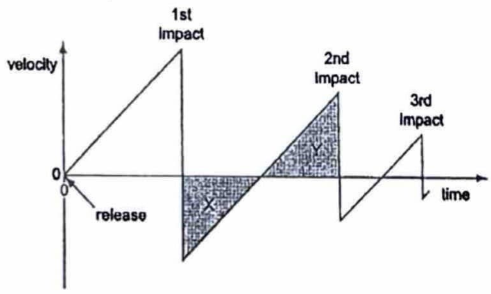
C The acceleration is positive at the start, increases, then decreases to zero.

D The acceleration starts from zero, increases, then decreases to zero.

N09/I/3

SP4

A ball is released from rest above a horizontal surface. The graph shows the variation with time of its velocity.



Why are areas X and Y equal?

A For one impact, the speed at which the ball hits the surface equals the speed at which it leaves the surface.

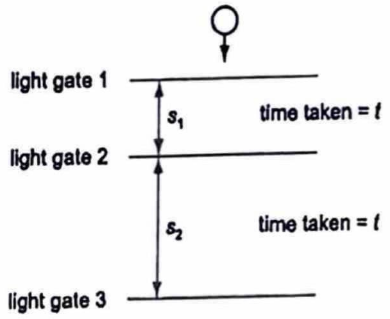
B The ball rises and falls through the same distance between impacts.

C The ball's acceleration is the same during its upward and downward motion.

D The speed at which the ball leaves the surface after an impact is equal to the speed at which it returns to the surface for the next impact.

N06/I/4

SP5 An object falls freely with constant acceleration a from above three light gates. It is found that it takes a time t to fall between the first two light gates a distance of s_1 apart. It then takes an additional time, also t, to fall between the second and third light gates a distance s_2 apart.



What is the acceleration in terms of s_1, s_2 and t?

A $\frac{(s_2 - s_1)}{t^2}$

B $\frac{(s_2 - s_1)}{2t^2}$

C $\frac{2(s_2 - s_1)}{3t^2}$

D $\frac{2(s_2 - s_1)}{t^2}$

N10/I/5

SP6 A ball is thrown vertically upward with an initial speed of 20 m s^{-1}. Assuming negligible air resistance, calculate

(a) the time taken to return to the thrower,

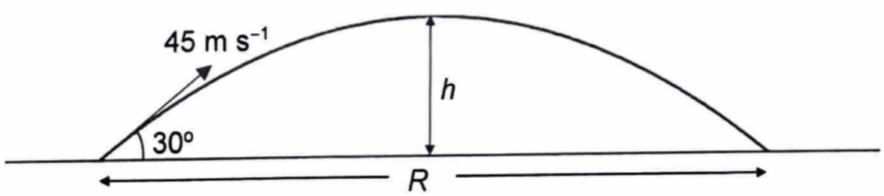
(b) the maximum height reached.

SP7 An aeroplane, flying horizontally with a speed of 140 m s^{-1} at a constant height of 300 m above the ground, releases a package. Determine

(a) the time taken for the package to reach the ground,

(b) the horizontal distance travelled by the package.

SP8 A shell is fired from a gun with a speed of 45 m s^{-1} at an angle of 30° to the horizontal.



(a) Calculate, for the shell, the initial values of

(i) the horizontal component of the velocity,

(ii) the vertical component of the velocity.

