PJC 2011 JC2 H2 Mathematics End of Year Paper 2 Solution

1

Let sequence, $T_r = ar^3 + br^2 + cr + d$

then $a+b+c+d=1$

$8a+4b+2c+d=2$

$27a+9b+3c+d=4$

$64a+16b+4c+d=8$

Using GC: $a=\frac{1}{6}, \quad b=-\frac{1}{2}, \quad c=\frac{4}{3}, \quad d=0$

$T_r = \frac{1}{6}r^3 - \frac{1}{2}r^2 + \frac{4}{3}r$

2(i)

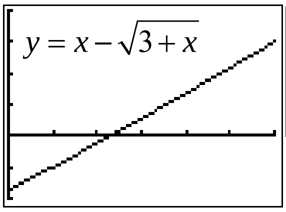
Let $\lim\limits_{n\to\infty} X_n = \alpha$ i.e as $n \to \infty$, $X_n \to \alpha$ and $X_{n+1} \to \alpha$

then $\alpha = \sqrt{3+\alpha}$

$\alpha^2 - \alpha - 3 = 0$

$\alpha = \frac{1 \pm \sqrt{1+12}}{2}$

$= \frac{1+\sqrt{13}}{2}$ or $\frac{1-\sqrt{13}}{2}$ (NA since x_n>0)



Note: Graph is not a straight line

from graph of $y = x - \sqrt{3+x}$

for $0 < x < \alpha$, $y < 0$

$x_n - \sqrt{3+x_n} < 0$

$x_n - x_{n+1} < 0$

$x_{n+1} > x_n$

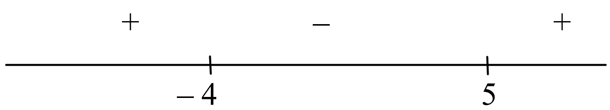
3

$\frac{2x-1}{x+4} \le 1$

$\frac{2x-1}{x+4} - 1 \le 0$

$\frac{x-5}{x+4} \le 0$

$-4 < x \le 5$ ---------------(1)



$\frac{(2-e^x) \div e^x}{(1+4e^x) \div e^x} \le 1$

$\frac{2e^{-x} - 1}{e^{-x} + 4} \le 1$

Replace x in (1) with e^{-x}

then $\quad -4 < \mathrm{e}^{-x} \le 5$

since $\mathrm{e}^{-x} > 0$ for all x

$0 < \mathrm{e}^{-x} \le 5$

$-x \le \ln 5 \quad \rightarrow \quad x \ge -\ln 5$

4(i)

$$\overrightarrow{\mathrm{OL}} = \frac{1}{2}\left(\overrightarrow{OC} + \overrightarrow{OD}\right) = \frac{1}{2}\left[\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}\right] = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

4(ii)

area $\Delta OAL = \frac{1}{2}(OL)(OA)\sin\angle AOL = \frac{1}{2}\left|\overrightarrow{OL} \times \overrightarrow{OA}\right|$

$$= \frac{1}{2}\left|\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}\right|$$

$$= \frac{1}{2}\left|\begin{pmatrix} 0 \\ 12 \\ -12 \end{pmatrix}\right|$$

$$= \frac{1}{2}\sqrt{288} = 6\sqrt{2}$$

4(iii)

using $\overrightarrow{ML}$ dot $\overrightarrow{OD}$

$$\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \sqrt{14}\sqrt{14}\cos\theta$$

$$\cos\theta = \frac{6}{14}$$

$$\theta = 64.6^0$$

5(i)

$$x\left[\frac{1}{2}\frac{-2x}{\sqrt{4-x^2}}\right] + \sqrt{4-x^2} + 4\left(\frac{1}{2}\right)\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

$$= \frac{-x^2 + 4 - x^2}{\sqrt{4-x^2}} + \frac{4}{\sqrt{4-x^2}}$$

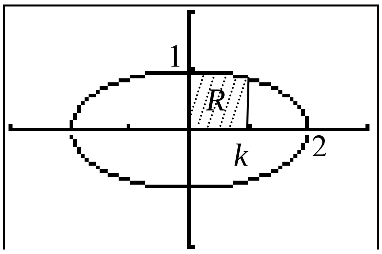
$$= 2\sqrt{4-x^2}$$

5(ii)

