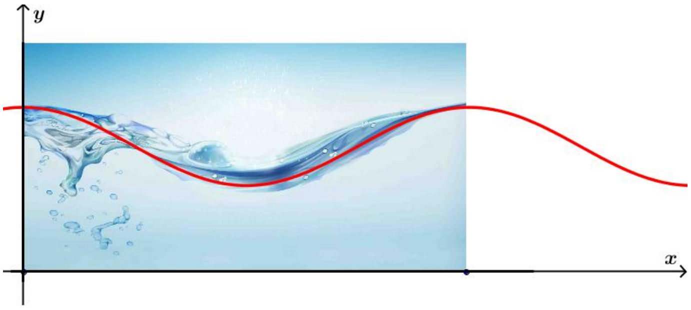
National Junior College
2016 – 2017 H2 Mathematics

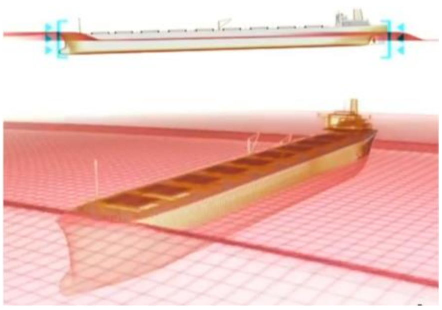
Revision: Trigonometry

Key Questions to Answer:

- What are the three basic and three non-basic trigonometric functions?
- What is meant by the basic or reference angle?
- How do I determine the sign (+ or –) of a trigonometric expression expressed either in degree or radians?
- What are the key characteristics of the graph of the three basic trigonometric functions?
- Why is it important to define principal values or principal range of inverse trigonometric functions?
- Are inverse trigonometric functions the same as the reciprocal of trigonometric functions?
- What are the trigonometric values of special angles $(30°, 45°, 60°)$?
- How do you obtain the value of a trigonometric function from another, i.e., given $\sin A = 0.4$ where A is an acute (or obtuse) angle, what is $\cos A$?
- What are some uses for each of the following:
  - → Pythagorean identities,
  - → Addition Formulae,
  - → Double Angle Formulae,
  - → Sum-to-Product/Factor Formulae,
  - → Product-to-Sum Formulae,
  - → R-formulae?
- In what situations do we use the sine/cosine rule to find the angle and/or length of one side of a triangle?

§1 Introduction





Waves occur in many natural phenomena such as seismic waves in earthquakes and ocean waves. Due to their periodic behavior, these waves are commonly modeled by the two trigonometric functions, sine and cosine.

In one computer generated visual simulation, it showed that MV Derbyshire, the biggest British registered merchant ship ever to have been lost at sea, was sunk by typhoon Orchid because the ocean waves at the time of incident were just slightly longer than 294 metres, the length of the ship. This meant that her bow would have been submerged before she had the chance to rise above the waves.

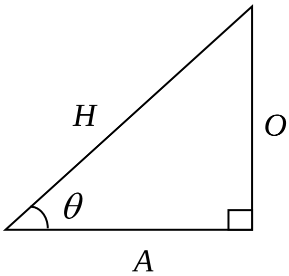
Source: http://www.liverpoolmuseums.org.uk/maritime/exhibitions/derbyshire/

§2 The Three Basic Trigonometric Functions

Radians vs Degrees for angles

π radians is equivalent to $180°$, θ in degrees is equivalent to $\left(\dfrac{\theta}{180°}\right)\times\pi$ radians.

For an acute angle θ, i.e. $0<\theta<\dfrac{\pi}{2}$, we can use a right-angled triangle to define the three basic trigonometric functions: sine, cosine and tangent.