(b) Assuming negligible air resistance, determine

(i) the maximum vertical height h reached by the shell,

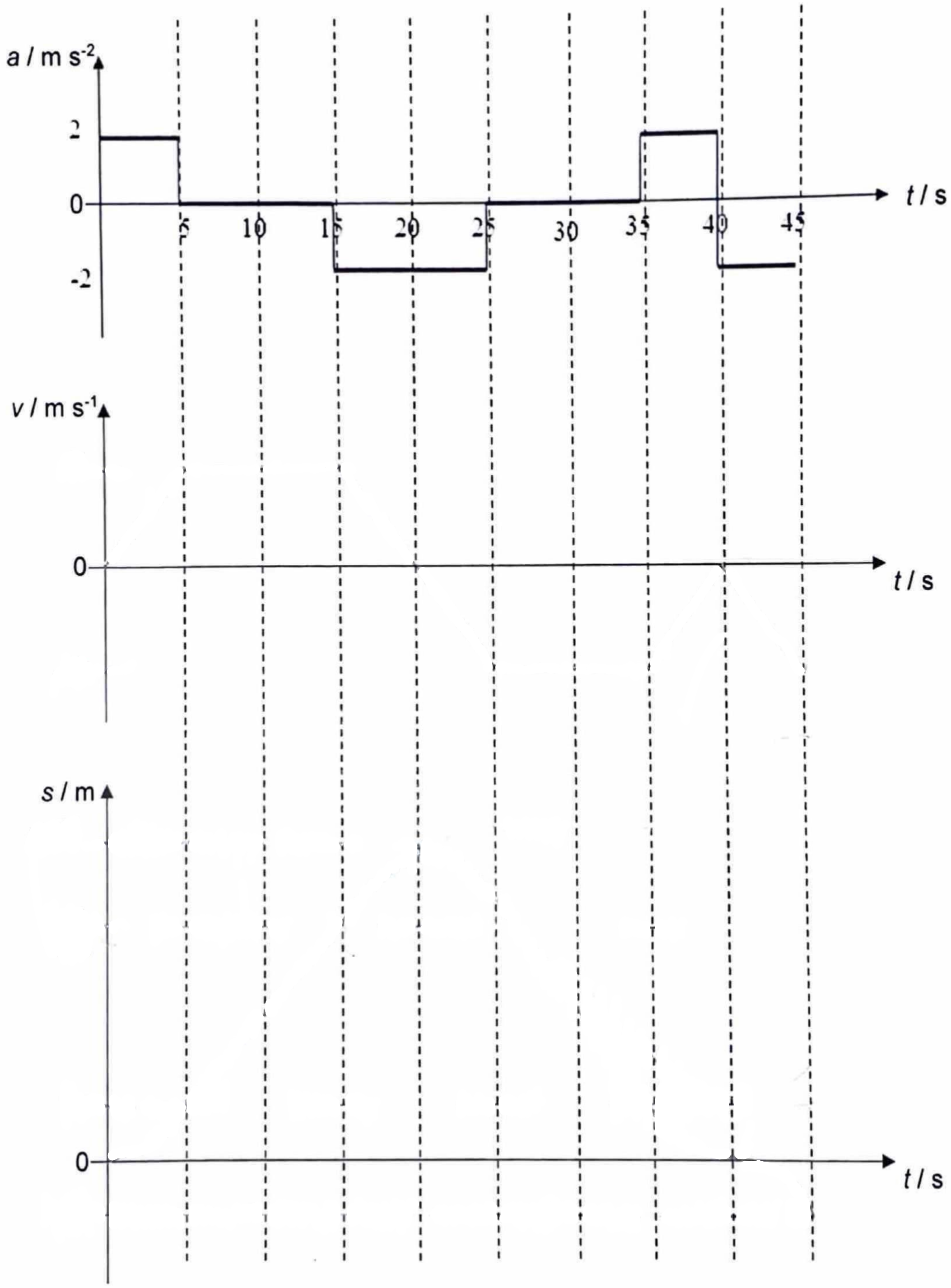
(ii) the time of flight of the shell for its entire trajectory,

(iii) the horizontal range R of the shell.

Discussion Questions

Rectilinear Motion

D1 The graph below shows the variation with time *t* of the acceleration *a* of a body moving along the x-axis. Sketch the graphs of the variation with time *t* of its velocity *v* and displacement *s*, given that at $t = 0$, $s = v = 0$. Indicate the relevant values on the vertical axis.



