Measurement

1 Importance of Measurement

Scientific knowledge is powerful because it is not just untested theory and hypotheses. Scientists demand that the theories be supported with empirical evidence or measurements. For example, Einstein's General Relativity theory suggested that gravity can bend the path of light but our ultimate confidence in the theory is whether the bending can be measured and checked against the amount predicted by the theory.

Measurement is about quantifying things and the ability to quantify things allows calculations, analyses and deductions which in turn lead to new knowledge or theories. Once a theory is well supported by empirical data, it then can be used for predicting outcomes or results. To understand what the big deal about prediction is, let's just consider the construction of a high rise building which costs millions of dollars. An architect will carry out calculations to make sure the designed building can withstand the expected loading and maybe possible earthquake. Those calculations are done using theoretical formulae that have been verified with prior measurements!

2 SI System of Quantities & Units

For measurements to be useful, they need to be expressed in terms of appropriate units which are internationally accepted.

* See Joint Committee for Guides in Metrology (JCGM), International Vocabulary of Metrology, Basic and General Concepts and Associated Terms (VIM), III ed., Pavillon de Breteuil : JCGM 200:2012.

International System of Quantities* consists of	International System of Units* consists of
7 base quantities which cannot be defined in terms of other quantities: length, mass, time, electric current, thermodynamic temperature, amount of substance, luminous intensity	7 base units corresponding to the base quantities: metre (m), kilogram (kg), second (s), ampere (A), kelvin (K), mole (mol) and candela (cd) (Note: candela is not required in syllabus)
Derived quantities are defined in terms of the base quantities. e.g. Force $F = ma$ but $a = \dfrac{\Delta v}{\Delta t}$ and $v = \dfrac{\Delta s}{\Delta t}$ so F is finally defined in terms of the base quantities length, mass and time.	Derived units are defined in terms of the base units. The notation* [Q] means 'unit of Q': $[F] = [ma]$ $= [m]\,[a] = [m]\,[v/t]$ $= [m]\,[s/t]/[t]$ $=$ kg m s^{-2} The derived unit kg m s^{-2} is given a more convenient short form N for newton.

There are 7 base quantities and corresponding base units in the SI system of quantities and units.

All other quantities are called derived quantities as they are ultimately defined in terms of the 7 base quantities using defining equations. The *same equations* are also used to work out the corresponding derived units.

In the SI system, the same set of equations is used to express derived quantities or units in terms of the base quantities or units.

Note also that *quantities* are always defined in terms of *quantities* and *units* are always defined in terms of *units* i.e. quantities and units are *different* entities. Hence it would be wrong to define *speed*(a quantity) as 'the *distance*(a quantity) travelled per *second*(a unit)'.

Quantities cannot be defined in terms of units and vice versa.

Examples of SI Derived Units

Quantities	Derived in terms of base	and short form
Energy	kg m^2 s^{-2}	J
Power	m^2 kg s^{-3}	W
Electric potential difference	m^2 kg s^{-3} A^{-1}	V
Electric resistance	m^2 kg s^{-3} A^{-2}	Ω
Specific heat capacity	m^2 kg s^{-2} K^{-1}	J kg^{-1} K^{-1}
Electric field strength	m kg s^{-3} A^{-1}	V m^{-1} or N C^{-1}

SI derived units are often given short forms which one must learn to recognise.

Multiples and submultiples.

Frequently, there are situations where it is more convenient to use smaller or bigger units than the standard base and derived units. For that purpose, prefixes for multiples and submultiples are used together with the base or derived units.

For example, when molecular size objects are being measured, it is more convenient to use nano-metre or nm in recordings. Another example is the use of mega-watt or MW when referring to power output of power stations.

Factor	Prefix	
	Name	Symbol
10^{-12}	pico	p
10^{-9}	nano	n
10^{-6}	micro	μ
10^{-3}	milli	m
10^{-2}	centi	c
10^{-1}	deci	d
10^{3}	kilo	k
10^{6}	mega	M
10^{9}	giga	G
10^{12}	tera	T

Only those in syllabus

Prefixes for multiples and submultiples are often used and need to be learnt.

