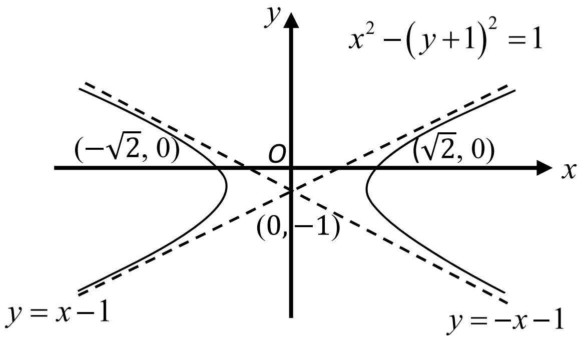
2023 MI H2 Math Pre-U 1 Exam Solutions

Qn	Solution
1(i) **[2]**	When $n \geq 2$, $u_n = S_n - S_{n-1}$ $= n^2 + 4n - \left[(n-1)^2 + 4(n-1)\right]$ $= n^2 + 4n - \left(n^2 - 2n + 1 + 4n - 4\right)$ $= 2n + 3$ When $n = 1$, $u_1 = S_1 = 1^2 + 4(1)$ $= 5$ $= 2(1) + 3$ Thus, u_1 follows the form of $u_n = 2n + 3$ when $n = 1$. So, $u_n = 2n + 3$
1(ii) **[2]**	$u_n - u_{n-1} = 2n + 3 - [2(n-1) + 3]$ $= 2n + 3 - 2n - 1$ $= 2$ Since $u_n - u_{n-1} = 2$ is a constant independent of n, the sequence is an arithmetic progression.
2(i) **[3]**	Let \$$x$, \$$y$ and \$$z$ be the price of a short, tall and grande cup of coffee respectively. $3x + 12y + 8z = 147.9$ $4x + 8y + 7z = 120.10$ $2x + 5y + 4z = 70$ Using GC, $x = 4.50$, $y = 6.20$, $z = 7.50$ The price of a short, tall and grande cup of coffee is \$4.50, \$6.20 and \$7.50 respectively.
2(ii) **[2]**	The required amount $= 0.9\left(4.50 + 4 \times 6.20\right) + 2(7.50) = 41.37$ Andrew pays \$41.37.

3(i) **[3]**	
3(ii) **[3]**	$x^2-(y+1)^2=1 \xrightarrow{\text{I: replace } x \text{ by } x/4} \left(\frac{x}{4}\right)^2-(y+1)^2=1$ $\left(\frac{x}{4}\right)^2-(y+1)^2=1 \xrightarrow{\text{II: replace } y \text{ by } \frac{y}{1/2}} \left(\frac{x}{4}\right)^2-(2y+1)^2=1$ $\left(\frac{x}{4}\right)^2-(2y+1)^2=1 \xrightarrow{\text{III: replace } y \text{ by } y-2} \left(\frac{x}{4}\right)^2-\left[2(y-2)+1\right]^2=1$ $\Rightarrow \left(\frac{x}{4}\right)^2-(2y-3)^2=1$

