Nanyang Junior College

JC1 H2 Mathematics 2019

Lecture Notes

Chapter	Topic
7	Differentiation Techniques
8	Applications of Differentiation

Name:____________________

CT: 19__

Chapter 7

Differentiation Techniques

At the end of this chapter, students should be able to:

(a) understand the derivative of $f(x)$ as the gradient of the tangent to the graph of $y = f(x)$ and applying the First Principles $\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$ to find the derivative of $f(x)$;

(b) apply the following rules of differentiation: Product Rule, Quotient Rule, Chain Rule;

(c) find the approximate value of a derivative at a given point using a graphical calculator;

(d) find the derivative of x^n for any rational n, a^x, $\log_a x$, together with constant multiples, sums and differences;

(e) understand the derivative of $\sin x$, $\cos x$, $\tan x$ and their inverse functions, together with constant multiples, sums and differences;

(f) find and use the derivative of simple functions defined implicitly.

(g) find the derivative of functions defined parametrically.

7.1 Rules of Differentiation (Independent reading)

Let u and v be functions of x with n, c and d being constants.

a. Derivative of a constant:

$$\boxed{\frac{d}{dx}(c) = 0} \qquad \text{e.g. } \frac{d}{dx}(50) = 0$$

b. Derivative of $c(u)$:

$$\boxed{\frac{d}{dx}(cu) = c\frac{d}{dx}(u) = c\frac{du}{dx}} \qquad \text{e.g. } \frac{d}{dx}(5x) = 5\frac{d}{dx}(x) = 5(1) = 5$$

c. Derivative of x^n, $n \in \mathbb{R}$:

$$\boxed{\frac{d}{dx}(cx^n) = c\frac{d}{dx}(x^n) = c\left(nx^{n-1}\right) = cnx^{n-1}} \qquad \text{e.g. } \frac{d}{dx}(3x^5) = 3\frac{d}{dx}(x^5) = 3\left(5x^4\right) = 15x^4$$

d. Sum and Difference Rules:

$$\boxed{\frac{d}{dx}[cu \pm dv] = c\frac{du}{dx} \pm d\frac{dv}{dx}}$$

e.g.
$$\begin{aligned}\frac{d}{dx}(12x^{10} + 3x^{1/2}) &= 12\frac{d}{dx}(x^{10}) + 3\frac{d}{dx}(x^{1/2}) \\ &= 12\left(10x^9\right) + 3\left(\frac{1}{2}x^{-1/2}\right) \\ &= 120x^9 + \frac{3}{2}x^{-1/2}\end{aligned}$$

e. Product Rule:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

e.g. $$\frac{d}{dx}(3x^2)(2x+1) = (2x+1)\frac{d}{dx}(3x^2) + (3x^2)\frac{d}{dx}(2x+1)$$
$$= (2x+1)(6x) + (3x^2)(2)$$
$$= 18x^2 + 6x$$

f. Quotient Rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$ where $v \neq 0$.

e.g. $$\frac{d}{dx}\left(\frac{x^3}{x^2+1}\right) = \frac{(x^2+1)\frac{d}{dx}(x^3) - x^3\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$
$$= \frac{(x^2+1)(3x^2) - x^3(2x)}{(x^2+1)^2}$$
$$= \frac{x^2(3x^2+3-2x^2)}{(x^2+1)^2}$$
$$= \frac{x^2(x^2+3)}{(x^2+1)^2}$$

$$\frac{d}{dx}x^n = \lim_{\delta x \to 0} \frac{(x+\delta x)^n - x^n}{\delta x}$$
$$= \lim_{\delta x \to 0} \frac{\left(x^n + \binom{n}{1}x^{n-1}\delta x^1 + \binom{n}{2}x^{n-2}\delta x^2 + \cdots \binom{n}{n}\delta x^n\right) - x^n}{\delta x}$$
$$= \lim_{\delta x \to 0} \frac{\binom{n}{1}x^{n-1}\delta x^1 + \binom{n}{2}x^{n-2}\delta x^2 + \cdots \binom{n}{n}\delta x^n}{\delta x}$$
$$= \lim_{\delta x \to 0} \frac{\delta x\left[\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}\delta x^1 + \cdots \binom{n}{n}\delta x^{n-1}\right]}{\delta x}$$
$$= nx^{n-1}$$

g. Chain Rule:

We can apply chain rule to differentiate composite functions using the result

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx},$$ where y and u are both functions of x.

e.g. Given that $y = 5(x^2+4x)^3$, we want to differentiate y w.r.t x.

$$y = 5(x^2+4x)^3$$
$$\frac{dy}{dx} = 5\frac{d}{dx}(x^2+4x)^3$$
$$= 5\left[3(x^2+4x)^2\right]\frac{d}{dx}(x^2+4x)$$
$$= 15(x^2+4x)^2(2x+4)$$

In general, $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \times f'(x), \quad n \in \mathbb{R}$.

Example 1

Find $\frac{dy}{dx}$ for each of the following:

(a) $y=(3x^2+7x+1)^5$ (b) $y=\frac{1}{\sqrt{(x^2+1)}}$ (c) $y=\left(x^2+2\right)^{e}$

(d) $y=2x^3\left(1-x^2\right)^{-\frac{3}{2}}$ (e) $y=\frac{2x^3+7}{x^4+1}$ (f) $y=\sqrt{\frac{x}{x^2+1}}$

Solution:

(a) $y=(3x^2+7x+1)^5$

$$\frac{dy}{dx}=5(3x^2+7x+1)^4\frac{d}{dx}(3x^2+7x+1)$$

$$=5(3x^2+7x+1)^4(6x+7)$$

(b) $y=\frac{1}{\sqrt{x^2+1}}=(x^2+1)^{-\frac{1}{2}}$

$$\frac{dy}{dx}=-\frac{1}{2}(x^2+1)^{-\frac{3}{2}}(2x)$$

$$=-x(x^2+1)^{-\frac{3}{2}}$$

Why is it desirable to write $\frac{1}{\sqrt{(x^2+1)}}$ as $(x^2+1)^{-\frac{1}{2}}$ when differentiating?

(c) $y=(x^2+2)^{e}$

$$\frac{dy}{dx}=e\,(x^2+2)^{e-1}(2x)$$

$$=2ex(x^2+2)^{e-1}$$

(d) $y=2x^3\left(1-x^2\right)^{-\frac{3}{2}}$

$$\frac{dy}{dx}=6x^2(1-x^2)^{-\frac{3}{2}}+2x^3\left[-\frac{3}{2}(1-x^2)^{-\frac{5}{2}}(-2x)\right]$$

$$=6x^2(1-x^2)^{-\frac{3}{2}}+6x^4(1-x^2)^{-\frac{5}{2}}$$

$$=6x^2(1-x)^{-\frac{5}{2}}\left(1-x^2+x^2\right)$$

$$=6x^2(1-x)^{-\frac{5}{2}}$$

(e) $y=\frac{2x^3+7}{x^4+1}$

$$\frac{dy}{dx}=\frac{6x^2\left(x^4+1\right)-\left(2x^3+7\right)\left(4x^3\right)}{\left(x^4+1\right)^2}$$

$$=\frac{2x^2\left(3x^4+3-4x^4-14x\right)}{\left(x^4+1\right)^2}$$

$$=\frac{2x^2\left(3-14x-x^4\right)}{\left(x^4+1\right)^2}$$

(f) $y=\sqrt{\dfrac{x}{x^2+1}}$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{x}{x^2+1}\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(\frac{x}{x^2+1}\right)$$

$$=\frac{1}{2}\left(\frac{x}{x^2+1}\right)^{-\frac{1}{2}}\times\frac{(x^2+1)\cdot 1-x(2x)}{(x^2+1)^2}$$

$$=\frac{1}{2}\left(\frac{x^2+1}{x}\right)^{\frac{1}{2}}\times\frac{1-x^2}{(x^2+1)^2}=\frac{1-x^2}{2\sqrt{x}\left(\sqrt{x^2+1}\right)^3}$$