D2 A car starts from rest with a constant acceleration of 2.0 m s^{-2} north at the moment (t = 0 s) a lorry with a constant velocity of 18 m s^{-1} north, passes 60 m ahead of it. Sketch velocity-time graphs for the lorry and the car using the same axes, and use them to find the time at which the car overtakes the lorry, and the change of displacement of the car at that time.

D3 As a motorist travelling at 13 m s^{-1} approaches a traffic light, it turns red when he is 25 m away from the stop-line. His ***reaction time*** (i.e. the interval between seeing the red light and applying the brakes) is 0.70 s, and the condition of the road and his tyres is such that the car cannot slow down at a rate of more than 4.5 m s^{-2}. If he brakes fully, how far from the stop-line will he stop, and on which side of it?

D4 A hot air balloon is travelling vertically upward at a constant velocity of 5.0 m s^{-1}. When it is 21.0 m above the ground, a package is released from the balloon.

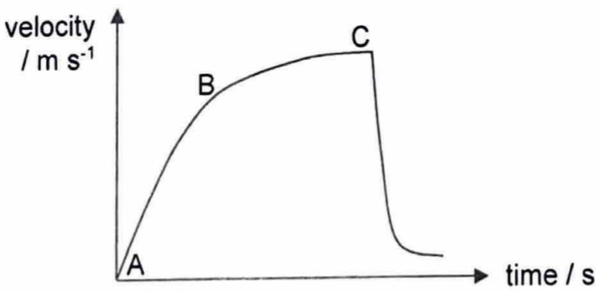
(a) How long after being released is the package in the air?

(b) What is the velocity of the package just before it hits the ground?

D5 A parachutist jumps from a hovering helicopter at an altitude of 1.5 km. For the first 10 s, he falls freely, and then he pulls the ripcord. The parachute opens and he falls with an upward acceleration of 20 m s^{-2} until the downward speed is 5.0 m s^{-1}. Thereafter, he falls at a constant velocity until he touches down.

(a) Determine the duration of the entire trip.

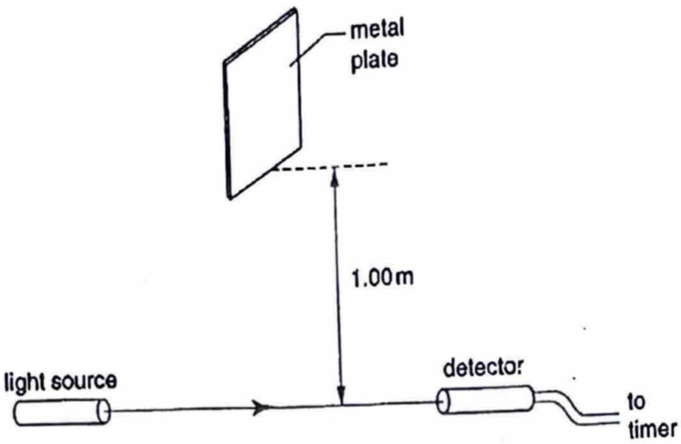
(b) The graph below shows the variation of the parachutist's actual velocity with time.



Regions A, B and C of the graph show the velocity before the parachute is opened. Explain why the graph has this shape in the regions marked A, B and C.

D6 [N05/2/1]

A student wishes to measure the length of a metal plate. The only equipment available is an electronic timer controlled by a light beam and a rod 1.00 m long.
Using the rod, the student positions the plate so that its lower edge is 1.00 m above the light beam, as shown in the following figure.



The metal plate is released and the timer starts to record as the light beam is cut. The total time for the plate to pass through the beam is 0.052 s. The student is told that the local value for the acceleration of free fall is 9.79 m s^{-2}.

