Qn	Solution
1	**Mathematical Induction**
(a)	$\mathbf{A}=\begin{pmatrix}1 & 1\\ 0 & 0.5\end{pmatrix}$ $\mathbf{A}^2=\begin{pmatrix}1 & 1.5\\ 0 & 0.25\end{pmatrix}=\begin{pmatrix}1 & 2-0.5\\ 0 & 0.5^2\end{pmatrix}$ $\mathbf{A}^3=\begin{pmatrix}1 & 1.75\\ 0 & 0.125\end{pmatrix}=\begin{pmatrix}1 & 2-0.5^2\\ 0 & 0.5^3\end{pmatrix}$ Conjecture: $\mathbf{A}^n=\begin{pmatrix}1 & 2-(0.5)^{n-1}\\ 0 & (0.5)^n\end{pmatrix}$
(b)	Let P_n be the proposition that $\mathbf{A}^n=\begin{pmatrix}1 & 2-(0.5)^{n-1}\\ 0 & (0.5)^n\end{pmatrix}$ for all positive integers n. When $n=1$ LHS $=\mathbf{A}=\begin{pmatrix}1 & 1\\ 0 & 0.5\end{pmatrix}$ RHS $=\begin{pmatrix}1 & 2-(0.5)^0\\ 0 & 0.5^1\end{pmatrix}=\begin{pmatrix}1 & 1\\ 0 & 0.5\end{pmatrix}$ Since LHS = RHS, P_1 is true. Assume P_k is true for some positive integer k i.e. $\mathbf{A}^k=\begin{pmatrix}1 & 2-(0.5)^{k-1}\\ 0 & (0.5)^k\end{pmatrix}$ When $n=k+1$ $\mathbf{A}^{k+1}=\mathbf{A}\mathbf{A}^k$ $=\begin{pmatrix}1 & 1\\ 0 & 0.5\end{pmatrix}\begin{pmatrix}1 & 2-(0.5)^{k-1}\\ 0 & (0.5)^k\end{pmatrix}$ $=\begin{pmatrix}1 & 2-(0.5)^{k-1}+(0.5)^k\\ 0 & (0.5)^{k+1}\end{pmatrix}$ $=\begin{pmatrix}1 & 2-(0.5)^k\\ 0 & (0.5)^{k+1}\end{pmatrix}$ $\therefore P_k$ true $\Rightarrow P_{k+1}$ true. Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for all positive integers n.

Qn	Solution

2	**Complex Numbers**
(a)	$\cos 5\theta + \mathrm{i}\sin 5\theta = (\cos\theta + \mathrm{i}\sin\theta)^5$ (by De Moivre's Theorem) $= \cos^5\theta + \mathrm{i}5\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta$ $-\mathrm{i}10\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta$ $\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta}$ By comparing real and imaginary parts, $\tan 5\theta = \dfrac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$ $= \dfrac{5x - 10x^3 + x^5}{1 - 10x^2 + 5x^4}$ (dividing throughout by $\cos^5\theta$)
(b)	$\tan\left(5\left(\dfrac{\pi}{5}\right)\right) = 0$ $5x - 10x^3 + x^5 = 0$ $x = 0$ or $x^2 = \dfrac{10 \pm \sqrt{100 - 4(5)}}{2}$ $= 5 \pm 2\sqrt{5}$ Since $\tan\left(5\left(\dfrac{k\pi}{5}\right)\right) = 0$ for $k = -2, -1, 0, 1, 2$, $\tan^2\left(\dfrac{\pi}{5}\right)$ is the smaller value. $\tan^2\left(\dfrac{\pi}{5}\right) = 5 - 2\sqrt{5}$

Qn	Solution
3	**Recurrence Relations**
(a)	$x^2-3x+1=0$ $x=\dfrac{3\pm\sqrt{9-4(1)(1)}}{2}=\dfrac{3\pm\sqrt{5}}{2}$ $\alpha=\dfrac{3-\sqrt{5}}{2}$ and $\beta=\dfrac{3+\sqrt{5}}{2}$
(b)	Note that α and β are the limits of the sequence if the sequence converges. Consider $\alpha<x_n<\beta$: $(x_{n+1})^2-(x_n)^2=3x_n-1-(x_n)^2$ $=-(x_n-\alpha)(x_n-\beta)$ Since $\alpha<x_n<\beta$, $(x_n-\alpha)>0$ and $(x_n-\beta)<0$. Thus, $(x_{n+1})^2-(x_n)^2>0$ Also, $(x_{n+1})^2-(x_n)^2=(x_{n+1}-x_n)(x_{n+1}+x_n)$ Since x_n and x_{n+1} are positive, then $(x_{n+1}+x_n)>0$ Thus, $(x_{n+1}-x_n)>0 \Rightarrow x_{n+1}>x_n$ Since $\alpha<x_n<\beta$, we have $(x_{n+1})^2=3x_n-1<3\beta-1=\beta^2$. Thus, $x_{n+1}<\beta$ since $x_{n+1}>0$ Therefore if $\alpha<x_n<\beta$, then $\alpha<x_n<x_{n+1}<\beta$ for all $n\in\mathbb{Z}^+$. Thus, the sequence increases and converges to β.