7.2 Differentiation of Basic Trigonometric Functions (Independent reading)

$y=f(x)$	Derivative $\frac{dy}{dx}$	$y=f(x)$	Derivative $\frac{dy}{dx}$
$\sin x$	$\cos x$	$\sin f(x)$	$f'(x)\times\cos f(x)$
$\cos x$	$-\sin x$	$\cos f(x)$	$-f'(x)\times\sin f(x)$
$\tan x$	$\sec^2 x$	$\tan f(x)$	$f'(x)\times\sec^2 f(x)$
$\sec x$	$\sec x\tan x$ [MF 26]	$\sec f(x)$	$f'(x)\times\sec f(x)\tan f(x)$
$\operatorname{cosec} x$	$-\operatorname{cosec} x\cot x$ [MF 26]	$\operatorname{cosec} f(x)$	$-f'(x)\times\operatorname{cosec} f(x)\cot f(x)$
$\cot x$	$-\operatorname{cosec}^2 x$	$\cot f(x)$	$-f'(x)\times\operatorname{cosec}^2 f(x)$

Table 7.1

$\frac{d}{d\theta}\sin\theta^\circ=\frac{d}{d\theta}\sin\frac{\theta\pi}{180}$

Note:

1. When differentiating trigonometric functions, x must be in **radians**. $=\frac{\pi}{180}\cos\frac{\theta\pi}{180}$
2. You could write $\sec x$ as $(\cos x)^{-1}$ or $\dfrac{1}{\cos x}$ and obtain the derivative by applying

$\frac{d}{dx}[f(x)]^n=n[f(x)]^{n-1}\times f'(x)$ (Chain Rule) or $\frac{d}{dx}\left(\frac{u}{v}\right)=\dfrac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ (Quotient Rule).

The same method applies to differentiating $\operatorname{cosec} x$ and $\cot x$.

Important: $(\cos x)^{-1}$ is not to be written as $\cos^{-1}x$.

$y=\cos^{-1}x$ is the inverse function of $y=\cos x$.

Example 2

(i) Prove that $\frac{d}{dx}(\tan x) = \sec^2 x$.

(ii) Differentiate the following with respect to x:

(a) $\cos^2(3x^2)$ (b) $\sec^5(5x)$ (c) $x^2 \sin^2 x$ (d) $\tan(x^2) - x^2$

Solution:

(i) $$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos^2 x - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

(ii)(a) $$\frac{d}{dx}\cos^2(3x^2) = \frac{d}{dx}\left[\cos(3x^2)\right]^2$$

$$= 2\left[\cos(3x^2)\right]^{2-1}\frac{d}{dx}[\cos(3x^2)]$$

$$= 2\cos(3x^2)\left[-\sin(3x^2)\frac{d}{dx}(3x^2)\right] = -12x\sin(3x^2)\cos(3x^2) = -6x\sin(6x^2)$$

(b) $$\frac{d}{dx}\sec^5(5x) = 5[\sec(5x)]^4[5\sec(5x)\tan(5x)] = 25\sec^5(5x)\tan(5x)$$

(c) $$\frac{d}{dx}(x^2\sin^2 x) = 2x\sin^2 x + x^2(2\sin x\cos x) = 2x\sin x(\sin x + x\cos x)$$

(d) $$\frac{d}{dx}(\tan(x^2) - x^2) = \sec^2(x^2)\frac{d}{dx}(x^2) - 2x$$

$$= 2x\sec^2(x^2) - 2x$$

$$= 2x(\sec^2(x^2) - 1) \qquad [\text{since } 1 + \tan^2\theta = \sec^2\theta]$$

$$= 2x\tan^2(x^2)$$

7.3 Using the Graphic Calculator (GC) to find the approximate value of the derivative at a particular point

Given $y = f(x)$, we can use the graphic calculator to calculate the numerical value of the derivative $\frac{dy}{dx}$ or slope of the tangent line at a given point on the graph of $f(x)$.

Example 3

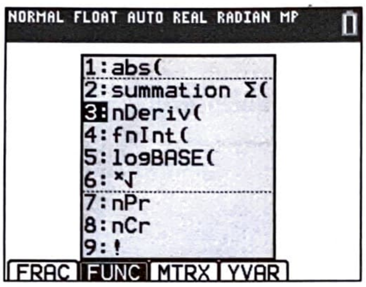
Use the GC to find $\frac{dy}{dx}$ at the point where $x = 2$ given that $y = 3x^4 + 5$.