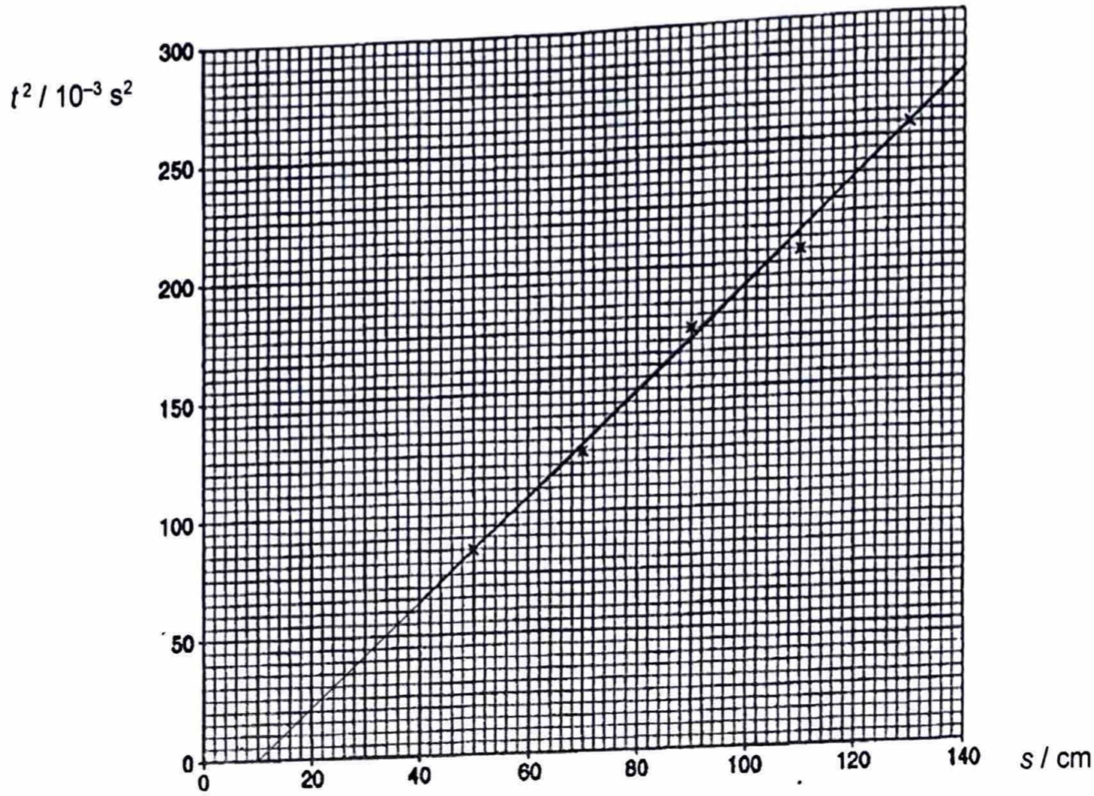
(a) **(i)** Show that the time for the bottom edge of the plate to reach the light beam is 0.452 s. [2]

(ii) Calculate the length of the metal plate, giving your answer to an appropriate number of significant figures. [2]

(b) Suggest two reasons why the time for the bottom edge of the plate to reach the light beam may differ from that quoted in **(a)(i)**. [3]

D7 [N06/II/1(part)]

A student takes measurements to determine the acceleration of free fall. He determines the time t taken for a ball to fall a distance s from rest. The following figure shows the variation with distance s of t^2.



(a) Use the figure to determine

(i) the gradient of the line, [1]

(ii) a value for the acceleration of free fall. [1]