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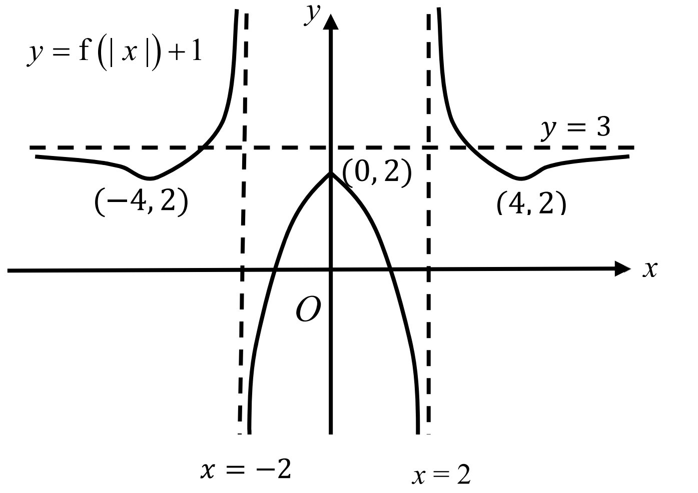
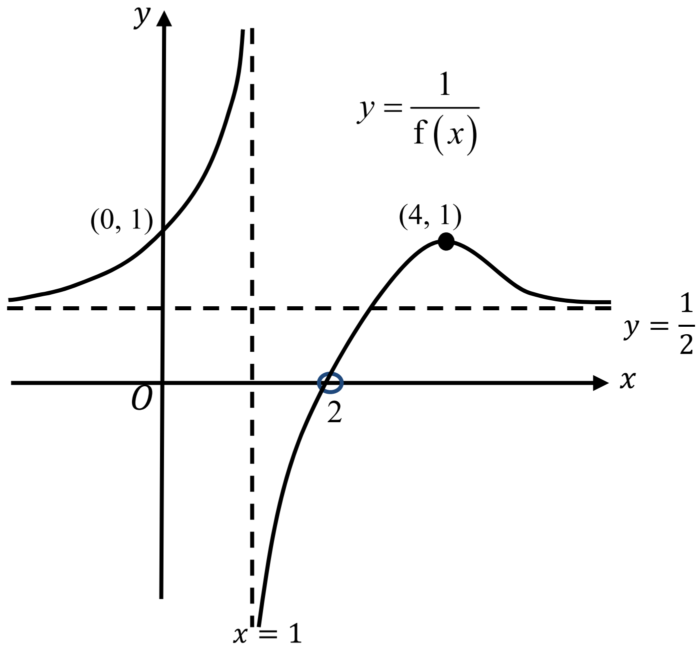
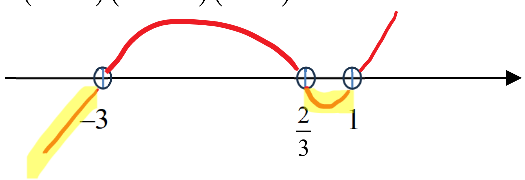
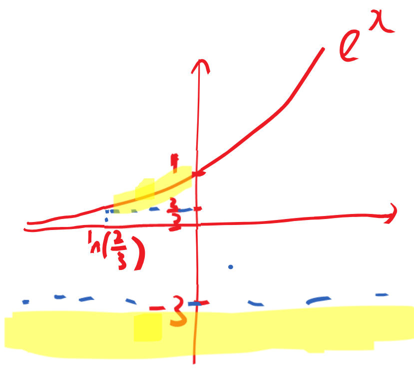
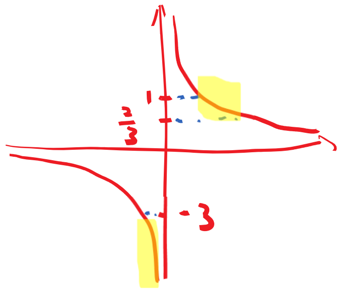
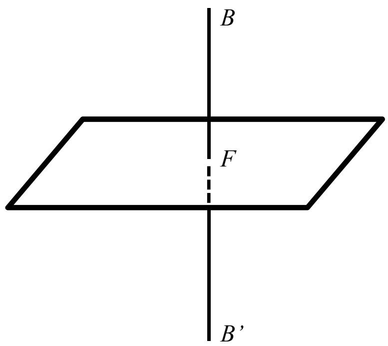
4 (i) [3]	
4 (ii) [3]	
5(i) [2]	$\overrightarrow{OA} = (1-\lambda)\overrightarrow{OB} + \lambda\overrightarrow{OC}$ $\overrightarrow{OA} = \overrightarrow{OB} - \lambda\overrightarrow{OB} + \lambda\overrightarrow{OC}$ $\overrightarrow{OA} - \overrightarrow{OB} = \lambda\overrightarrow{OC} - \lambda\overrightarrow{OB}$ $\overrightarrow{BA} = \lambda\overrightarrow{BC}$ A, B and C are collinear, B is the common point (shown)