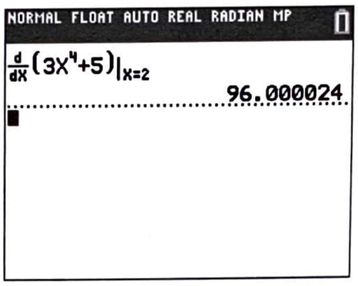
Method 1: We can enter the function into the graphic calculator and find the value of the derivative of function at the point where $x = 2$.

▮ Solution:

Step 1: Press ALPHA WINDOW to call the command **3:nDeriv(** on the screen.

Step 2: Enter the function $y = 3x^4 + 5$ and $x = 2$ into the GC as shown below.





$\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^4+5\right)\big|_{x=2}$ is the value of $\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^4+5\right)$ when $x = 2$.

Check for yourself that the derivative obtained is the same as using computation by hand.

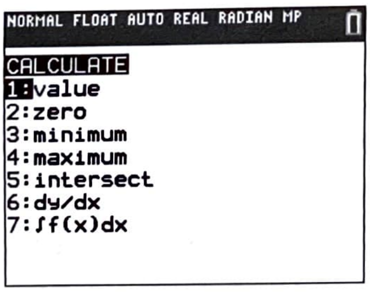
Method 2: We can also find the derivative by graphing the function using the GC.

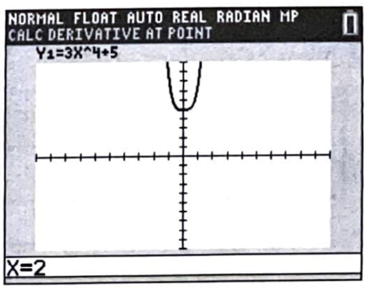
▮ Solution:

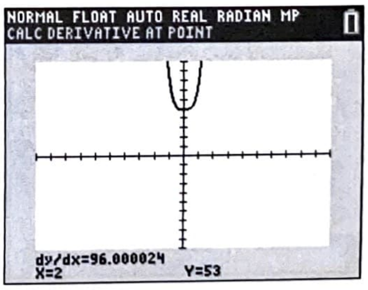
Step 1: Enter the function $y = 3x^4 + 5$ into Y_1 = and press GRAPH to graph the function.

Step 2: Press 2nd TRACE 6 to select the command **6:dy/dx,** the numeric derivative option. This will return the user to the graph screen.

Step 3: Enter the X value by pressing 2, and you will see X=2 on the graph screen. Then press ENTER. This will give you the value of $\frac{dy}{dx}$ at $x = 2$.







Note:

When using the GC, there are some restrictions in it giving numerical answers instead of exact answers.

7.4 Differentiation by first principles

The gradient at any point on the curve is defined as the gradient of the tangent at that point on the curve.

Consider a curve $y = f(x)$. Let $P(x, y)$ be an arbitrary fixed point chosen on the curve. Our objective is to find the gradient of the curve at P.

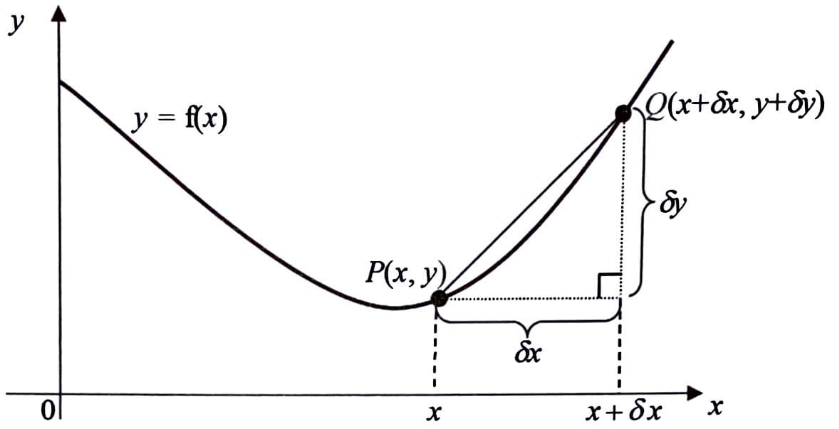


Figure 7.1

Let $Q(x+\delta x, y+\delta y)$ be another point on the curve near P where δx (read as 'delta x') denotes the small increment of x and δy (delta y) denotes the corresponding small increment of y.