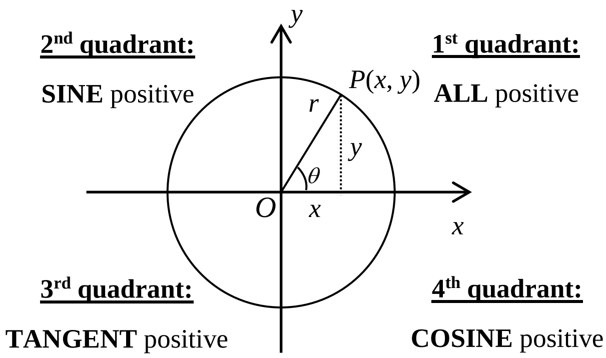
$$\sin\theta=\frac{\text{opposite}}{\text{hypotenuse}}=\frac{O}{H},\quad \cos\theta=\frac{\text{adjacent}}{\text{hypotenuse}}=\frac{A}{H},\quad \tan\theta=\frac{\text{opposite}}{\text{adjacent}}=\frac{O}{A}.$$

Mnemonics: Go ask your Grandma (Ah SOH) with the big foot (TOA CAH) for help!

TOA-CAH-SOH

For any angle θ, we use a point P on the circumference of a circle of radius r centred at the origin O of the Cartesian place to define the three basic trigonometric functions. Let θ be the angle through which the line OP has rotated **anti-clockwise** from the **positive *x*-axis**:

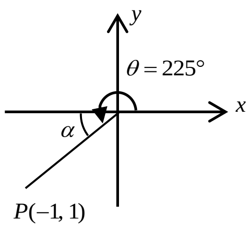
$$\sin\theta = \frac{y}{r},\quad \cos\theta = \frac{x}{r},\quad \tan\theta = \frac{y}{x}.$$



Acronym for ASTC: **A**ll **S**cience **T**eachers are **C**razy, **A**dd **S**ugar **T**o **C**offee…

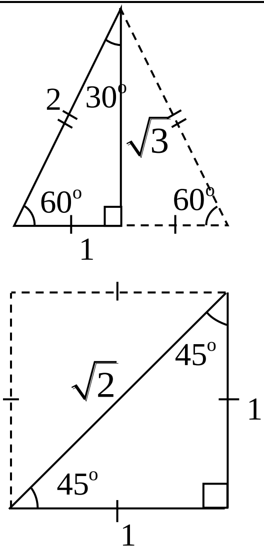
The positive acute angle between OP and the x-axis is known as the **basic** or **reference angle**, α. When $0<\theta<2\pi$, the following table gives the relationship between θ and α for each quadrant that P lies in.

Quadrant that P lies in	Basic angle, α
1st	θ
2nd	$\pi-\theta$
3rd	$\theta-\pi$
4th	$2\pi-\theta$



2.1 Trigonometric Values of Special Angles

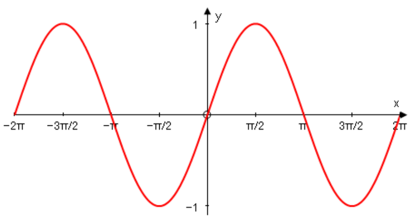
Angle θ	degree	0^o	30^o	45^o	60^o	90^o
	radian	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$		0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$		1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$		0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Not defined



Refer to **Appendix A** for methods to memorize and/or to derive the above results.

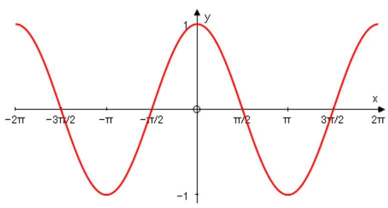
2.2 Graphs of Basic Trigonometric Functions

(i) Graph of $y = \sin x$,



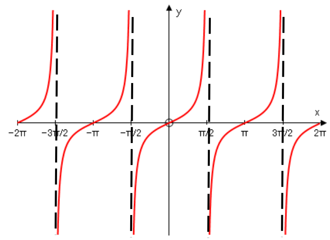
- x is continuous over $\mathbb{R}$.
- y lies between -1 and 1 (inclusive).
- It is 2π-periodic, i.e. $\sin(x+2\pi) = \sin x$.
- Odd function, i.e. $\sin(-x) = -\sin x$.