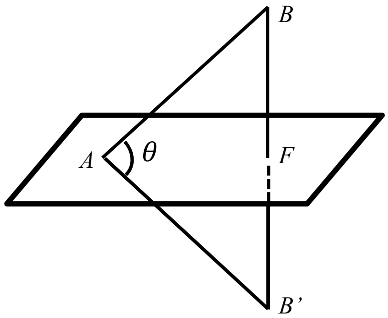
5 (ii) [3]	Given $\lambda = \frac{1}{6}$, $\overrightarrow{OA} = \frac{5}{6}\overrightarrow{OB} + \frac{1}{6}\overrightarrow{OC}$ $6\overrightarrow{OA} = 5\overrightarrow{OB} + \overrightarrow{OC}$ $\overrightarrow{OC} = 6\overrightarrow{OA} - 5\overrightarrow{OB}$ **<u>Method 1: Sub and Eliminate $\overrightarrow{OC}$</u>** By Ratio Theorem, $\overrightarrow{OD} = \frac{\overrightarrow{OC} + 4\overrightarrow{OA}}{5}$ $\overrightarrow{OD} = \frac{\left(6\overrightarrow{OA} - 5\overrightarrow{OB}\right) + 4\overrightarrow{OA}}{5}$ $\overrightarrow{OD} = 2\overrightarrow{OA} - \overrightarrow{OB}$ $\overrightarrow{OD} = 2\begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ $\overrightarrow{OD} = \begin{pmatrix} -1 \\ -5 \\ 2 \end{pmatrix}$ **<u>Method 2: Find $\overrightarrow{OC}$ and Sub in</u>** $\overrightarrow{OC} = 6\overrightarrow{OA} - 5\overrightarrow{OB} = 6\begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} - 5\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -5 \\ -17 \\ 10 \end{pmatrix}$ By Ratio Theorem, $\overrightarrow{OD} = \frac{\overrightarrow{OC} + 4\overrightarrow{OA}}{5}$ $\overrightarrow{OD} = \frac{1}{5}\left[\begin{pmatrix} -5 \\ -17 \\ 10 \end{pmatrix} + 4\begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}\right]$ $\overrightarrow{OD} = \begin{pmatrix} -1 \\ -5 \\ 2 \end{pmatrix}$

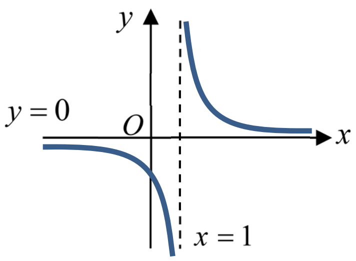
5 (iii) [3]	$\overrightarrow{AB}=\begin{pmatrix}1\\1\\-2\end{pmatrix}-\begin{pmatrix}0\\-2\\0\end{pmatrix}=\begin{pmatrix}1\\3\\-2\end{pmatrix}$ $\overrightarrow{AE}=\begin{pmatrix}1\\-1\\2\end{pmatrix}-\begin{pmatrix}0\\-2\\0\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}$ Area of the triangle *ABE* $=\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AE}\right\|$ $=\frac{1}{2}\left\|\begin{pmatrix}1\\3\\-2\end{pmatrix}\times\begin{pmatrix}1\\1\\2\end{pmatrix}\right\|$ $=\frac{1}{2}\left\|\begin{pmatrix}8\\-4\\-2\end{pmatrix}\right\|$ $=\left\|\begin{pmatrix}4\\-2\\-1\end{pmatrix}\right\|$ $=\sqrt{21}$ units^2
5 (iv) [2]	Length of projection of $\overrightarrow{OE}$ onto $\overrightarrow{AB}$. $=\left\|\overrightarrow{OE}\bullet\frac{\overrightarrow{AB}}{\left\|\overrightarrow{AB}\right\|}\right\|$ $=\frac{1}{\sqrt{14}}\left\|\begin{pmatrix}1\\-1\\2\end{pmatrix}\bullet\begin{pmatrix}1\\3\\-2\end{pmatrix}\right\|$ $=\frac{1}{\sqrt{14}}\left\|(-6)\right\|$ $=\frac{3\sqrt{14}}{7}$ units