The gradient of the chord $PQ = \dfrac{\delta y}{\delta x}$.

Now imagine Q to be a "bead" threaded to the curve and is gradually sliding down towards P. As Q approaches P, we see that the gradient of the chord PQ approaches the gradient of the tangent to the curve at P.

Gradient of the curve at P = $\lim\limits_{Q \to P}$ (gradient of the chord PQ)

$$= \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \quad \text{(since } Q \to P \text{ as } \delta x \to 0\text{)}$$

Since $\delta y = f(x+\delta x) - f(x)$, the gradient of tangent at P is

$$\boxed{\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}}$$

This process of taking the limit of $\dfrac{\delta y}{\delta x}$ as $\delta x \to 0$ to find the gradient of the tangent at a point on the curve is known as **differentiation by first principles**.

The symbol '$\dfrac{dy}{dx}$' is used to denote the more cumbersome notation $\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$, i.e. $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

The symbol '$\dfrac{dy}{dx}$' is called the **derivative of y with respect to x** or equivalently, the **gradient function** corresponding to the curve $y = f(x)$.

Remarks:

1. If $y = f(x)$, we can also denote $\frac{dy}{dx}$ by $f'(x)$ (read as 'f prime of x').

2. '$\frac{d}{dx}$' is an **operator**, <u>not</u> a fraction. $\frac{dy}{dx}$ is read as 'the result when $\frac{d}{dx}$ **operates** on y' or 'differentiate y with respect to x'.

Input	**Operator**	**Output**
$y = f(x)$ →	$\frac{d}{dx}$	→ $\frac{dy}{dx}$ or $\frac{d}{dx}[f(x)]$ or $f'(x)$
Function		**Derivative**

Example 4

Use 'Differentiation by First Principles' to find $f'(x)$ given that $f(x) = x^2$.

Solution:

Using first principles,

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{(x+\delta x)^2 - x^2}{\delta x} = \lim_{\delta x \to 0} \frac{x^2 + 2x\delta x + (\delta x)^2 - x^2}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2x\delta x + (\delta x)^2}{\delta x} = \lim_{\delta x \to 0} \frac{\delta x[2x + (\delta x)]}{\delta x} = \lim_{\delta x \to 0} (2x + \delta x) = 2x$$

Example 5

Use 'Differentiation by First Principles' to find $f'(x)$ given that $f(x) = \frac{1}{x}$.

Solution:

Using first principles,

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{\frac{1}{x+\delta x} - \frac{1}{x}}{\delta x} = \lim_{\delta x \to 0} \frac{\frac{x-(x+\delta x)}{x(x+\delta x)}}{\delta x} = \lim_{\delta x \to 0} \frac{\frac{-\delta x}{x(x+\delta x)}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-1}{x(x+\delta x)} = -\frac{1}{x^2}$$

Example 6

Use 'Differentiation by First Principles' to find $f'(x)$ given that $f(x) = \sin x$.

[Take $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$]

Solution:

Using first principles, $f'(x)=\lim\limits_{\delta x\to 0}\dfrac{f(x+\delta x)-f(x)}{\delta x}$ $=\lim\limits_{\delta x\to 0}\dfrac{\sin(x+\delta x)-\sin x}{\delta x}$	Recall that in [MF 26]: $\sin P-\sin Q=2\cos\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$
$=\lim\limits_{\delta x\to 0}\dfrac{2\cos\frac{1}{2}(2x+\delta x)\sin\frac{1}{2}(\delta x)}{\delta x}$ $=\lim\limits_{\delta x\to 0}\dfrac{\cos\frac{1}{2}(2x+\delta x)\sin\frac{1}{2}(\delta x)}{\frac{1}{2}(\delta x)}$	
$=\lim\limits_{\delta x\to 0}\cos\frac{1}{2}(2x+\delta x)\lim\limits_{\delta x\to 0}\dfrac{\sin\frac{1}{2}(\delta x)}{\frac{1}{2}(\delta x)}$	Applying theorem of limits: Given that $\lim\limits_{x\to a}f(x)$ and $\lim\limits_{x\to a}g(x)$ exists, $\lim\limits_{x\to a}\left[f(x)g(x)\right]=\lim\limits_{x\to a}f(x)\lim\limits_{x\to a}g(x)$
$=\left[\cos\frac{1}{2}(2x)\right](1)$ $=\cos x$	Using $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$

Self-Review 1

Use 'Differentiation by First Principles' to find $f'(x)$ given that $f(x)=x^3$.