(ii) Graph of $y = \cos x$,



- x is continuous over $\mathbb{R}$.
- y lies between -1 and 1 (inclusive).
- It is 2π-periodic, i.e. $\cos(x+2\pi) = \cos x$.
- Even function, i.e. $\cos(-x) = \cos x$.

(iii) Graph of $y = \tan x$.



- Undefined at $x = (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}$. (c.f. vertical asymptotes).
- y can take any real values.
- It is π-periodic, i.e. $\tan(x+\pi) = \tan x$.
- Odd function, i.e. $\tan(-x) = -\tan x$.

§3 Inverse of Trigonometric Functions

$$y = \sin x \quad \Leftrightarrow \quad x = \sin^{-1} y.$$
$$y = \cos x \quad \Leftrightarrow \quad x = \cos^{-1} y.$$
$$y = \tan x \quad \Leftrightarrow \quad x = \tan^{-1} y.$$

Note that $\sin^{-1} x$ denotes the **inverse** of sin x. It is **NOT** the reciprocal of sin x, i.e.

$$\sin^{-1} x \neq \frac{1}{\sin x} = (\sin x)^{-1}, \qquad (\tan^{-1} x)^2 \neq \tan^{-2} x = \frac{1}{\tan^2 x}.$$

3.1 Principal Values of Basic Trigonometric Functions

From the graph of $y = \sin x$, we observe that several intervals of x correspond to the interval $-1 \le y \le 1$ (which are all the possible values of $\sin x$), such as

$$-\frac{\pi}{2} \le x \le \frac{\pi}{2}, \quad \frac{\pi}{2} \le x \le \frac{3\pi}{2}, \quad \frac{3\pi}{2} \le x \le \frac{5\pi}{2}, \ldots$$

The interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ is <u>chosen</u> to be the **principal values** of x, or $\sin^{-1} y$.

Example 3.1

List down a few values of x that satisfies $\sin x = \frac{1}{2}$. What is the principal value of $\sin^{-1}\left(\frac{1}{2}\right)$?

Solution:

$$x = \ldots, -2\pi + \frac{\pi}{6}, -2\pi + \frac{5\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, 2\pi + \frac{\pi}{6}, 2\pi + \frac{5\pi}{6}, \ldots$$

Principal value of $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

Note: Your calculator will only give the principal values of inverse trigonometric functions.

The principal values of the three basic trigonometric functions are provided in MF-15. They are given as follows:

$$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2} \qquad (|x| \le 1)$$

$$0 \le \cos^{-1} x \le \pi \qquad (|x| \le 1)$$

$$-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}.$$

IMPORTANT: Principal value and basic angle are two **different** concepts. The next example illustrates this.

Example 3.2

Solve $\sin x = -\frac{1}{2}$, where $0 \le x \le 2\pi$. What is the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$?

Solution:

Basic angle = $\sin^{-1}\left|-\frac{1}{2}\right| = \frac{\pi}{6}$.

The modulus sign is **absolutely** necessary to find basic angle.

By "ASTC", $\sin x$ is negative in the 3rd and 4th quadrants.

Thus $x = \pi + \frac{\pi}{6}$ or $2\pi - \frac{\pi}{6}$

$= \frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

Principal value of $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

3.2 Relationships between Trigonometric Functions and Inverse Trigonometric Functions

The general procedure to obtain expressions involving trigonometric functions of inverse trigonometric functions is to consider the geometry of an appropriate right-angled triangle (for acute angles).

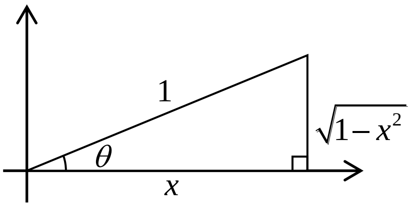
Example 3.3

Given that $\cos\theta = x$ such that $0 < \theta < 90°$, find $\tan\theta$ in terms of *x*.

Solution:

This question is equivalent to simplifying $\tan\left(\cos^{-1} x\right)$.

Using the right-angled triangle, $\tan\theta = \dfrac{\sqrt{1-x^2}}{x}$.