Qn	Solution
4	**Recurrence Relation**
(a)	$\frac{1}{1-x}=(1-x)^{-1}$ $=1+x+x^2+\cdots$ $=\sum_{r=0}^{\infty}x^r$
(b)	$\mathrm{f}(x)=\sum_{r=0}^{\infty}u_r x^r$ $x\mathrm{f}(x)=\sum_{r=0}^{\infty}u_r x^{r+1}=\sum_{r=1}^{\infty}u_{r-1}x^r$ $x^2\mathrm{f}(x)=\sum_{r=0}^{\infty}u_r x^{r+2}=\sum_{r=2}^{\infty}u_{r-2}x^r$ $\mathrm{f}(x)-x\mathrm{f}(x)-x^2\mathrm{f}(x)=\sum_{r=0}^{\infty}u_r x^r-\sum_{r=1}^{\infty}u_{r-1}x^r-\sum_{r=2}^{\infty}u_{r-2}x^r$ $=u_0x^0+u_1x^1-u_0x^1+\sum_{r=2}^{\infty}\left[u_r-u_{r-1}-u_{r-2}\right]x^r$ $=x$ Thus, $\mathrm{f}(x)\left(1-x-x^2\right)=x$ $\mathrm{f}(x)=\frac{-x}{x^2+x-1}$
	$\mathrm{f}(x)=\frac{-x}{x^2+x-1}=\frac{A}{x-\alpha}+\frac{B}{x-\beta}$ where $\alpha=\frac{-1-\sqrt{5}}{2}$ and $\beta=\frac{-1+\sqrt{5}}{2}$. Solving: $A=-\frac{-1-\sqrt{5}}{2}\div\left(\frac{-1-\sqrt{5}}{2}-\frac{-1+\sqrt{5}}{2}\right)=\frac{-1-\sqrt{5}}{2\sqrt{5}}$ $B=-\frac{-1+\sqrt{5}}{2}\div\left(\frac{-1+\sqrt{5}}{2}-\frac{-1-\sqrt{5}}{2}\right)=-\frac{-1+\sqrt{5}}{2\sqrt{5}}$

	$\mathrm{f}(x)=\frac{A}{x-\alpha}+\frac{B}{x-\beta}$ $=-\frac{A}{\alpha}\left(\frac{1}{1-\left(\frac{x}{\alpha}\right)}\right)-\frac{B}{\beta}\left(\frac{1}{1-\left(\frac{x}{\beta}\right)}\right)$ $=-\frac{A}{\alpha}\sum_{r=0}^{\infty}\left(\frac{x}{\alpha}\right)^r-\frac{B}{\beta}\sum_{r=0}^{\infty}\left(\frac{x}{\beta}\right)^r$ using result from part $(\mathbf{a})$ $=\sum_{r=0}^{\infty}\left(-\frac{A}{\alpha}\left(\frac{1}{\alpha}\right)^r-\frac{B}{\beta}\left(\frac{1}{\beta}\right)^r\right)x^r$ Comparing with $\mathrm{f}(x)=\sum_{r=0}^{\infty}u_r x^r$, we get $u_r=-\frac{A}{\alpha}\left(\frac{1}{\alpha}\right)^r-\frac{B}{\beta}\left(\frac{1}{\beta}\right)^r$ $u_n=-\frac{1}{\sqrt{5}}\left(\frac{2}{-1-\sqrt{5}}\right)^n+\frac{1}{\sqrt{5}}\left(\frac{2}{-1+\sqrt{5}}\right)^n$