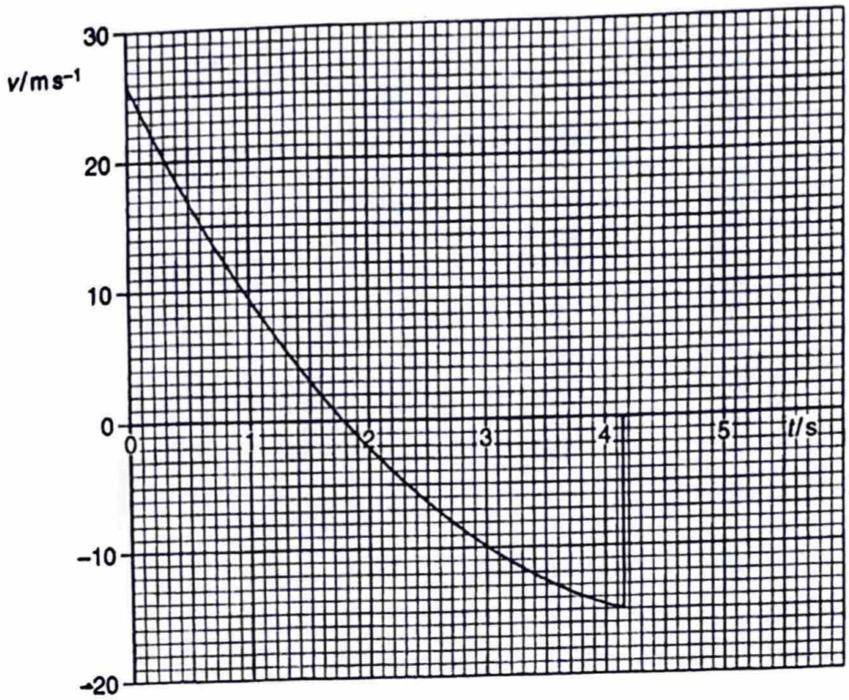
(b) The data collected by the student consisted of values of s and t. Plotting the variation with s of t^2 was expected to give a straight line. [2]

(i) State what other variation could be plotted to give a straight line graph from these data. [1]

(ii) Explain why the graph in **(b)(i)** gives a straight line. [1]

D8. [N03/III/1(part)]

The following graph shows the variation with time *t* of the velocity *v* of a ball from the moment it is thrown with a velocity of 26 m s^{-1} vertically upwards.



(a) State the time at which the ball reaches its maximum height.

(b) Just after the ball leaves the thrower's hand, it has a downward acceleration of approximately 20 m s^{-2}. Explain how this is possible.

(c) State the time at which the acceleration is *g*. Explain why the acceleration has this value only at this particular time.

(d) Sketch an acceleration-time graph for the motion. Show the value of *g* on the acceleration axis.

(e) Explain why, for all real vertical throws, the time taken to reach maximum height must be shorter than the time taken to return to the starting point.

Non-linear Motion

D9 Consider an object projected from the ground with speed *u* at an angle of θ upwards. It reaches a maximum height of *H* at t_H and hits the ground at time *t*.

(a) Obtain an expression in terms of *u* and θ for

(i) the object's time of flight *t*,

(ii) the time taken for the object to reach its maximum height t_H and.

(iii) the object's horizontal displacement *R*.

(b) Using your answer in **(a)(iii)**, show that a maximum horizontal displacement is achieved when $\theta = 45^\circ$.

(Hint: $\sin 2\theta = 2\sin\theta\cos\theta$)

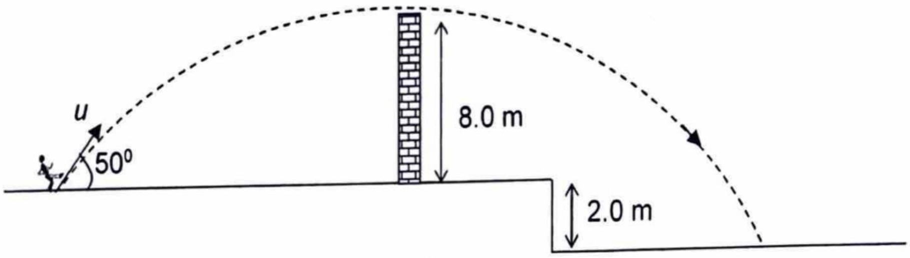
D10 A dive bomber, diving at an angle of 60° with the vertical, releases a bomb at an altitude of 600 m. The bomb hits the ground 5.0 s after being released.

(a) What is the speed of the bomber?

(b) How far did the bomb travel horizontally during its flight?