Examples of Non-SI units & Conversion Factors

Length	- inch (″)	1 ″ = 0.0254 m or 2.54 cm
Speed	- knot (kn)	1 kn = 0.514444 m s^{-1}
Energy	- electronvolt (eV)	1 eV = 1.6×10^{-19} J
Pressure	- bar (bar)	1 bar = 10^5 Pa
Time	- minutes (min or ′)	1 min = 60 s
Time	- hour (h or hr)	1 h = 3600 s
Volume	- litre (L or l)	1 L = 10^{-3} m^3 or 1000 cm^3

The factors for multiples, submultiples and for converting non-SI to SI units are important in calculations.

Why are units important?

Without one, a measurement value is a pure number that does not tell us how much of what quantity we have and thus is a useless number. In addition, failure to pay attention to the kind of units used is a very common reason for students getting wrong values in calculations. Some people paid dearly for such mistakes. In 1999, NASA's Mars orbiter smashed into the planet's atmosphere because their engineers failed to convert English pound of force into SI newton in their calculations.

How to handle units in equations?

Rule 1 – All *terms* in an equation must have the same units. *Terms* refer to quantities forming a group by multiplication or division and each group is separated from others by + - or = sign. For example, $s = ut + ½ at^2$ is made up of three terms s, ut and $½ at^2$ and if s is to be in cm, then numerical values substituted must yield cm for both terms ut and $½ at^2$.

All *terms* in an equation must have the same units.

Rule 2 – Attention must be paid to the conversion factors when *multiples* or *submultiples* of base units or when *non-SI units* are used in calculations so that all terms will have consistent units. In contrast, using *only SI base* and *SI derived* units does not introduce any numerical factor into the equations.

Examples:

The net force required to give a 2 kg mass an acceleration of 3 m s^{-2} is calculated by $F_{net} = ma = 2(3) = 6$ N but the net force required to give a 2 kg mass an acceleration of 3 cm s^{-2} is definitely not calculated by $F_{net} = ma = 2(3) = 6$ N. The latter case's numerical value of 6 is not 6 N of force but should be 6 kg cm s^{-2} of force and 1 N is not the same as 1 kg cm s^{-2}.

Given a particle of mass 1.7×10^{-27} kg with KE 3 eV and desiring to find the speed in m s^{-1} using $KE = ½ mv^2$, one must convert eV into J using the appropriate conversion factor.

For calculations using numerical values in *multiples* or *submultiples of units* and *non-SI units*, their conversion factors should be used to make the units on both sides of the equation the same.

3 Errors and Uncertainties

Definitions

Measurement error = Measured value – True value

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In the majority of cases, the true value is unknown. However, many reference values have been established through careful measurements so that engineers and scientists can compare their measured values against these reference values instead.

Measurement error is made up of two components: Systematic and Random.

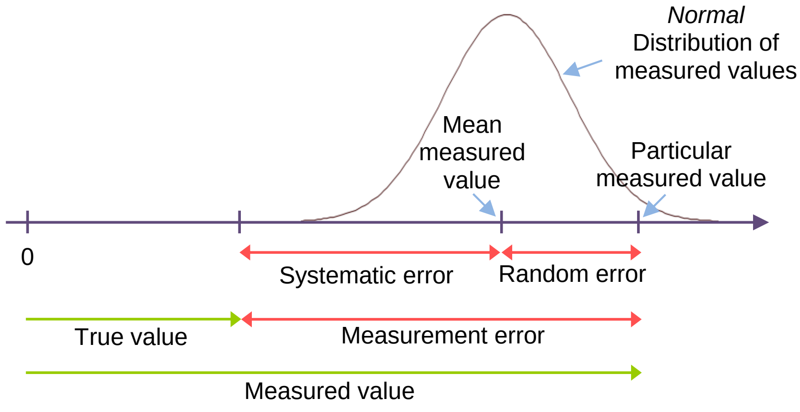
Measurement error = Systematic error + Random error

Systematic error is one which is constant or varies in a predictable way when the measurement is repeated.