Qn	Solution
5	**Numerical Methods**
(a)	Note that f is continuous for all $x \in \mathbb{R}$. $\mathrm{f}(0) = -\frac{1}{3} < 0$ $\mathrm{f}(1) = 0.825 > 0$ $\mathrm{f}'(x) = 2 - \cos x > 0$ for all $x \in \mathbb{R}$ Since f is continuous and is a strictly increasing function, and $\mathrm{f}(0)\mathrm{f}(1) < 0$, C cuts the x-axis exactly once in the interval $0 < x < 1$.
(b)	Using $x_{n+1} = x_n - \frac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$ with $x_0 = 0$, $x_1 = 0.333333$ $x_2 = 0.327515$ $x_3 = 0.327509$ $x_4 = 0.327509$ To verify: $\mathrm{f}(0.32745) = -6.29 \times 10^{-5} < 0$ $\mathrm{f}(0.32755) = 4.24 \times 10^{-5} > 0$ Thus, x-intercept of C $= 0.3275$ (4 d.p.)
(c)	**Possible Explanation 1:** For $y = \lvert \mathrm{f}(x) \rvert$, there will be a kink at $x = 0.3275$ for the first quadratic segment from $x = 0$ to $x = 0.5$ when using Simpson's Rule with four strips. Thus, the estimation might not be good. **Possible Explanation 2:** For $y = \mathrm{f}(x)$, the curve is below the x-axis from $x = 0$ to $x = 0.3275$ and above the x-axis from $x = 0.3275$ to $x = 1$. The use of Simpson's Rule with four strips will split the region into 2 parts: from $x = 0$ to $x = 0.5$ and from $x = 0.5$ to $x = 1$ for estimation. Thus, this estimation might not be good as the regions under and above the x-axis are not properly accounted for using Simpson's Rule with four strips.
(d)	**Method 1:** Using $y = \lvert \mathrm{f}(x) \rvert$ A $= \int_0^1 \lvert \mathrm{f}(x) \rvert \, \mathrm{d}x$ $\approx \frac{1}{3}\left(\frac{1-0}{6}\right)\left[\lvert \mathrm{f}(0)\rvert + 4\left\lvert \mathrm{f}\left(\frac{1}{6}\right)\right\rvert + 2\left\lvert \mathrm{f}\left(\frac{1}{3}\right)\right\rvert + 4\left\lvert \mathrm{f}\left(\frac{1}{2}\right)\right\rvert + 2\left\lvert \mathrm{f}\left(\frac{2}{3}\right)\right\rvert + 4\left\lvert \mathrm{f}\left(\frac{5}{6}\right)\right\rvert + \lvert \mathrm{f}(1)\rvert\right]$ $= 0.31774$ (5 d.p.)

	Method 2: Using $y=\mathrm{f}\left(x\right)$ A $\approx -\int_{0}^{\frac{1}{3}} \mathrm{f}\left(x\right)\ \mathrm{d}x+\int_{\frac{1}{3}}^{1} \mathrm{f}\left(x\right)\ \mathrm{d}x$ $\approx -\frac{1}{3}\left(\frac{1-0}{6}\right)\left[\mathrm{f}\left(0\right)+4\mathrm{f}\left(\frac{1}{6}\right)+\mathrm{f}\left(\frac{1}{3}\right)\right]$ $+\frac{1}{3}\left(\frac{1-0}{6}\right)\left[\mathrm{f}\left(\frac{1}{3}\right)+4\mathrm{f}\left(\frac{1}{2}\right)+2\mathrm{f}\left(\frac{2}{3}\right)+4\mathrm{f}\left(\frac{5}{6}\right)+\mathrm{f}\left(1\right)\right]$ $=0.31705 \qquad \left(5 \text{ d.p.}\right)$