(c) What is the magnitude and direction of the velocity of the bomb just before striking the ground?

D11 A ball was kicked over a 8.0 m wall as shown with a velocity u at an angle of 50° above the horizontal. At the highest part of the trajectory, the ball managed to just go over the wall. It landed into a depression 2.0 m deep.



Deduce

(a) the initial speed of the ball, [2]

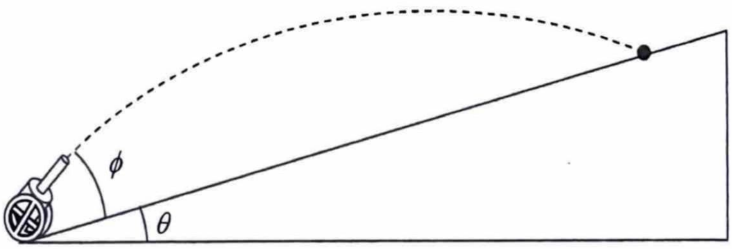
(b) the velocity of the ball just before it hit the depression, [4]

(c) the time of flight of the ball. [2]

Challenging Questions

C1 A fighter bomber is flying 50 m s^{-1} horizontally at an altitude of 600 m. Its target is a tank that is 700 m ahead and moving at 15 m s^{-1} in the same direction. Assuming negligible air resistance, how many seconds later should it drop a bomb so that it will hit the tank?

C2 A cannon fires a shot at an angle of ϕ up a plane inclined to the horizontal at an angle θ as shown. In terms of θ, at what angle ϕ, should the shot be fired in order to maximize the distance s up the inclined plane? Explain clearly how you obtain your answer.



Answers

D2 20.9 s, 436 m
D3 2.9 m
D4 2.64 s, 20.9 m s^{-1} downwards
D5 169 s
D6 0.24 m
D7 0.212 $s^2\ m^{-1}$, 9.43 m s^{-2}
D8 1.80 s, 1.80 s
D10 190 m s^{-1}, 830 m, 220 m s^{-1} at an angle of 41° below the horizontal
D11 16.4 m s^{-1}, 17.5 m s^{-1} at 53.1° below the horizontal, 2.7 s
C1 8.9 s **C2** $\phi = \pi/4 - \theta/2$

Tutorial 2 Kinematics Suggested Solutions

S1 Displacement is the distance moved in a specified direction from a reference point. It is a vector quantity.

The instantaneous speed of an object is defined as the rate of change of distance travelled with respect to time. It is a scalar quantity.

The instantaneous velocity of an object is defined as the rate of change of displacement with respect to time. It is a vector quantity.

The instantaneous acceleration of an object is defined as the rate of change of velocity with respect to time. It is a vector quantity.

S2

	Gradient	**Area Under**
Displacement-Time	Instantaneous velocity	No significance
Velocity-Time	Instantaneous acceleration	Change in displacement
Acceleration-Time	No significance	Change in velocity

S3 Refer to Section 2.4 of lecture notes for derivation.

Conditions for equations to be valid:

1. Acceleration must be constant.
2. Motion must be in a straight line.

S4 When the object is released, it will accelerate. As it gains velocity, the amount of drag force acting on it will increase (since drag force is dependent on velocity of a body). When velocity increases to a value such that the drag force acting on the object is equal in magnitude (and opposite in direction) to the weight of the object, there will be no net force acting on the object. Hence, the acceleration of the object becomes zero, and the object continues to fall with a constant velocity. This is known as the terminal velocity of the object.

S5 When an object is thrown near the earth's surface and describes a curved path in vertical plane (due to the presence of gravity), it undergoes projectile motion.

In analyzing projectile motion, we assume that

the effects of air resistance are negligible,

- acceleration in the horizontal direction is always zero ($a_x = 0$).
- acceleration in the vertical direction is only due to gravity and directed downwards ($a_y = g$).

S6 The path described by a projectile is known as its trajectory.