7.5 Implicit Differentiation

Suppose we are given $y=f(x)$. Then to find $\frac{dy}{dx}$, we simply differentiate $f(x)$ directly. However, there are instances when y cannot be written explicitly as a function of x. In this case, y is said to be an **implicit function** of x. In such cases, we will use the **chain rule** to find $\frac{dy}{dx}$. This process is known as **implicit differentiation.** So to differentiate a function in y with respect to x, we have $\frac{d}{dx}[g(y)]=\frac{dg(y)}{dy}\frac{dy}{dx}$. For example $\frac{d}{dx}y^2=\frac{d}{dy}(y^2)\frac{dy}{dx}=2y\frac{dy}{dx}$.

To obtain the derivative of an implicit function, **differentiate** the given equation **term by term** with respect to the variable required.

We can also differentiate the given equation with respect to another variable. For example:

Differentiating $y^2 + x = 1$ with respect to y, we get: $2y\dfrac{dy}{dy} + \dfrac{dx}{dy} = 0$

Note: $\dfrac{dx}{dx}$ and $\dfrac{dy}{dy}$ are both equal 1.

Example 7

(a) Find $\dfrac{dy}{dx}$ in each of the following in terms of x and y:

(i) $y + y^2 + xy = 1$ (ii) $\sin y = e^x$.

(b) Without the use of a Graphing Calculator, find $\dfrac{dy}{dx}$ at the point where $x = 0$ given that $y = \sqrt{\dfrac{1+x^2}{1-x^2}}$ $(|x|<1)$.

Solution:

(a)(i) $y + y^2 + xy = 1$

Differentiate the equation implicitly w.r.t x,

$$\frac{d}{dx}(y) + \frac{d}{dx}(y^2) + \frac{d}{dx}(xy) = \frac{d}{dx}(1)$$

$$\frac{dy}{dx} + 2\frac{dy}{dx}y + x\frac{dy}{dx} + y(1) = 0$$

$$\frac{dy}{dx}(2y + x + 1) + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{2y + x + 1}$$

(a)(ii) $\sin y = e^x$

Implicitly differentiate w.r.t. x,

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(e^x) \Rightarrow \cos y\frac{dy}{dx} = e^x$$

$$\frac{dy}{dx} = \frac{e^x}{\cos y}$$

(b) $y = \sqrt{\dfrac{1+x^2}{1-x^2}}$ $(|x|<1)$

Squaring both sides of the equation,

$$y^2 = \frac{1+x^2}{1-x^2}.$$

Differentiating implicitly, we obtain

Note:

(a)(i) $\dfrac{d}{dx}(x^2) = 2x$, but $\dfrac{d}{dx}(y^2) \neq 2y$.

As $\dfrac{d}{dx}[g(y)] = g'(y)\dfrac{dy}{dx}$

So, $\dfrac{d}{dx}(y^2) = 2y\dfrac{d}{dx}(y) = 2y\dfrac{dy}{dx}$

By the product rule,

$$\frac{d}{dx}(xy) = y\frac{d}{dx}(x) + x\frac{d}{dx}(y)$$
$$= y + x\frac{dy}{dx}$$

Note that $\dfrac{dy}{dx}$ is in terms of x and y.

(a)(ii)

$$\frac{d}{dx}(\sin f(x)) = \cos f(x)\frac{d}{dx}(f(x))$$

(b) Why must $|x|<1$?

Another method is to differentiate directly, i.e. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\sqrt{\dfrac{1+x^2}{1-x^2}}\right)$

Compare with the method of

$$2y\frac{dy}{dx}=\frac{2x\left(1-x^2\right)-(-2x)\left(1+x^2\right)}{\left(1-x^2\right)^2}$$

$$=\frac{2x\left(1-x^2+1+x^2\right)}{\left(1-x^2\right)^2}=\frac{4x}{\left(1-x^2\right)^2}.$$

$$\frac{dy}{dx}=\frac{2x}{y\left(1-x^2\right)^2}$$

When $x=0$, $y=1$ gives $\frac{dy}{dx}=0$

squaring both sides, which method is easier?