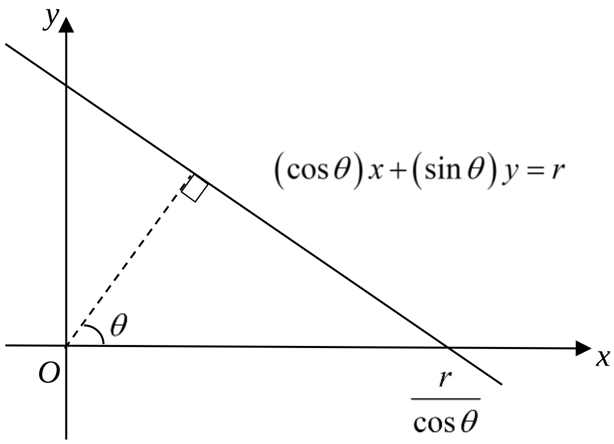
Qn	Solution
6	**Matrices and Linear Spaces**
(a)(i)	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & k_1 & 0 & k_1 \\ 0 & 0 & k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & ak_1 & bk_2 & ak_1+bk_2 \\ 0 & ck_1 & dk_2 & ck_1+dk_2 \end{bmatrix}$ $\therefore O'(0,0)$ $P'(ak_1, ck_1)$ $Q'(bk_2, dk_2)$ $R'(ak_1+bk_2, ck_1+dk_2)$
(ii)	Note that $O'P'R'Q'$ is a parallelogram Area of parallelogram $O'P'R'Q'$ $= \left\| \begin{pmatrix} ak_1 \\ ck_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} bk_2 \\ dk_2 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 0 \\ adk_1k_2 - bck_1k_2 \end{pmatrix} \right\|$ $= \|adk_1k_2 - bck_1k_2\|$ $= \|ad-bc\|\|k_1k_2\|$ $= \|\det(\mathbf{A})\| \times$ area of rectangle $OPRQ$
(b)(i)	Let $\mathbf{A}$ and $\mathbf{B}$ be orthogonal matrices. $(\mathbf{AB})(\mathbf{AB})^T$ $= \mathbf{ABB}^T\mathbf{A}^T$ $= \mathbf{AIA}^T$ $= \mathbf{I}$ $(\mathbf{AB})^T(\mathbf{AB})$ $= \mathbf{B}^T\mathbf{A}^T\mathbf{AB}$ $= \mathbf{B}^T\mathbf{IB}$ $= \mathbf{I}$ Therefore, $\mathbf{AB}$ is an orthogonal matrix.
(ii)	$\det(\mathbf{MM}^T) = \det(\mathbf{I}) = 1$ $\det\mathbf{M}\det\mathbf{M}^T = 1$ $(\det\mathbf{M})^2 = 1$ $\det\mathbf{M} = 1$ or $\det\mathbf{M} = -1$
(iii)	$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\det\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = -1$ but $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

	Hence, $\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$ is not orthogonal
(iv)	$\begin{bmatrix}\cos\theta & 0 & \sin\theta\\0 & 1 & 0\\-\sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\0 & 1 & 0\\\sin\theta & 0 & \cos\theta\end{bmatrix}$ $=\begin{bmatrix}\cos^2\theta+\sin^2\theta & 0 & -\cos\theta\sin\theta+\cos\theta\sin\theta\\0 & 1 & 0\\-\cos\theta\sin\theta+\cos\theta\sin\theta & 0 & \sin^2\theta+\cos^2\theta\end{bmatrix}$ $=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ **<u>Alternative</u>** $\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\0 & 1 & 0\\\sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & 0 & \sin\theta\\0 & 1 & 0\\-\sin\theta & 0 & \cos\theta\end{bmatrix}$ $=\begin{bmatrix}\cos^2\theta+\sin^2\theta & 0 & \cos\theta\sin\theta-\cos\theta\sin\theta\\0 & 1 & 0\\\cos\theta\sin\theta-\cos\theta\sin\theta & 0 & \sin^2\theta+\cos^2\theta\end{bmatrix}$ $=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$
	$\mathbf{B}^{-1}=\mathbf{B}^T=\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\0 & 1 & 0\\\sin\theta & 0 & \cos\theta\end{bmatrix}$ Since $\mathbf{BB}^T=\mathbf{I}$ and $\mathbf{BB}^{-1}=\mathbf{I}$ imply that $\mathbf{B}^{-1}=\mathbf{B}^T$.