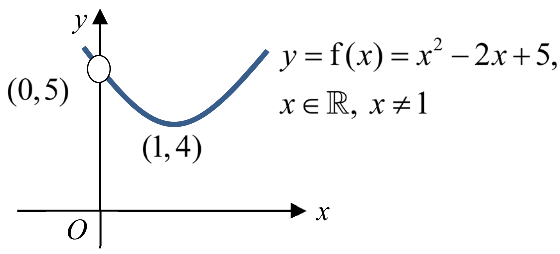
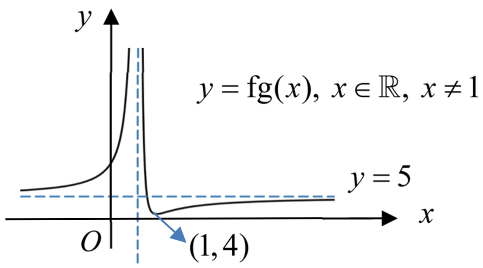
Qn	Solution
6(i) **[4]**	$\dfrac{3x^2+9x-8}{x-1}<2$ $\dfrac{\left(3x^2+9x-8\right)-2\left(x-1\right)}{x-1}<0$ $\dfrac{3x^2+7x-6}{x-1}<0$ $\dfrac{\left(x+3\right)\left(3x-2\right)}{x-1}<0$ $\left(x+3\right)\left(3x-2\right)\left(x-1\right)<0$  $x<-3$ or $\dfrac{2}{3}<x<1$
6(ii) **[2]**	$\dfrac{3x^2+9x-8}{x-1}<2$ Replace x with e^x $\dfrac{3\mathrm{e}^{2x}+9\mathrm{e}^x-8}{\mathrm{e}^x-1}<2$ $\Rightarrow\quad \mathrm{e}^x<-3$ or $\dfrac{2}{3}<\mathrm{e}^x<1$ no solution $\quad \ln\dfrac{2}{3}<x<\ln 1$ $\ln\dfrac{2}{3}<x<0$ 

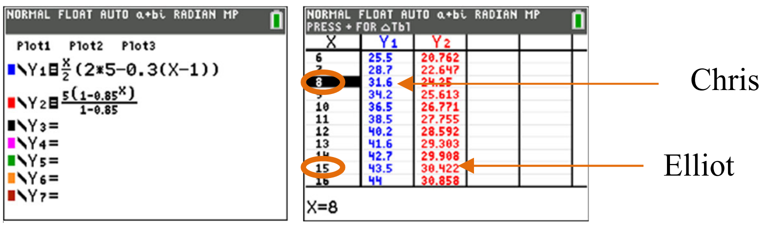
Qn	Solution
6(iii) **[3]**	$\dfrac{3x^2+9x-8}{x-1}<2$ Replace x by $\dfrac{1}{x}$ $\dfrac{3\left(\frac{1}{x}\right)^2+9\left(\frac{1}{x}\right)-8}{\frac{1}{x}-1}<2$ $\dfrac{3+9x-8x^2}{x^2}\div\dfrac{1-x}{x}<2$ $\dfrac{3+9x-8x^2}{x^2}\times\dfrac{x}{1-x}<2$ $\dfrac{3+9x-8x^2}{x-x^2}<2$ So, $\dfrac{1}{x}<-3$ or $\dfrac{2}{3}<\dfrac{1}{x}<1$ $-\dfrac{1}{3}<x<0$ or $1<x<\dfrac{3}{2}$ 
7(i) **[3]**	$\mathbf{n}=\begin{pmatrix}-2\\2\\1\end{pmatrix}\times\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}4\\5\\-2\end{pmatrix}$ $\mathbf{r}\bullet\begin{pmatrix}4\\5\\-2\end{pmatrix}=\begin{pmatrix}3\\7\\1\end{pmatrix}\bullet\begin{pmatrix}4\\5\\-2\end{pmatrix}$ $\mathbf{r}\bullet\begin{pmatrix}4\\5\\-2\end{pmatrix}=45$ Let $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $4x+5y-2z=45$ (shown)