$$\frac{1}{2}\int_0^k \sqrt{4-x^2}\,dx = \frac{1}{2}\left[\frac{1}{2}\left(x\sqrt{4-x^2}+4\sin^{-1}\left(\frac{x}{2}\right)\right)\right]_0^k$$

$$= \frac{1}{4}\left[k\sqrt{4-k^2}+4\sin^{-1}\left(\frac{k}{2}\right)\right] \quad \Rightarrow \quad a=\frac{k}{4}$$

5(iii)



$4y^2+x^2=4$

$y^2+\dfrac{x^2}{2^2}=1$

$R=\dfrac{1}{2}\displaystyle\int_0^k \sqrt{4-x^2}\,dx$

5(iv)

Required area $=4R$ with $k=1$

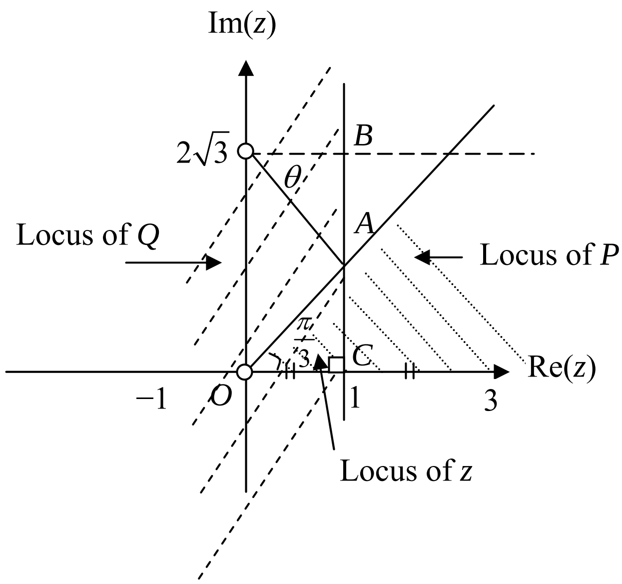
$$=\sqrt{3}+4\sin^{-1}\left(\frac{1}{2}\right)$$

$$=\sqrt{3}+4\left(\frac{\pi}{6}\right)$$

$$=\sqrt{3}+\frac{2\pi}{3}$$

6(i) $|z-3|\geq|z+1|$

6(ii)



$-\dfrac{\pi}{6}\leq \arg\left(\sqrt{3}+i\right)-\arg z\leq\dfrac{\pi}{6}$

$-\dfrac{\pi}{6}\leq\dfrac{\pi}{6}-\arg z\leq\dfrac{\pi}{6}$

$-\dfrac{\pi}{3}\leq-\arg z\leq 0$

$0\leq\arg z\leq\dfrac{\pi}{3}$

$|z-3|\geq|z+1|$

z lies on OAC excluding point 'O'

6(iv)

$AB=2\sqrt{3}-\sqrt{3}=\sqrt{3}$

$\tan\theta=\dfrac{AB}{1}=\sqrt{3}$

$\theta=\dfrac{\pi}{3}$

Hence $-\dfrac{\pi}{2}<\arg\left(z-2\sqrt{3}i\right)\leq-\dfrac{\pi}{3}$

7(i) $N = 90$, $n = 10$, so $k = 9$. List all the faculty staff by their names in alphabetical order and choose a random number from 1 to 9. Say we have chosen the number 4, then we select the 4^{th}, 13^{th}, 22^{nd}, 31^{st}, 40^{th}, 49^{th}, 58^{th}, 67^{th}, 76^{th}, 85^{th} names from the list to form the sample of 10.