> **Explore:** Solve Example 3.3 using an algebraic approach.
>
> **(***Hint*: You will need one of the Pythagorean identities given in the next section.)

§4 Trigonometric Identities

A trigonometric identity expressed in terms of *x* is an equality that involves trigonometric functions, and is true for all values of *x* (except possibly at singular points where the trigonometric function is not defined).

The basic trigonometric identities are as follows:

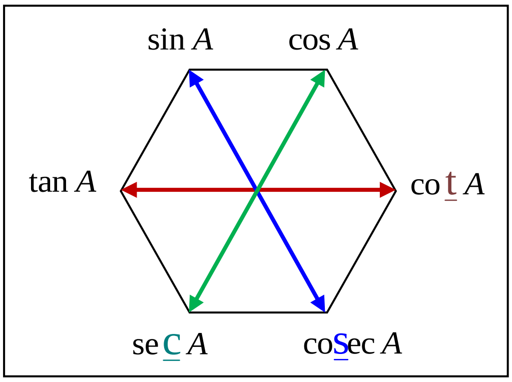
1 RATIO IDENTITIES $\tan A = \dfrac{\sin A}{\cos A}$, $\cos A \neq 0$. $\cot A = \dfrac{\cos A}{\sin A}$, $\sin A \neq 0$.	2 RECIPROCAL IDENTITIES $\sec A = \dfrac{1}{\cos A}$, $\cos A \neq 0$. $\operatorname{cosec} A = \dfrac{1}{\sin A}$, $\sin A \neq 0$. $\cot A = \dfrac{1}{\tan A}$, $\tan A \neq 0$.
3 NEGATIVE ANGLE IDENTITIES $\sin(-A) = -\sin A$. $\cos(-A) = \cos A$. $\tan(-A) = -\tan A$.	4 PYTHAGOREAN IDENTITIES $\sin^2 A + \cos^2 A = 1$. $1 + \tan^2 A = \sec^2 A$. $1 + \cot^2 A = \operatorname{cosec}^2 A$.

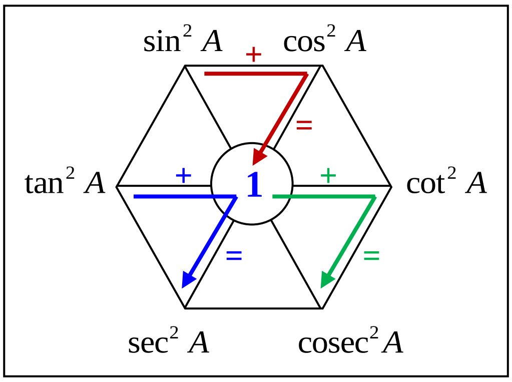
Remarks:

(1) Use the third letter of the "non-basic" trigonometric function to deduce its reciprocal. For example, the reciprocal of se**C** *A* is **C**os *A*.

(2) The negative angle identities can be deduced from the ASTC diagram.

(3) The following diagram can be used to obtain the Pythagorean identities.





4.1 Addition Formulae (Given in MF-15)

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B.$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$$

Example 4.1

Express $\sin\left(\frac{\pi}{6} + x\right)$ in terms of $\sin x$ and $\cos x$. Hence find $\sin\left(\frac{5\pi}{12}\right)$ in exact form.

Solution:

$$\sin\left(\frac{\pi}{6} + x\right) = \sin\frac{\pi}{6}\cos x + \cos\frac{\pi}{6}\sin x = \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x.$$

$$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \sin\frac{\pi}{6}\cos\frac{\pi}{4} + \cos\frac{\pi}{6}\sin\frac{\pi}{4} = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right) + \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}.$$

4.2 Double Angle Formulae

$$\sin 2A = 2\sin A \cos A.$$

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 1 - 2\sin^2 A$$
$$= 2\cos^2 A - 1.$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}.$$

$$\sin^2 A = \frac{1 - \cos 2A}{2}.$$

$$\cos^2 A = \frac{1 + \cos 2A}{2}.$$

Not given in formula list.
Useful in integration techniques.