7(ii) **[3]**	Line *BF*: $\mathbf{r}=\begin{pmatrix}2\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}4\\5\\-2\end{pmatrix},\ \lambda\in\mathbb{R}$ $\overrightarrow{OF}=\begin{pmatrix}2\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}4\\5\\-2\end{pmatrix}$ for some λ Since $\overrightarrow{OF}$ lies on π, $\overrightarrow{OF}\bullet\begin{pmatrix}4\\5\\-2\end{pmatrix}=45$ $\left[\begin{pmatrix}2\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}4\\5\\-2\end{pmatrix}\right]\bullet\begin{pmatrix}4\\5\\-2\end{pmatrix}=45$ $4(2+4\lambda)+5(5\lambda)-2(4-2\lambda)=45$ $45\lambda=45$ $\lambda=1$ $\overrightarrow{OF}=\begin{pmatrix}2\\0\\4\end{pmatrix}+\begin{pmatrix}4\\5\\-2\end{pmatrix}=\begin{pmatrix}6\\5\\2\end{pmatrix}$
7(iii) **[2]**	$\overrightarrow{OF}=\frac{\overrightarrow{OB}+\overrightarrow{OB}'}{2}$ $\overrightarrow{OB}'=2\overrightarrow{OF}-\overrightarrow{OB}$ $=2\begin{pmatrix}6\\5\\2\end{pmatrix}-\begin{pmatrix}2\\0\\4\end{pmatrix}$ $=\begin{pmatrix}10\\10\\0\end{pmatrix}$ 

7(iv) **[3]**	**<u>Method 1</u>** $\overrightarrow{AB}=\begin{pmatrix}2\\0\\4\end{pmatrix}-\begin{pmatrix}3\\7\\1\end{pmatrix}=\begin{pmatrix}-1\\-7\\3\end{pmatrix}$ $\overrightarrow{AB'}=\begin{pmatrix}10\\10\\0\end{pmatrix}-\begin{pmatrix}3\\7\\1\end{pmatrix}=\begin{pmatrix}7\\3\\-1\end{pmatrix}$  Let angle $BAB' = \theta$ $\theta=\cos^{-1}\left(\frac{\overrightarrow{AB}\bullet\overrightarrow{AB'}}{\lvert\overrightarrow{AB}\rvert\lvert\overrightarrow{AB'}\rvert}\right)$ $\theta=\cos^{-1}\left(\frac{1}{\sqrt{59}\sqrt{59}}\left[\begin{pmatrix}-1\\-7\\3\end{pmatrix}\bullet\begin{pmatrix}7\\3\\-1\end{pmatrix}\right]\right)$ $\theta=\cos^{-1}\left(\frac{-31}{\sqrt{59}\sqrt{59}}\right)$ $\theta=121.697°$ $\theta=121.7°$ (1 d.p.) **<u>Method 2</u>** $\overrightarrow{AB}=\begin{pmatrix}2\\0\\4\end{pmatrix}-\begin{pmatrix}3\\7\\1\end{pmatrix}=\begin{pmatrix}-1\\-7\\3\end{pmatrix}$ Let angle $BAB' = \theta$ $\theta=2\sin^{-1}\left(\frac{\lvert\overrightarrow{AB}\bullet\mathbf{n}\rvert}{\lvert\overrightarrow{AB}\rvert\lvert\mathbf{n}\rvert}\right)$ $\theta=2\sin^{-1}\left(\frac{1}{\sqrt{59}\sqrt{45}}\left[\left\lvert\begin{pmatrix}-1\\-7\\3\end{pmatrix}\bullet\begin{pmatrix}4\\5\\-2\end{pmatrix}\right\rvert\right]\right)$ $\theta=2\sin^{-1}\left(\frac{45}{\sqrt{59}\sqrt{45}}\right)$ $\theta=121.697°$ $\theta=121.7°$ (1 d.p.)

	Method 3 Let angle *BAB'* $= \theta$ $\overrightarrow{AB} = \begin{pmatrix}2\\0\\4\end{pmatrix} - \begin{pmatrix}3\\7\\1\end{pmatrix} = \begin{pmatrix}-1\\-7\\3\end{pmatrix}$ $\overrightarrow{AF} = \begin{pmatrix}6\\5\\2\end{pmatrix} - \begin{pmatrix}3\\7\\1\end{pmatrix} = \begin{pmatrix}3\\-2\\1\end{pmatrix}$ $\theta = 2\cos^{-1}\left(\frac{\overrightarrow{AB} \bullet \overrightarrow{AF}}{\lvert\overrightarrow{AB}\rvert\lvert\overrightarrow{AF}\rvert}\right)$ $\theta = 2\cos^{-1}\left(\frac{1}{\sqrt{59}\sqrt{14}}\left[\begin{pmatrix}-1\\-7\\3\end{pmatrix} \bullet \begin{pmatrix}3\\-2\\1\end{pmatrix}\right]\right)$ $\theta = 2\cos^{-1}\left(\frac{14}{\sqrt{59}\sqrt{14}}\right)$ $\theta = 121.697°$ $\theta = 121.7°$ (1 d.p.)
8(i) **[2]**	 $R_g = (-\infty, 0) \cup (0, \infty)$. $D_f = (-\infty, \infty)$. Since $R_g \subseteq D_f$, fg exists.
8 (ii) **[2]**	$\text{fg}(x) = \text{f}[\text{g}(x)] = \text{f}\left(\frac{1}{x-1}\right)$ $= \left(\frac{1}{x-1}\right)^2 - 2\left(\frac{1}{x-1}\right) + 5$ $= \frac{1}{(x-1)^2} - \frac{2}{x-1} + 5$ $D_{fg} = D_g = (-\infty, 1) \cup (1, \infty)$