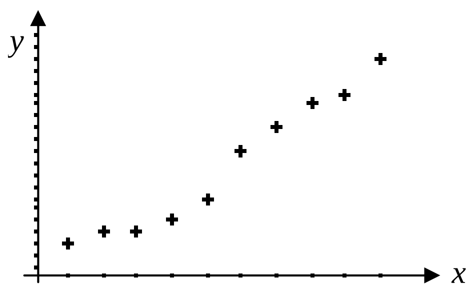
7(ii) Choose the number of Professor, Senior Lecturers and Lecturers in the following:

	Professor	Senior Lecturer	Lecturer	Total
Population	9	18	63	90
Sample	1	2	7	10

Compile 3 lists of names – Professor, Senior Lecturers and Lecturers. Perform simple random sampling from the separate lists of Professor, Senior Lecturers and Lecturers to select 1 Professor, 2 Senior Lecturers and 7 lecturers to form a sample of 10.

Stratified sampling is preferred as the sample selected is more representative of the faculty.

8(i) $r = 0.9785$



8(ii) $r = 0.9785$ indicates that there is a strong positive linear correlation but from the scatter diagram, there is non-linear relation between x and y. So a linear model may not be appropriate.

8(iii) From GC, $\ln y = 1.35553 + 0.132498\,x$

8(iv) From the description, x is the controlled variable while y is the response. So we will use $\ln y$ on x to estimate the amount: $\ln 15 = 1.35553 + 0.132498x$, so $x = 10.2$ mg

9(i)

Let A ~ students own IPod B~ students own IPhone
Given $n(A) = 25$, $n(B) = 40$ and $n(A \cup B) - n(A \cap B) = 35$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$35 + n(A \cap B) = 25 + 40 - n(A \cap B)$, so $n(A \cap B) = 15$
$P(A \cap B) = \dfrac{15}{100} = \dfrac{3}{20}$

OR

$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$

$\frac{35}{100} + \text{P}(A \cap B) = \frac{25}{100} + \frac{40}{100} - \text{P}(A \cap B)$

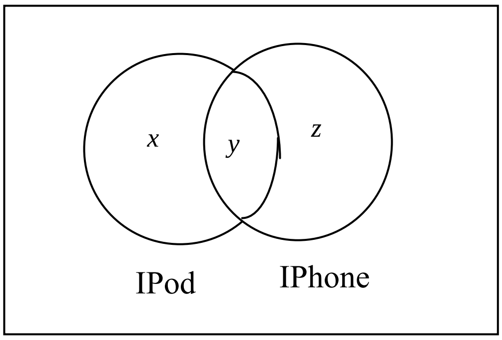
$\text{P}(A \cap B) = \frac{15}{100} = \frac{3}{20}$

9(ii) From (i) $\text{n}(A \cup B) = 50$

So the prob required $= 1 - \frac{50}{100} = \frac{1}{2}$

9(iii) $\text{P}(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{15}{40} = \frac{3}{8}$

Alternative: Use venn diagram to consider the numbers:



Given $x + y = 25$ ----- (1)

$y + z = 40$ ----- (2)

$x + z = 35$ ----- (3)

(1)+(2): $x + 2y + z = 65$, so $y = 15$

(1) + (2) + (3): $2x+2y+2z = 100$

So $x + y + z = 50$

(i) Prob req $= \frac{15}{100} = \frac{3}{20}$

(ii) Prob req $= \frac{50}{100} = \frac{1}{2}$

10(i) Number of way required = $10^4 = 100\,00$

10(ii) Number of ways for last digit = 5

Number of ways required = $9 \times 8 \times 7 \times 6 \times 5 \times 5 = 75600$

10(iii) Number of ways for 3 odd digits = ${}^5C_3 = 10$

Number of ways for 3 even digits = ${}^5C_3 = 10$

Number of ways required = $10 \times 10 \times 6! = 72000$

11(i) X ~ number of defective pen, out of 10

$X \sim \text{B}(10, 0.01)$

$\text{P}(X > 1) = 1 - \text{P}(X \le 1) = 0.0042662 \approx 0.00427$

11(ii) Y~ number of boxes being rejected, out of 1000

$Y \sim \text{B}(1000, 0.0042662)$

Since n is large, $np = 4.2662 < 5$, $Y \sim \text{Po}(4.2662)$

At least 990 boxes being accepted is the same as at most 10 boxes being rejected

$\text{P}(Y \le 10) = 0.995$

11(iii) W~ number of boxes being rejected out of 2000
W~B(2000, 0.0042662)
Since n is large, $np = 8.5324 > 5$ and $n(1-p) = 1991.4676 > 5$
W~N(8.5324, 8.495999) approx
$P(W \le 12) = P(W < 12.5) = 0.913$