Example 4.2

Prove the following identities:

(a) $\dfrac{\sin\theta+\sin 2\theta}{1+\cos 2\theta+\cos\theta}=\tan\theta.$

(b) $\cos^4\alpha+\sin^4\alpha=\dfrac{1}{2}(1+\cos^2 2\alpha).$

(c) $\cot x+\tan x=\sec x\ \operatorname{cosec} x.$

Solution:

To prove an identity, the strategy is to start from the more complicated side, and perform manipulation till we reach the other side.

(a)
$$\begin{aligned}\text{LHS}&=\frac{\sin\theta+\sin 2\theta}{1+\cos 2\theta+\cos\theta}\\&=\frac{\sin\theta+2\sin\theta\cos\theta}{1+\left(2\cos^2\theta-1\right)+\cos\theta}\\&=\frac{\sin\theta\left(1+2\cos\theta\right)}{\cos\theta\left(1+2\cos\theta\right)}\\&=\tan\theta\\&=\text{RHS.}\end{aligned}$$

(b)
$$\begin{aligned}\text{LHS}&=\left(\cos^2\alpha\right)^2+\left(\sin^2\alpha\right)^2\\&=\left(\frac{1+\cos 2\alpha}{2}\right)^2+\left(\frac{1-\cos 2\alpha}{2}\right)^2\\&=\frac{1+2\cos 2\alpha+\cos^2 2\alpha}{4}+\frac{1-2\cos 2\alpha+\cos^2 2\alpha}{4}\\&=\frac{1}{2}(1+\cos^2 2\alpha)\\&=\text{RHS.}\end{aligned}$$

(c)
$$\begin{aligned}\text{LHS}&=\frac{1}{\tan x}+\tan x\\&=\frac{1+\tan^2 x}{\tan x}\\&=\frac{\sec^2 x}{\tan x}\\&=\sec x\left(\frac{\sec x}{\tan x}\right)\\&=\sec x\left(\frac{1}{\cos x}\times\frac{\cos x}{\sin x}\right)\\&=\text{RHS.}\end{aligned}$$

Warning:

To prove an identity, you must **NOT** begin by assuming that the identity is true, and then perform manipulation, such as,

$$\begin{aligned}\frac{\sin\theta+\sin 2\theta}{1+\cos 2\theta+\cos\theta}=\tan\theta\ &\Rightarrow\ \sin\theta+\sin 2\theta=\left(\tan\theta\right)\left(1+\cos 2\theta+\cos\theta\right)\\&\Rightarrow\ \vdots\\&\Rightarrow\ 0=0.\end{aligned}$$

This is because you cannot use a result that has yet to be proven true.

4.3 Factor/Sum-to-Product Formulae (Given in MF-15)

$$\sin P+\sin Q=2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q).$$
$$\sin P-\sin Q=2\cos\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q).$$
$$\cos P+\cos Q=2\cos\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q).$$
$$\cos P-\cos Q=-2\sin\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q).$$

Note: A good way to remember these **factor formulae** is to quote them verbally as:

Shop + **S**hop = **S**hopping **C**entre.

Shop – Shop = **C**lose **S**hop.

Cash + **C**ash = **C**redit **C**ard.

Cash – **C**ash = **M**ust **S**ave **S**ave.

The sum-to-product/factor formulae express a **sum** of sine and cosine terms as a **product** to sine and cosine terms. They are particularly useful in solving trigonometric equations.

Example 4.3

Solve $\sin 5x+\sin x=0,$ where $0\le x\le 2\pi.$

Solution: $\sin 5x + \sin x=0$ where $0\le x\le 2\pi.$

$\Rightarrow\ 2\sin(3x)\cos(2x)=0$

$\Rightarrow\ \sin(3x)=0$ or $\cos(2x)=0$

$\Rightarrow\ 3x=0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ 5\pi,\ 6\pi$ or $2x=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2}$

$\Rightarrow\ x=0,\frac{\pi}{3},\frac{2\pi}{3},\pi,\frac{4\pi}{3},\frac{5\pi}{3},2\pi,\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},$ or $\frac{7\pi}{4}.$