Qn	Solution
7	**Matrices and Linear Spaces**
(a)	Since $\begin{pmatrix}12\\-3\\1\end{pmatrix}$ is an eigenvector of **M**, $\begin{pmatrix}3&1&3\\-1&-1&k\\1&3&k\end{pmatrix}\begin{pmatrix}12\\-3\\1\end{pmatrix}=\lambda\begin{pmatrix}12\\-3\\1\end{pmatrix}$ for some $\lambda\in\mathbb{R}$. $\begin{pmatrix}3&1&3\\-1&-1&k\\1&3&k\end{pmatrix}\begin{pmatrix}12\\-3\\1\end{pmatrix}=\lambda\begin{pmatrix}12\\-3\\1\end{pmatrix}$ $\begin{pmatrix}36\\-9+k\\3+k\end{pmatrix}=\lambda\begin{pmatrix}12\\-3\\1\end{pmatrix}$ Solving, $\lambda=3, k=0.$ Solve for $\lambda\mathbf{I}-\mathbf{M}=\mathbf{0}$, $\left\lvert\begin{pmatrix}\lambda-3&-1&-3\\1&\lambda+1&0\\-1&-3&\lambda\end{pmatrix}\right\rvert=0$ $(\lambda-3)(\lambda+1)(\lambda)+(-3)(1)(-3)-(-3)(-1)(\lambda+1)-(1)(-1)(\lambda)=0$ $\lambda^3-2\lambda^2-5\lambda+6=0$ $\lambda=-2$ or $\lambda=1$ or $\lambda=3$ Solve for $\mathbf{Mx}=\lambda\mathbf{x}$, When $\lambda=-2$ $\left(\begin{array}{ccc\|c}-5&-1&-3&0\\1&-1&0&0\\-1&-3&-2&0\end{array}\right)\xrightarrow{\text{Gauss-Jordan Elimination}}\left(\begin{array}{ccc\|c}1&0&1/2&0\\0&1&1/2&0\\0&0&0&0\end{array}\right)$ $\mathbf{x}=s\begin{pmatrix}-1\\-1\\2\end{pmatrix},\ s\in\mathbb{R}.$ $\therefore$ one possible eigenvector for the eigenvalue -2 is $\begin{pmatrix}-1\\-1\\2\end{pmatrix}$. When $\lambda=1$, $\left(\begin{array}{ccc\|c}-2&-1&-3&0\\1&2&0&0\\-1&-3&1&0\end{array}\right)\xrightarrow{\text{Gauss-Jordan Elimination}}\left(\begin{array}{ccc\|c}1&0&2&0\\0&1&-1&0\\0&0&0&0\end{array}\right)$

	$$\mathbf{x} = t\begin{pmatrix}-2\\1\\1\end{pmatrix},\ t \in \mathbb{R}.$$ $\therefore$ one possible eigenvector for the eigenvalue 1 is $\begin{pmatrix}-2\\1\\1\end{pmatrix}$.
(b)	Since the plane P intersects the line $\mathbf{r} = \begin{pmatrix}1\\0\\3\end{pmatrix} + \mu\begin{pmatrix}1\\1\\-2\end{pmatrix}$ at a unique point, P is not parallel to $\begin{pmatrix}1\\1\\-2\end{pmatrix}$. Since O is invariant under $\mathbf{M}$, $$\text{Equation of } P: \mathbf{r} = \begin{pmatrix}0\\0\\0\end{pmatrix} + s\begin{pmatrix}-2\\1\\1\end{pmatrix} + t\begin{pmatrix}12\\-3\\1\end{pmatrix},\ s,t \in \mathbb{R}.$$ $$\mathbf{n}_p = \begin{pmatrix}-2\\1\\1\end{pmatrix} \times \begin{pmatrix}12\\-3\\1\end{pmatrix} = 2\begin{pmatrix}2\\7\\-3\end{pmatrix}$$ Equation of $P: \mathbf{r} \cdot \begin{pmatrix}2\\7\\-3\end{pmatrix} = 0$ $2x + 7y - 3z = 0$

Qn	Solution
8	**Polar Curves + Definite Integral**
(a)	Note that for trivial cases where $\theta = k\pi, k \in \mathbb{Z},$ it is easily observed that the shortest distance from origin to the line is *r*. Suppose $\theta \neq k\pi, k \in \mathbb{Z}.$ Differentiate $(\cos\theta)x + (\sin\theta)y = r$ ---(1) with respect to *x*, $$(\cos\theta) + (\sin\theta)\frac{dy}{dx} = 0$$ $$\frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta}$$ Equation of the line perpendicular to $(\cos\theta)x + (\sin\theta)y = r,$ passing through *O*, is $$y = -\frac{1}{\left(-\frac{\cos\theta}{\sin\theta}\right)}x$$ $$= \frac{\sin\theta}{\cos\theta}x \qquad \text{---(2)}$$ Sub (2) into (1), $$(\cos\theta)x + \left(\frac{\sin^2\theta}{\cos\theta}\right)x = r$$ $$x = r\cos\theta$$ Sub into (2), $$y = r\sin\theta$$ $$\therefore \text{shortest distance} = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = r$$ Since the gradient of the line represented by equation (2) is $y = \frac{\sin\theta}{\cos\theta} = \tan\theta$, θ is the <u>angle the normal makes with the positive *x*-axis</u>. **<u>Alternative method</u>** Since gradient of line is $-\cot\theta$, gradient of normal to the line is $\tan\theta$. Angle that normal makes with the positive *x*-axis is θ.