Random error is one which varies in unpredictable manner when the measurement is repeated.

Measurement error = Systematic error (one which is constant or varies in a predictable way when the measurement is repeated) + Random error (one which varies in unpredictable manner when the measurement is repeated)

Relationship between one particular measured value, true value and the two component errors:



Sources of Systematic and Random Errors

Origins of random error could be:

1. Apparatus have unpredictable fluctuations which in turn may be due to random fluctuations in environmental conditions. e.g. vibrations, fluctuating temperature, pressure, electric and magnetic fields.
2. Experimenter's randomness in interpreting reading or carrying out procedures. e.g. reaction time in using stopwatch, not placing load at exactly the same spot as required, not measuring length from exact same point perhaps due to difficulty in placing ruler right next to it.
3. Property to be measured has random nature. e.g. radioactive decay, voltage inherently fluctuates due to fluctuating environmental conditions, diameter of wire varies along wire.

Origins of systematic error could be:

1. Apparatus have predictable errors. e.g zero error, a fixed extension of string for a given load, faulty stopwatch which consistently runs fast, calibration markings which are closer than what they should be, electrical circuit heats up after prolonged usage.
2. Experimenter may interpret reading or carry out procedure with a consistent error. e.g. measuring length of pendulum from suspension point to the tip instead of centre of bob, marking out a wrong height on a ramp to release a ball, leaving a mercury thermometer bulb at the bottom of a beaker close to the heat source and not stirring.

There are many sources contributing to the final systematic or random error.

Handling Measurement Errors

Understanding how the errors arise allows us to better manage them. As seen above, each type of error can have a number of contributing factors. In principle, the total systematic error can be eliminated if we can identify all the causes. The problem is that they are difficult to detect especially if true or reference values are not available.

Systematic error can be eliminated once the causes are known.

On the other hand, total random error cannot be completely eliminated but only reduced. Some contributions such as experimenter's inconsistent way of carrying out the steps or the poor set up of equipment might be eliminated or reduced with better technique. For example, parallax error when a ruler is not placed right next to the start and end points but separated by a gap. The gap causes the experimenter to estimate the start and end points on the ruler, thereby causing random error. When all that can be done has been done, the component of random error that cannot be removed would be the error that is due to the interpolation between scale markings.

Random error cannot be totally eliminated. Some contributions can be eliminated or reduced. The contribution associated with the smallest division of the instrument will always be present.

For a single measurement without a true value for reference, we can never know the amount of random error except the contribution linked to the smallest scale marking. With repeated measurements, we get a better idea of the amount of random error. With many measured values, the individual random errors will follow a normal distribution as shown above. The *representative random error* or *uncertainty* is taken to be the *standard deviation σ of the errors*. As calculation of σ is not required at 'A' level, it is enough to know that σ varies as $1/\sqrt{N-1}$ which means that the larger the number of readings N the smaller the *uncertainty* σ. For 'A' level, it is enough to *estimate σ based on the amount of fluctuation from the mean*.

The table shows 6 pairs of measurements of some length, average values and deviations ΔL from the means. Ideally more repeated measurements is better but minimum expectation for 'A' level is to repeat once. Even then, the ΔL can still give a very rough idea of σ.

L_1 / cm	L_2 / cm	L_{ave} / cm	ΔL / cm
4.6	5.1	4.85	0.25
6.5	6.0	6.25	0.25
7.9	8.3	8.10	0.20
8.9	9.2	9.05	0.15
10.1	10.8	10.45	0.35
11.9	12.5	12.20	0.30

We see that ΔL is from 0.15 to 0.35, pointing to random error contributions besides the 0.1 cm associated with the ruler's smallest division. Estimation of σ is about 0.25 or 0.3 cm. To conclude, when repeated measurements are available, we take the mean value to be a representative or reliable value of the measured quantity and σ to be the uncertainty; where σ is like the mean of the random errors.

Making more repeated measurements lead to a more reliable mean value with a representative random error called the uncertainty.