If $0\le x\le 2\pi,$ then
$0\le 2x\le 4\pi$
$0\le 3x\le 6\pi.$

4.4 Product-to-Sum Formulae (<u>NOT</u> Given in MF-15)

$$2\sin A\cos B=\sin(A+B)+\sin(A-B).$$
$$2\cos A\sin B=\sin(A+B)-\sin(A-B).$$
$$2\cos A\cos B=\cos(A+B)+\cos(A-B).$$
$$-2\sin A\sin B=\cos(A+B)-\cos(A-B).$$

The product-to-sum formulae do the reverse: it expresses a **product** of sine and cosine terms as a **sum** of sine and cosine terms. These formulae are useful in the future topic of integration, e.g. integrating a product of trigonometric terms as shown below.

Example 4.4

Evaluate $\int 2\sin 3x\cos x\,\mathrm{d}x.$

Solution:

$$2\sin 3x\cos x=\sin(3x+x)+\sin(3x-x)=\sin 4x+\sin 2x$$

$$\Rightarrow\ \int 2\sin 3x\cos x\,\mathrm{d}x=\int \sin 4x+\sin 2x\,\mathrm{d}x=-\frac{1}{4}\cos 4x-\frac{1}{2}\cos 2x+C.$$

4.5 R- Formulae (<u>Not</u> Given in MF-15)

For a, b > 0,

$$a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha)$$
$$a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$$

where $R=\sqrt{a^2+b^2}$, $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$ and α is acute.

Example 4.5

Express $3\sin x - 4\cos x$ in the form $R\sin(x-\alpha)$, where $R>0$ and α is an acute angle. Hence find the maximum and minimum value of $3\sin x - 4\cos x$.

Solution:

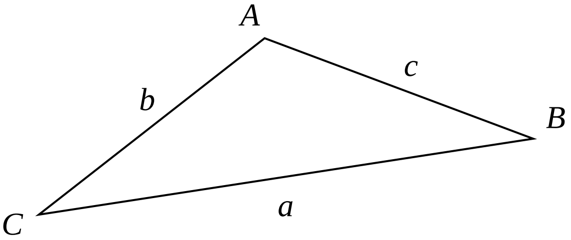
$$3\sin x - 4\cos x = R\sin(x-\alpha) = 5\sin\left(x - \tan^{-1}\frac{4}{3}\right).$$

Maximum value of $3\sin x - 4\cos x$ = Maximum of $5\sin(x-\alpha)$ = 5.

Minimum value of $3\sin x - 4\cos x$ = –5.

§5 Sine and Cosine Rules (<u>Not</u> Given in MF-15)

Sine and cosine rules can be used on any triangle, not just right-angled triangles. Before applying the rules, ensure that each pair of capital letter and small letter (A, a), (B, b) and (C, c) are assigned the correct orientation as shown below.

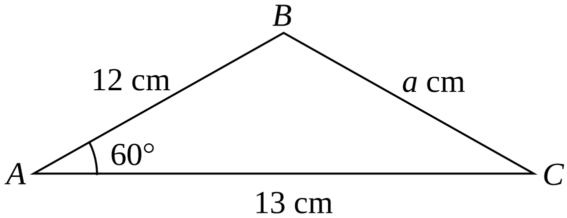


<u>Sine Rule:</u> $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ $\left(\text{or } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}\right)$.

<u>Cosine Rule:</u> $a^2 = b^2 + c^2 - 2bc\cos A$.

Example 5.1

Find the length a, and the angles B and C.



Solution:

By cosine rule: $a^2 = 12^2 + 13^2 - 2(12)(13)\cos 60°$

$\Rightarrow a = \sqrt{157} \approx 12.5$ cm.

By sine rule: $\dfrac{\sin 60°}{\sqrt{157}} = \dfrac{\sin B}{13} \Rightarrow \sin B = 0.89851... \Rightarrow B \approx 64.0°$ or $116°$ (rejected).