The horizontal distance on the plane between the point of projection and the point of impact is known as the range.

S7 (a) At maximum height, $v_y = 0$

(b) The horizontal component of velocity remains unchanged at all points throughout the flight (since horizontal acceleration is zero).

S8
1. The trajectory of the projectile is no longer symmetrical,
2. the maximum height reached is lower and
3. the horizontal range is shorter.

SP1 Area of triangle = additional distance covered by the car in time T as compared to the lorry

$\frac{1}{2}(T)(26 - 10) = 17 + 7$

$T = 3.0$ s

Answer: D

SP2 Note: deep tank with viscous oil => steel sphere reaches terminal velocity after some time

Therefore, the spacing between two dots (indicating velocity since the time interval is the same) will increase initially (showing increase in velocity). When terminal velocity is achieved, the spacing between two dots will be equal (showing constant velocity, i.e terminal velocity)

Answer: B

SP3 Recall gradient of the *v-t* graph gives the acceleration.

Answer: C

SP4 (From the graph, downward is the positive direction)

Negative area X gives the upward displacement as the ball bounces up from its 1st impact to reach its maximum height (zero velocity). Positive area Y gives the downward displacement as it heads down to make its second impact.

Answer: B

SP5

$$s_1 = ut + \frac{1}{2}gt^2 \quad \cdots\cdots (1)$$

$$s_1 + s_2 = u(2t) + \frac{1}{2}g(2t)^2 \quad \cdots\cdots (2)$$

$$(2) - 2\times(1) \Rightarrow s_2 - s_1 = gt^2$$

Answer: A

SP6 (a) Taking upwards as positive,

$$v = u + at$$
$$0 = 20 - 9.81t$$
$$t = 2.04 \text{ s}$$

t is time taken to reach maximum height.
Time to return to thrower's hand
$= 2t = 2 \times 2.04 = 4.08$ s

or:

$$v = u + at$$
$$-20 = 20 - 9.81t'$$
$$t' = 4.08 \text{ s}$$

t' is total time (ball returns to thrower at speed 20 m s^{-1} in opposite direction)

(b) Consider upward journey,

$$v^2 = u^2 + 2as$$
$$0 = 20^2 + 2(-9.81)s$$
$$s = 20.4 \text{ m}$$

Max height = 20.4 m

or:

$$s = ut + \frac{1}{2}at^2$$
$$s = 20(2.039) + \frac{1}{2}(-9.81)(2.039)^2$$
$$= 20.4 \text{ m}$$

SP7 (a) Using $s_y = u_y t + \frac{1}{2}a_y t^2$ and substituting s_y = 300 m, u_y = 0 and a_y = 9.81 m s^{-2},

$$300 = \tfrac{1}{2}gt^2$$
$$t = \sqrt{\frac{2 \times 300}{9.81}} = 7.8 \text{ s}$$

(b) Horizontal distance travelled by the package = $u_x t$ = 140 × 7.82 = 1100 m = 1.1 km

SP8 (a) (i) Horizontal component of the initial velocity = 45 cos 30° = 39.0 m s^{-1}

(ii) Vertical component of the initial velocity = 45 sin 30° = 22.5 m s^{-1}

(b) (i) Consider vertical motion:

Substituting v_y = 0 into $v_y^2 = u_y^2 + 2a_y s_y$,

$$0 = 22.5^2 + 2(-9.81)h$$
$$h = \frac{22.5^2}{2(9.81)} = 25.8 \text{ m}$$

(ii) Substituting s_y = 0 into $s_y = u_y t + \frac{1}{2}a_y t^2$,

$$0 = u_y t - \tfrac{1}{2}gt^2$$
$$t\left(u_y - \tfrac{1}{2}gt\right) = 0$$
$$t = 0 \text{ (N.A.)} \quad \text{or } t = \frac{2 \times 22.5}{9.81} = 4.59 \text{ s}$$

(iii) Horizontal range $R = u_x t$ = 39.0 × 4.59 = 179 m