Note that the uncertainty σ is not exact; hence there is no sense in quoting its value with more than 1 significant figure (s.f.). Thus knowing the uncertainty allows us to record the value of our measured quantity with a meaningful number of s.f. For example, the measured quantity in row 1 is represented by the mean value 4.85 cm but since the error σ is 0.3, the digit 8 is uncertain. The digit 5 has place value of 0.01 compared to place value 0.1 for digit 8. Digit 5 is thus less significant than 8. With digit 8 being uncertain, we should only record the final measurement as 4.9 ± 0.3 cm.

Uncertainty should be given to 1 s.f.

4 Precision Vs Accuracy

Precision describes measured values or an instrument. A smaller precision means a smaller random errors and closer clustering of measured values.

Precision is a measure of the closeness of measured values when a measurement is repeated.

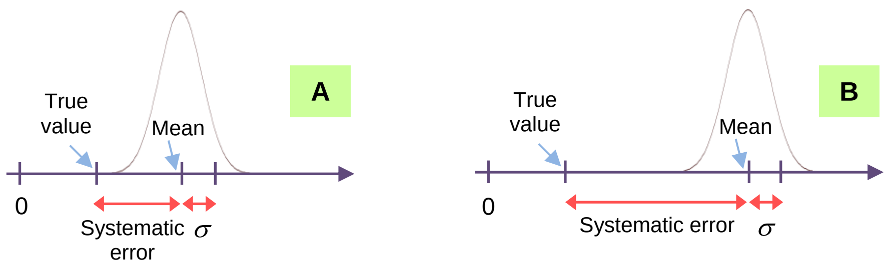
The word precision is used to describe *measured values* as well as the *instrument*. A higher precision corresponds to a smaller spread of values about the mean i.e. smaller random errors or σ.

Accuracy is a measure of the closeness of a measured value to the true value or reference value.

Accuracy is thus poor if the systematic error is large.

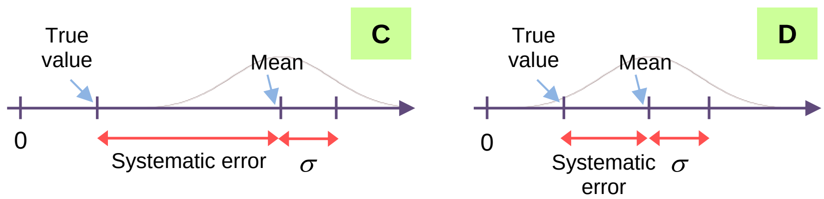
Accuracy describes the closeness of measured values from the true or reference value. Better accuracy means smaller systematic error.

Mean measured value in **A** is more accurate than in **B** but equally precise

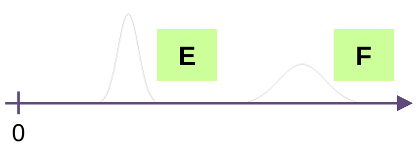


Mean measured value in **A** is more accurate and more precise than in **C**

Mean measured value in **B** is more precise but less accurate than in **D**



For **E** and **F**, can compare precision but not accuracy



No assessment of accuracy is possible without knowing the true value or reference value.

5 Propagation of Random Errors in Calculations

Measured values are often called raw data. Assuming that systematic errors have been eliminated, there are still random errors that are impossible to eliminate, and we would expect the uncertainty to propagate to the calculated values. So how do uncertainties propagate in calculations? The answer is that it depends on the type of mathematical operations involved.

There are two basic types of math operations (Δa, Δb, Δc are uncertainties of the raw data and R is the calculated result):

+ & –	× & ÷
Basic rule: Then $\boxed{R = a + b \quad \text{or} \quad R = a - b \qquad \Delta R = \Delta a + \Delta b}$	Basic rule: Then $\boxed{R = a \times b \div c \qquad \dfrac{\Delta R}{R} = \dfrac{\Delta a}{a} + \dfrac{\Delta b}{b} + \dfrac{\Delta c}{c}}$
Variations from the basic rule: For $R = 2a = a + a$, $\Delta R = 2\Delta a$. In general, for $R = na$, whether n is integer or not, $\Delta R = n\Delta a$	Variations from the basic rule: For $R = a^2$ or $R = a^{-2}$, $\dfrac{\Delta R}{R} = \dfrac{\Delta a}{a} + \dfrac{\Delta a}{a}$, $\dfrac{\Delta R}{R} = \dfrac{2\Delta a}{a}$. In general, for $R = a^{\pm n}$, whether n is integer or not, $\dfrac{\Delta R}{R} = \dfrac{n\Delta a}{a}$