Therefore $C \approx 180° - (60° + 64.0°) = 56.0°$.

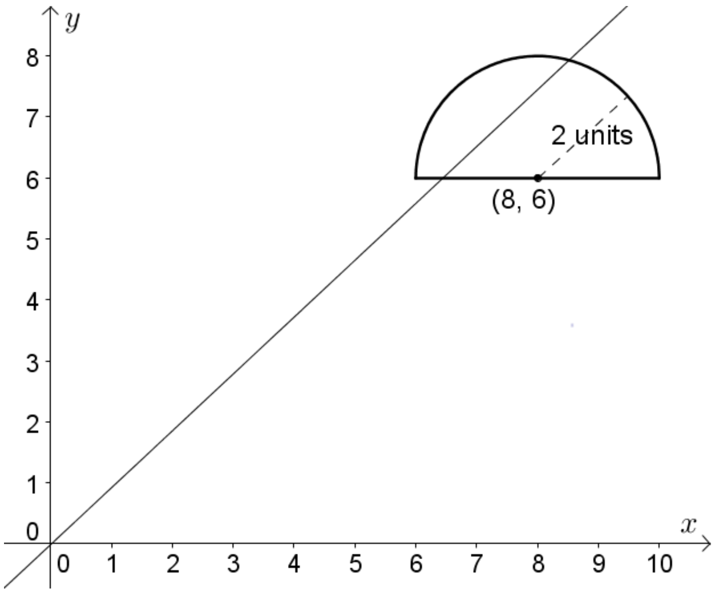
Why? Hint: Compare this with an equilateral triangle.

§6 Application of Trigonometry to Geometry

Refer to e-learning video on LMS**.**

Example 6.1

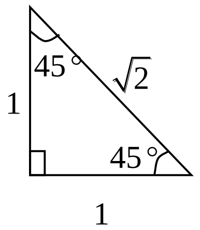
The following diagram shows a semi-circle centred at (8, 6) with radius 2 units, and a line that passes through the origin *O*.

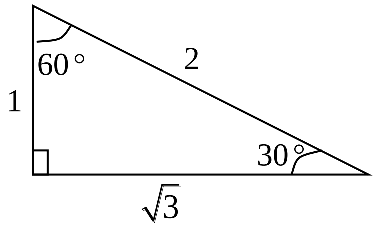


Given that θ is the acute angle made by the line with the **positive** *x*-axis, find the range of values of θ such that the line intersects the semi-circle at least once, giving your answer in radians.

Appendix A: How to remember the trigonometric ratios involving special angles?

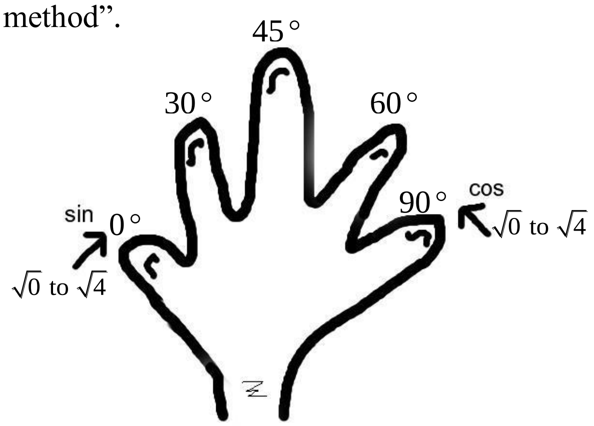
Method 1: Use right-angled triangles.





Method 2: Use calculator (for verification only).
Do NOT give numerical answers when exact answers are required.

Method 3: "Spiderman method".



For the values of the three basic trigonometric functions at the special angles, we can use the following:

$$\sin(*) = \frac{\sqrt{\text{no. of fingers counting from thumb}}}{2},$$

$$\cos(*) = \frac{\sqrt{\text{no. of fingers counting from little finger}}}{2},$$

$$\tan(*) = \frac{\sqrt{\text{no. of fingers counting from thumb}}}{\sqrt{\text{no. of fingers counting from little finger}}}.$$