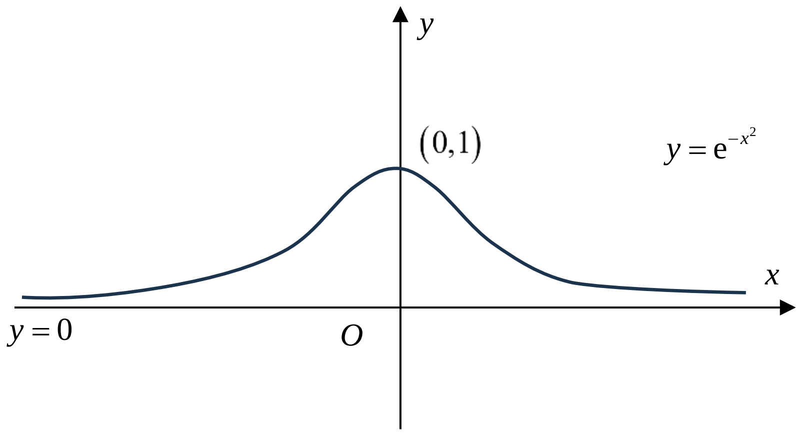
	 Shortest distance from origin to the line $=\dfrac{r}{\cos\theta}\cos\theta=r$ **Alternative method** Since gradient of line is $-\cot\theta$, gradient of normal to the line is $\tan\theta$. Angle that normal makes with the positive *x*-axis is θ. $(\cos\theta)x+(\sin\theta)y=r$ $\Rightarrow(\cos\theta)x+(\sin\theta)y-r=0$ Distance from O to line *L* $=\dfrac{\lvert(\cos\theta)(0)+(\sin\theta)(0)-r\rvert}{\sqrt{\cos^2\theta+\sin^2\theta}}$ $=r$
(b)	Differentiate $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with respect to *x*, $\dfrac{2x}{a^2}+\dfrac{2y\left(\frac{dy}{dx}\right)}{b^2}=0$ $\dfrac{dy}{dx}=-\dfrac{b^2x}{a^2y}$ Equation of tangent at $P(x_0,y_0)$, $y-y_0=-\dfrac{b^2x_0}{a^2y_0}(x-x_0)$ $\dfrac{yy_0}{b^2}-\dfrac{{y_0}^2}{b^2}=\dfrac{{x_0}^2}{a^2}-\dfrac{xx_0}{a^2}$ $\dfrac{xx_0}{a^2}+\dfrac{yy_0}{b^2}=\dfrac{{x_0}^2}{a^2}+\dfrac{{y_0}^2}{b^2}$ $\dfrac{xx_0}{a^2}+\dfrac{yy_0}{b^2}=1$ ---(3) (Since $P(x_0,y_0)$ lies on C)
(c)	Comparing (3) with (1),

	$$\frac{x_0}{a^2}=\frac{\cos\theta}{r} \quad \text{and} \quad \frac{y_0}{b^2}=\frac{\sin\theta}{r}$$ $$x_0=\frac{a^2\cos\theta}{r} \quad \text{and} \quad y_0=\frac{b^2\sin\theta}{r}$$ where r is the is the shortest distance from O to X, and θ is the angle it makes with the positive x-axis. Since (x_0, y_0) lies on C, $$\frac{\left(\frac{a^2\cos\theta}{r}\right)^2}{a^2}+\frac{\left(\frac{b^2\sin\theta}{r}\right)^2}{b^2}=1$$ $$a^2\cos^2\theta+b^2\sin^2\theta=r^2 \quad \text{(Shown)}$$
(d)	Since $a=3$ and $b=2$, the polar equation of the pedal curve is $$9\cos^2\theta+4\sin^2\theta=r^2.$$ Differentiate both sides with respect to θ, $$-18\cos\theta\sin\theta+8\sin\theta\cos\theta=2r\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)$$ $$\frac{-5\sin\theta\cos\theta}{\sqrt{9\cos^2\theta+4\sin^2\theta}}=\frac{\mathrm{d}r}{\mathrm{d}\theta}$$ $$\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2=\frac{25\sin^2\theta\cos^2\theta}{9\cos^2\theta+4\sin^2\theta}$$ Required arc length $$=\int_0^{2\pi}\sqrt{\left(9\cos^2\theta+4\sin^2\theta\right)+\left(\frac{25\sin^2\theta\cos^2\theta}{9\cos^2\theta+4\sin^2\theta}\right)}\,\mathrm{d}\theta$$ $=16.484$ ≈ 16.5 **<u>Alternative method</u>** Since $a=3$ and $b=2$, the polar equation of the pedal curve is $$9\cos^2\theta+4\sin^2\theta=r^2$$ $$\therefore r=\pm\sqrt{9\cos^2\theta+4\sin^2\theta}.$$ Required arc length $$=\int_0^{2\pi}\sqrt{\left(9\cos^2\theta+4\sin^2\theta\right)+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$ $=16.484$ ≈ 16.5