Combinations of + – × ÷

Example:

$R = 2ab = ab + ab$ or $2Q$ $\qquad Q = ab$

$\Delta R = 2\Delta Q$ $\qquad$ where $\qquad \dfrac{\Delta Q}{Q} = \dfrac{\Delta a}{a} + \dfrac{\Delta b}{b}$

$$\Delta R = 2\Delta Q = 2\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)Q$$

$$\frac{\Delta R}{2Q} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$$

$$\frac{\Delta R}{R} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$$

Hence, in general, for $R = nab$ where n is a number that is *not measured and without uncertainty*, $\dfrac{\Delta R}{R} = \dfrac{\Delta a}{a} + \dfrac{\Delta b}{b}$ i.e. use basic rule for × ÷ and simply ignore n.

Example:

$v = u + at$ $\qquad$ where u, a and t are measured and v is calculated

$\Delta v = \Delta u + \Delta R$ $\quad$ where $\quad \dfrac{\Delta R}{R} = \dfrac{\Delta a}{a} + \dfrac{\Delta t}{t} \Rightarrow \Delta R = \left(\dfrac{\Delta a}{a} + \dfrac{\Delta t}{t}\right)R$

$$\Delta v = \Delta u + \left(\frac{\Delta a}{a} + \frac{\Delta t}{t}\right)at$$

There are other mathematical operations such as log a, sin a, cos a and e^a but the propagation of uncertainties for these is not required for 'A' level.

Measurement errors propagate through calculations depending on the math operations. There are two basic categories of math operations. Each category has its basic rule for the calculation of propagated uncertainty.

For more complicated calculations, a combination of the basic rules apply.

Δa is called absolute uncertainty while $\dfrac{\Delta a}{a}$ is called fractional uncertainty and $\dfrac{\Delta a}{a} \times 100\%$ is called percentage uncertainty. Absolute uncertainty is given to 1 s.f. while both fractional and percentage uncertainties are frequently given up to 2 s.f. but not more.

Absolute uncertainty: Δa

Fractional uncertainty: $\dfrac{\Delta a}{a}$

Percentage uncertainty: $\dfrac{\Delta a}{a} \times 100\%$

It is of utmost importance to express the calculated quantity in terms of measured ones before applying the calculation rules. This will ensure that uncertainties of *measured* quantities are never subtracted from one another; only added. However, uncertainty of *measured* quantities can be subtracted from *calculated* quantity.

Always express calculated quantity in terms of measured ones before applying the calculation rules. This ensures uncertainties of *measured* quantities are never subtracted from one another; only added. However, uncertainty of *measured* quantities can be subtracted from *calculated* quantity.

<table>
<tr><td colspan="2">Given $T = 2\pi\sqrt{\dfrac{L}{g}}$ where T is the period of a pendulum of length L and g is the free fall acceleration. Calculate the fractional uncertainty of g given the measurements $T \pm \Delta T$ and $L \pm \Delta L$.</td></tr>
<tr><td>Calculation 1:
$$T = 2\pi\sqrt{\frac{L}{g}}$$ $$\Rightarrow \frac{\Delta T}{T} = \frac{1}{2}\left(\frac{\Delta L}{L}\right) + \frac{1}{2}\left(\frac{\Delta g}{g}\right)$$ $$\Rightarrow \frac{\Delta g}{g} = \frac{2\Delta T}{T} - \frac{\Delta L}{L}$$ ✗</td><td>Calculation 2:
Put g as subject first,
$$g = \frac{4\pi^2 L}{T^2}$$ $$\Rightarrow \frac{\Delta g}{g} = \left(\frac{\Delta L}{L}\right) + 2\left(\frac{\Delta T}{T}\right)$$ ✓</td></tr>
</table>

6 Estimation of Errors in Calculations

There are times when we do not need to know precisely the uncertainties in our calculated values. Then we would not apply the calculation rules in section 5. Instead, we rely on rules of thumb to help us estimate the uncertainties and then present the calculated values with a reasonable number of s.f. One such situation is in parts of 'A' level practical work.