8 (iii) **[2]**	**<u>Method 1: Mapping</u>** $D_g \xrightarrow{g} R_g \xrightarrow{f} R_{fg}$ $(-\infty,1)\cup(1,\infty) \quad (-\infty,0)\cup(0,\infty) \quad ?$  $R_{fg} = [4,\infty)$ **<u>Method 2: Graph $y = \text{fg}(x)$</u>**  $R_{fg} = [4,\infty)$
8 (iv) **[3]**	Let $y = \text{f}(x)$ $y = x^2 - 2x + 5$ $y = (x-1)^2 - 1^2 + 5$ $y = (x-1)^2 + 4$ $x = 1 - \sqrt{y-4}$ $\left[\text{reject } 1+\sqrt{y-4} \text{ as } x \le 1\right]$ $\therefore \text{f}^{-1}(x) = 1 - \sqrt{x-4}$ $D_{\text{f}^{-1}} = R_{\text{f}} = [4,\infty)$
8 (v) **[1]**	$\text{ff}^{-1}(x) = x \quad , \quad D_{\text{ff}^{-1}} = [4,\infty)$ Solving $\text{f}^{-1}\text{f}(x) = x, \; x \ge 4$
8 (vi) **[2]**	$\text{g}^{-1}(a) = b \quad \Rightarrow \quad a = \text{g}(b)$ $a = \dfrac{1}{b-1}$

9(i) **[4]**	$T_n = a + (n-1)d > 0$ $5 + (n-1)(-0.3) > 0$ $5 - 0.3n + 0.3 > 0$ $n < 17.7$ Chris pours 17 times before the simulation stops. $S_{17} = \frac{17}{2}\left[2(5) + (17-1)(-0.3)\right]$ $= 44.2$ The volume of water in the tank is 44.2 litres.
9(ii) **[2]**	Sum to infinity $= \frac{5}{1-0.85} = 33\frac{1}{3} < 33.5$ Elliot's comment that the target is unfair is justified.
9(iii) **[4]**	**<u>Method 1</u>** For Chris: $\frac{n}{2}\left[2(5) + (n-1)(-0.3)\right] \geq 30$ Using GC (Table), $n = 8$ Chris first reaches the target after pouring 8 times. For Elliot: $\frac{5\left(1-0.85^n\right)}{1-0.85} \geq 30$ Using GC, $n = 15$ Elliot first reaches target after pouring 15 times (> 8 times).  **<u>Method 2</u>** For Chris: $\frac{n}{2}\left[2(5) + (n-1)(-0.3)\right] \geq 30$ $-0.3n^2 + 10.3n - 60 \geq 0$ Using GC (Graph), $7.44 \leq n \leq 26.9$ Chris first reaches the target after pouring 8 times.

	For Elliot: $\dfrac{5\left(1-0.85^n\right)}{1-0.85} \geq 30$ $0.85^n \leq 0.1$ $n \geq \dfrac{\ln 0.1}{\ln 0.85}$ $n \geq 14.2$ Elliot first reaches target after pouring 15 times (> 8 times). Thus, Chris reaches the target first.
9(iv) **[2]**	$\dfrac{5\left(1-\left(\dfrac{r}{100}\right)^7\right)}{1-\dfrac{r}{100}} \geq 30$ Using GC (*Graph*), since $r \geq 0$, $r \geq 94.8$ Least $r = 95$