Qn	Solution
9	**Differential Equations**
(a)	$m\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=mg-K\frac{\mathrm{d}s}{\mathrm{d}t}$ $\frac{\mathrm{d}^2s}{\mathrm{d}t^2}+\frac{K}{m}\frac{\mathrm{d}s}{\mathrm{d}t}=g$ Homogeneous equation: $\frac{\mathrm{d}^2s}{\mathrm{d}t^2}+\frac{K}{m}\frac{\mathrm{d}s}{\mathrm{d}t}=0$ Auxiliary equation: $x^2+\frac{K}{m}x=0$ $\Rightarrow x\left(x+\frac{K}{m}\right)=0$ $\Rightarrow x=0$ or $x=-\frac{K}{m}$ Complementary function: $s_c=A+B\mathrm{e}^{-\frac{K}{m}t}$ Let particular integral be $s_p=Ct+D$ $\frac{\mathrm{d}^2s_p}{\mathrm{d}t^2}+\frac{K}{m}\frac{\mathrm{d}s_p}{\mathrm{d}t}=\frac{KC}{m}=g\Rightarrow C=\frac{mg}{K}$ $s=0$ at $t=0\Rightarrow D=0$ General Solution: $s=A+B\mathrm{e}^{-\frac{K}{m}t}+\frac{mg}{K}t$ $\frac{\mathrm{d}s}{\mathrm{d}t}=-\frac{K}{m}B\mathrm{e}^{-\frac{K}{m}t}+\frac{mg}{K}$ It is given that $s=0$ and $\frac{\mathrm{d}s}{\mathrm{d}t}=v_1$ at $t=0$. $A+B=0$ $v_1=-\frac{KB}{m}+\frac{mg}{K}$ $\therefore B=-\frac{m}{K}\left(v_1-\frac{mg}{K}\right)$ $A=\frac{m}{K}\left(v_1-\frac{mg}{K}\right)$ Particular solution: $s=\frac{m}{K}\left(v_1-\frac{mg}{K}\right)-\frac{m}{K}\left(v_1-\frac{mg}{K}\right)\mathrm{e}^{-\frac{K}{m}t}+\frac{mg}{K}t$ $s=\frac{mg}{K}\left[t+\left(\frac{v_1}{g}-\frac{m}{K}\right)\left(1-\mathrm{e}^{-\frac{K}{m}t}\right)\right]$
	Alternative (H2 Math method with substitution): $m\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=mg-K\frac{\mathrm{d}s}{\mathrm{d}t}$ $\frac{\mathrm{d}^2s}{\mathrm{d}t^2}+\frac{K}{m}\frac{\mathrm{d}s}{\mathrm{d}t}=g$ Let $v=\frac{\mathrm{d}s}{\mathrm{d}t}$

	$m\frac{\mathrm{d}v}{\mathrm{d}t}=mg-Kv$ $\int\frac{m}{mg-Kv}\ \mathrm{d}v=\int 1\ \mathrm{d}t$ $-\frac{m}{K}\ln\lvert mg-Kv\rvert=t+c$ $mg-Kv=A\mathrm{e}^{-\frac{K}{m}t}$ $v=\frac{1}{K}\left(mg-A\mathrm{e}^{-\frac{K}{m}t}\right)$ When $t=0,\ v=v_1,$ $v_1=\frac{1}{K}(mg-A)$ $A=mg-v_1K$ $v=\frac{1}{K}\left(mg-(mg-v_1K)\mathrm{e}^{-\frac{K}{m}t}\right)$ $s=\int\frac{1}{K}\left(mg-(mg-v_1K)\mathrm{e}^{-\frac{K}{m}t}\right)\mathrm{d}t$ $s=\frac{mg}{K}t+\frac{(mg-v_1K)}{K}\mathrm{e}^{-\frac{K}{m}t}+D$ $s=0$ at $t=0.$ $D=-\frac{(mg-v_1K)}{K}$ $s=\frac{mg}{K}t+\frac{(mg-v_1K)}{K}\mathrm{e}^{-\frac{m}{K}t}-\frac{(mg-v_1K)}{K}$ $s=\frac{mg}{K}\left[t+\left(\frac{v_1}{g}-\frac{m}{K}\right)\left(1-\mathrm{e}^{-\frac{K}{m}t}\right)\right]$
(b)	$m\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=mg-K\frac{\mathrm{d}s}{\mathrm{d}t}$ Terminal velocity is attained at $\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=0$. $mg-K\frac{\mathrm{d}s}{\mathrm{d}t}=0$ $\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{mg}{K}$ Terminal velocity $v_2=\frac{mg}{K}$