Rule of thumb 1 When multiplying or dividing two numbers, the number of s.f. in the answer should follow the input number with the least number of s.f.
Rule of thumb 2 When adding or subtracting two numbers, the number of d.p. (decimal places) in the answer should follow the input number with the least number of d.p.
Rule of thumb 3 When doing log a or ln a, the number of d.p. in the answer should be the number of s.f. of the input number.

If precise uncertainties are not needed, rules of thumb help us to retain a reasonable number of s.f. in calculated values.

The number of s.f. presented in a numerical value is the number of digits excluding the leading zeros but including the trailing zeros. The last digit is the one with uncertainty except for cases like last row's 320, which is ambiguous if the uncertainty is not stated. If the number is presented as 320 ± 2, then the number has 3 s.f. but if presented as 320 ± 10, then it has 2 s.f.

Number	Number of s.f.
315.2	4 s.f.
315.0	4 s.f.
0.3150	4 s.f
0.315	3 s.f.
320	2 or 3 s.f.

7 Estimation of Quantities

Estimating the value of a quantity in a given situation is very useful for quickly spotting mistakes in calculations.

Example:

A student is told that a sealed container of volume 0.25 m^3 contains 10 mol or 6.02×10^{22} molecules. He tried to calculate the diameter of each molecule as follows.

Volume occupied by each molecule $4\pi r^3/3 = 0.25/(6.02 \times 10^{22})$

radius $r = 10 \times 10^{-9}$ m

diameter $= 2.0 \times 10^{-8}$ m

If he had known that the typical size of an atom is about 10^{-10} m, then the calculated answer will immediately look wrong as it is 200 times bigger than 10^{-10} m.

No typical values specified by the syllabus but some suggested values are:

Quantity	Typical value
Size of atom	10^{-10} m
Height of Mt. Everest	8 km
Time for a human to sprint 100 m	10 s
Mass of a car	1000 – 2000 kg
Density of water	1 g cm^{-3}
Atmospheric pressure at sea level	10^5 Pa
Power of kettle	1 – 2 kW
Power consumption of filament bulb and LED	60 W, 50 mW
Household socket maximum current	13 A
Accelerating voltage in X-ray machine	10 kV
Visible light wavelength	400 – 800 nm

You can estimate some quantities indirectly if you know how they are related to other quantities. For example, the height of a HDB block is the number of storeys × height of each storey which can be estimated more easily. The mass of water in a 1 m^3 container = 10^6 cm^3 × 1 g cm^{-3} = 10^6 g or 1000 kg.

The ability to remember typical values of common quantities is a useful first line of defence against calculation mistakes.

8 Scalars and Vectors

A scalar is a quantity which only has a magnitude but no direction

A vector is a quantity which has a magnitude and a direction.

A scalar only has magnitude while a vector has both magnitude and direction.

Note that a vector cannot be specified with just a number, because a single number cannot tell us its direction. There are many ways to specify or describe a vector. Some examples are:

1. Add a description to a number e.g. 5 N pointing North, 2 m s^{-1} with bearing 120°, 6 m s^{-1} at an angle of 60° with respect to the horizontal.
2. Draw an arrow whose length tells us the magnitude and the direction is as indicated by the arrow.
3. If the vectors are all either pointing one way or directly opposite, then + and - sign can be used to indicate the two opposite directions. For example, if + 5 m displacement is to the right then - 5 m is to the left.

There are many ways to specify the direction of a vector. In particular, vector diagrams are very useful.