12(i)

Unbiased estimate for mean $= \bar{x} = \frac{\sum(x-30)}{24} + 30 = \frac{1266}{24} + 30 = 82.75$

Unbiased estimate for variance = s^2=

$$\frac{1}{23}\left(\sum(x-30)^2 - \frac{\left(\sum(x-30)\right)^2}{24}\right) = \frac{1}{23}\left(78060.5 - \frac{1266^2}{24}\right) = 490.391$$

12(ii)

$H_0: \mu = 80$ vs $H_1: \mu > 80$
Level of Sig = 3%
Test statistics $t = \frac{82.75-80}{\sqrt{490.391/24}}$ =0.60837

p-value = 0.27445 > 0.03

Since p-value is more than level of significance, do not reject H_0 and conclude that there is insufficient evidence to suggest that the mean speed is greater than 80 km/h at 3% level of significance.

The unbiased estimate of mean remains unchanged as 82.75

The unbiased estimate of pop variance

$$= \frac{1}{239}\left(\sum(x-30)^2 - \frac{\left(\sum(x-30)\right)^2}{240}\right) = \frac{1}{239}\left(780605 - \frac{12660^2}{240}\right) = 471.925$$

but the distribution $\bar{X} \sim N(80, \frac{471.925}{240})$ approximately under CLT

test statistics $z = \frac{82.75-80}{\sqrt{471.925/240}} = 1.9611$

p-value is 0.02493 < 0.03

Since p-value is less than the level of significance, we reject H_0 and conclude that there is sufficient evidence that the mean speed is greater than 80 km/h, so there is a change in conclusion.

13(i) X~ time taken by route A
$X \sim N(14, 2.5^2)$
$P(X > 12) = 0.788$

13(ii) Y ~ time taken by route B
$Y \sim N(12, 7.3)$

$X - Y \sim N(2, 2.5^2 + 7.3)$
$X - Y \sim N(2, 13.55)$
$P(|X - Y| \geq 3) = 1 - P(-3 \leq X - Y \leq 3) = 0.480$

Let $T = X_1 + X_2 + X_3 + Y_1 + Y_2 \sim N(42 + 24, 3(2.5^2)+2(7.3))=N(66,33.35)$
$T \sim N(66,33.35)$
$P(T < 70) = 0.756$

14 X~no of fish caught by angler in 1 hour
$X \sim Po(1.5)$
$P(X = 3) = 0.126$

14(i) Y~no of fish caught by angler's son in 1 hour
$Y \sim Po(1)$
$X + Y \sim Po(2.5)$
$P(X + Y \geq 2) = 1 - P(X + Y \leq 1) = 0.713$

14(ii) U~no of fish caught by angler in 2 hours
V~ no of fish caught by his son in 2 hours
$U \sim Po(3)$ $\quad$ $V \sim Po(2)$
$U + V \sim Po(5)$
Prob required

$$= P(U > V \mid U + V = 4) = \frac{P(U = 4)P(V = 0) + P(U = 3)P(V = 1)}{P(U + V = 4)}$$

$= 0.475$

Let F and S be the number of fish caught by angler and his son in n minutes respectively.

$F + S \sim Po(\frac{2.5n}{60})$
$P(F + S \geq 1) > 0.9$
$P(F + S = 0) < 0.1$
From the GC,
$n = 55$: $P(F + S = 0) = 0.1011 > 0.01$
$n = 56$: $P(F + S = 0) = 0.09697 < 0.01$
so least $n = 56$

Since $n = 50$ is large, By Central Limit Theorem
$\bar{X} \sim N(1.5, \frac{1.5}{50})$ approx
$P(\bar{X} > 2) = 0.00195$