(c)	$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{mg}{K}\left[1 + \frac{K}{m}\left(\frac{v_1}{g} - \frac{m}{K}\right)\left(e^{-\frac{K}{m}t}\right)\right]$ m = 75, g = 9.8, K = 110 and $v_1 = 54$. Using GC, $\frac{\mathrm{d}s}{\mathrm{d}t} < 1.02\left(\frac{75(9.8)}{110}\right)$ $\Rightarrow t > 4.0019506$ $\Rightarrow s > 58.911587$ Let a be altitude. $a < 941.088$ Altitude is ≤ 941 m.

Qn	Solution
10	**Conics**
(a)	 Volume of the display $=2\pi\int_0^r xy \ \mathrm{d}x$ $=2\pi\int_0^r x\mathrm{e}^{-x^2} \ \mathrm{d}x$ $=2\pi\left[-\frac{1}{2}\mathrm{e}^{-x^2}\right]_0^r$ $=2\pi\left[\frac{1}{2}-\frac{1}{2}\mathrm{e}^{-r^2}\right]$ As $r\to\infty$, volume of display $\to 2\pi\left(\frac{1}{2}-0\right)=\pi$. Therefore, the theoretical maximum volume of the solid ceramic display is π units^3. **OR** Volume of the display $=\pi\int_{\mathrm{e}^{-r^2}}^1 x^2 \ \mathrm{d}y$ $=\pi\int_{\mathrm{e}^{-r^2}}^1 -\ln y \ \mathrm{d}y$ $=-\pi\left\{\left[y\ln y\right]_{\mathrm{e}^{-r^2}}^1-\int_{\mathrm{e}^{-r^2}}^1 1 \ \mathrm{d}y\right\}$ $=-\pi\left\{\left[0-\mathrm{e}^{-r^2}\left(-r^2\right)\right]-\left[y\right]_{\mathrm{e}^{-r^2}}^1\right\}$ $=-\pi\left\{\left[0-\mathrm{e}^{-r^2}\left(-r^2\right)\right]-\left[1-\mathrm{e}^{-r^2}\right]\right\}$ $=\pi\left(1-r^2\mathrm{e}^{-r^2}-\mathrm{e}^{-r^2}\right)$ As $r\to\infty$, $r^2\mathrm{e}^{-r^2}\to 0$, $\mathrm{e}^{-r^2}\to 0$. $\therefore$ volume of display $\to \pi(1-0-0)=\pi$. Therefore, the theoretical maximum volume of the solid ceramic display is π units^3.

(b)(i)	Distance from vertex of the parabolic primary mirror to *O* $=80-\frac{100}{2}=30$. Equation of parabola: $y^2=4(80)(x+30)$ $y^2=320x+9600$
(b)(ii)	The distance from *O* to the vertex of the hyperbolic secondary mirror is 20cm, i.e., $a=20$. Since $F_1F_2=100\text{cm}$ and F_1 and F_2 are equidistant from *O*, $c=\frac{100}{2}=50.$ $\therefore e=\frac{c}{a}=\frac{50}{20}=\frac{5}{2}$
(b)(iii)	Cartesian equation of the hyperbola representing the cross-section of the secondary mirror: $\frac{x^2}{20^2}-\frac{y^2}{b^2}=1$ $a^2+b^2=c^2$ $20^2+b^2=50^2$ $b^2=2100$ $\therefore \frac{x^2}{400}-\frac{y^2}{2100}=1 \quad ---(1)$ Since the diameter of the parabolic primary mirror is 24cm, when $y=12$, $12^2=320x+9600$ $x=\frac{144-9600}{320}$ $x=-29.55$ The equation of the line passing through points $(-29.55,12)$ and $F_1(50,0)$: $\frac{y-0}{x-50}=\frac{12-0}{-29.55-50}$ $x=50-\frac{79.55}{12}y \quad ----(2)$ Sub (2) into (1), $\frac{\left(50-\frac{79.55}{12}y\right)^2}{400}-\frac{y^2}{2100}=1$ Using GC, $y=4.5108$ or $y=10.639$ (N.A.) $\therefore$ Minimum diameter $=2\times 4.5108\approx 9.02\text{cm}$ (3 s.f.)