There is 3rd method, which is to take ln both sides. We will discuss this method further later in Example 9(g).

Example 8 (MB N84/II/6)

A curve is defined by the equation $x^3+y^3+3xy-1=0$. Find the gradient of this curve at the point $(2, -1)$.

Solution:

Differentiate the equation with respect to x, we have

$$x^3+y^3+3xy-1=0$$

$$3x^2+3y^2\frac{dy}{dx}+(3y+3x\frac{dy}{dx})=0$$

$$\frac{dy}{dx}=\frac{-3x^2-3y}{3y^2+3x}=\frac{-x^2-y}{y^2+x}$$

At the point $(2, -1)$, $\frac{dy}{dx}=\frac{-(2)^2-(-1)}{(-1)^2+(2)}=\frac{-3}{3}=-1$

Therefore, gradient of this curve at the point $(2, -1)$ is -1.

Extension:

Obtain the coordinates of the point on the curve at which gradient is zero.

Ans: $(-1, -1)$

Self-Review 2

Find $\frac{dy}{dx}$ for $2\cos y+\sin x=y^2$ in terms of x and y. $\left[\frac{dy}{dx}=\frac{\cos x}{2(y+\sin y)}\right]$

$$\frac{d}{dx}2\cos y+\frac{d}{dx}\sin x=\frac{d}{dx}y^2$$

$$-2\sin y\frac{dy}{dx}+\cos x=2y\frac{dy}{dx}$$

$$(2y+2\sin y)\frac{dy}{dx}=\cos x$$

$$\frac{dy}{dx}=\frac{\cos x}{2y+2\sin y}$$

7.6 The Derivative of $\log_a x$ $(a > 0, a \neq 1, x > 0)$

We know that $\frac{d}{dx}(\log_e x) = \frac{d}{dx}(\ln x) = \frac{1}{x}$ (Or more generally, $\frac{d}{dx}\ln f(x) = \frac{1}{f(x)} \times f'(x)$)

Change of logarithm base:
$$\log_a b = \frac{\log_e b}{\log_e a} = \frac{\ln b}{\ln a}$$

So $\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{d}{dx}\left(\frac{1}{\ln a} \times \ln x\right)$

$$= \left(\frac{1}{\ln a}\right)\frac{d}{dx}(\ln x) = \left(\frac{1}{\ln a}\right)\left(\frac{1}{x}\right)$$

$$= \frac{1}{x \ln a}$$

$$\therefore \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

More generally, $$\boxed{\frac{d}{dx}\left[\log_a f(x)\right] = \frac{f'(x)}{(\ln a) f(x)}}$$

$$\frac{d}{dx}(\ln x) = \lim_{\delta x \to \infty} \frac{\ln(x+\delta x) - \ln x}{\delta x}$$

$$= \frac{\ln\left(\frac{x+\delta x}{x}\right)}{\delta x} = \ln\left(\frac{x+\delta x}{x}\right) \times \frac{1}{\delta x}$$

$$= \ln\left(\frac{x+\delta x}{x}\right) \times \ln\left(e^{\frac{1}{\delta x}}\right)$$

=

7.7 The Derivative of a^x $(a > 0, x \in \mathbb{R})$

We know that $\frac{d}{dx}(e^x) = e^x$.

More generally, by the chain rule, $$\boxed{\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}}$$

Let us consider the function $y = a^x$ in general.