Drawing accurate arrows on paper is a neat way to specify a vector. A well-drawn scale diagram can help us find the result of addition or subtraction of vectors. If only a rough sketch is drawn, it is still very useful as it allows us to make use of trigonometric functions, sine rule and cosine rule and Pythogoras Theorem to calculate the result of addition or subtraction.

Vector Subtraction

At 'A' level, there will be frequent encounters of vector subtraction because some common quantities are defined in terms of change of vectors. For example, acceleration is defined as rate of change of velocity $\frac{\Delta\vec{v}}{\Delta t}$. It is also very frequent that students wrongly subtract vectors as if they are just magnitudes. Since the *direction* associated with vectors cannot be subtracted numerically, the correct way is to use a vector diagram to keep track of the subtraction of both magnitude and direction. Vector subtraction is built upon the following two concepts:

1. negative of a vector
2. vector addition

Remembering which quantities are vectors is very important because vector addition and subtraction is not the same as for scalars.

Vector subtraction is built upon concept of
1. negative of a vector &
2. vector addition.

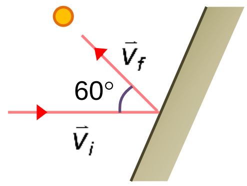
$\Delta\vec{v} = \vec{v}_f - \vec{v}_i$

$\Delta\vec{v} = \vec{v}_f + \vec{x}$

where $\vec{x} = -\vec{v}_i$

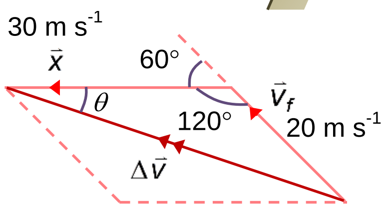
Example of Vector Subtraction

A ball approaches a slanted wall with a horizontal velocity of 30 m s^{-1}. After hitting the wall, it rebounds with a velocity of 20 m s^{-1} in a direction at an angle of 60° with the horizontal. Find the change in velocity.



By definition $\Delta\vec{v} = \vec{v}_f - \vec{v}_i$

$\Delta\vec{v} = \vec{v}_f + \vec{x}$ where $\vec{x} = -\vec{v}_i$



Using the vector diagram, the magnitude of $\Delta\vec{v}$ can be found using cosine rule:

$$|\Delta\vec{v}|^2 = 30^2 + 20^2 - 2(30)(20)\cos 120°$$

$$|\Delta\vec{v}|^2 = 43.6 \text{ m s}^{-1}$$

To find θ so that the direction of $\Delta\vec{v}$ can be specified, sine rule can be used:

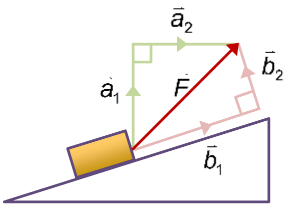
$$\frac{\sin\theta}{20} = \frac{\sin 120°}{|\Delta\vec{v}|}$$

$$\theta = 23°$$

$\therefore$ change in velocity is 44 m s^{-1}, at an angle of 23° with the horizontal.

Resolution of a Vector

Resolution of a 2D vector $\vec{R}$ refers to finding a pair of vectors which when added will produce the vector $\vec{R}$. There are actually an infinite number of pairs of vectors that when added will give $\vec{R}$. However, a particular kind that is most useful in Physics is a perpendicular pair. For example, the force $\vec{F}$ acting on a block on a slope can be *resolved* into *components* $\vec{a}_1$ & $\vec{a}_2$ or into the pair $\vec{b}_1$ & $\vec{b}_2$ or any other pair according to the *purpose* of resolution.



Components are frequently needed, so familiarity with resolution of a vector into perpendicular components is very important.

The reason for choosing perpendicular components is because they are *independent*, which allows analysis and calculations to be done for a specific direction involving only the relevant components. A simple example is the force $\vec{F}$ on the block above. The acceleration along the slope is only affected by the component $\vec{b}_1$ and not $\vec{b}_2$.

The directions of the components are chosen based on the *purpose* of resolution.

The magnitude of a component is frequently needed in terms of the magnitude R of the original vector. The components' magnitudes can be found using the definitions of sine and cosine.