Taking ln both sides gives $\ln y = x \ln a$

Differentiating w.r.t x, $\frac{1}{y}\frac{dy}{dx} = \ln a \Rightarrow \frac{dy}{dx} = y \ln a = a^x \ln a$

$$\frac{d}{dx}\left(a^x\right) = a^x \ln a$$

More generally, $$\boxed{\frac{d}{dx}\left(a^{f(x)}\right) = a^{f(x)} f'(x) \ln a}$$

Example 9

Differentiate the following with respect to x:

(a) $x^3 \log_2 x$ (b) $\sin\left(e^{x^2}\right)$ (c) $x^2 e^{\frac{1}{x}}$ (d) 10^{3x}

(e) $e^{\ln x^2}$ (f) $\ln\left(\frac{(2x+1)^2}{e^{-x}(1+x^3)}\right)$ (g) $\frac{(2x+1)^2}{e^{-x}(1+x^3)}$

Solution:

(a) $\dfrac{d}{dx}(x^3 \log_2 x) = \dfrac{d}{dx}(x^3 \dfrac{\ln x}{\ln 2}) = \dfrac{1}{\ln 2}\dfrac{d}{dx}(x^3 \ln x)$

$$= \frac{1}{\ln 2}(x^3 \cdot \frac{1}{x} + 3x^2 \ln x) = \frac{x^2}{\ln 2}(1 + 3\ln x)$$

(b) $\dfrac{d}{dx}\sin\left(e^{x^2}\right) = \left(\cos\left(e^{x^2}\right)\right)\dfrac{d}{dx}e^{x^2} = 2xe^{x^2}\cos\left(e^{x^2}\right)$

(c) $\dfrac{d}{dx}\left(x^2 e^{\frac{1}{x}}\right) = (2x)e^{\frac{1}{x}} + x^2\left(-\dfrac{1}{x^2}e^{\frac{1}{x}}\right) \quad = e^{\frac{1}{x}}(2x-1)$

(d) $\dfrac{d}{dx}\left(10^{3x}\right) = 10^{3x}(3)(\ln 10)$

(d) Remember to differentiate '$3x$' w.r.t. x.

(e) Using the result : $e^{\ln f(x)} = f(x)$

$\dfrac{d}{dx}(e^{\ln x^3}) = \dfrac{d}{dx}(x^3) = 2x$

(e) Recall that $e^{\ln f(x)} = f(x)$ where $f(x) > 0$.

(f) $\ln\left(\dfrac{(2x+1)^2}{e^{-x}(1+x^3)}\right) = \ln(2x+1)^2 - \ln\left(e^{-x}(1+x^3)\right)$

[redacted] $\dfrac{d}{dx}\left[\ln\left(\dfrac{(2x+1)^2}{e^{-x}(1+x^3)}\right)\right] = \dfrac{d}{dx}\left[\ln(2x+1)^2 - \ln(e^{-x}(1+x^3))\right]$

$\cancel{= \dfrac{2}{2x+1} - \dfrac{1}{e^{-x}(1+x^3)}\left(\dfrac{1}{e^{-x}}\right)(-1) + x^3\left(\dfrac{1}{e^{-x}}\right)(-1) + 3x^2 e^{-x}}$

$= \dfrac{d}{dx}[2\ln(2x+1) - \ln(e^{-x}) - \ln(1+x^3)] = \dfrac{d}{dx}(2\ln(2x+1) + x - \ln(1+x^3))$

$= \dfrac{4}{2x+1} + 1 - \dfrac{3x^2}{1+x^3}$

$$\frac{d}{dx}\ln\left(\frac{(2x+1)^2}{e^{-x}(1+x^3)}\right) = \frac{4}{2x+1} + 1 - \frac{3x^2}{1+x^3}$$

(f) Applying the rules of logarithm before diffentiating eases the differentiation

(g) Let $y = \dfrac{(2x+1)^2}{e^{-x}(1+x^3)} \Rightarrow \ln y = 2\ln(2x+1) + x - \ln(1+x^3)$

$\dfrac{d}{dx}\ln y = \dfrac{4}{2x+1} + 1 - \dfrac{3x^2}{1+x^3}$, differentiating implicitly

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{4}{2x+1} + 1 - \dfrac{3x^2}{1+x^3}$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{4}{2x+1} + 1 - \frac{3x^2}{1+x^3}\right) = \frac{(2x+1)^2}{e^{-x}(1+x^3)}\left(\frac{4}{2x+1} + 1 - \frac{3x^2}{1+x^3}\right)$$

(g) This is a typical example of using rules of logarithm followed by implicit differentialtion.

7.8 Differentiation of Inverse Trigonometric Functions

The Derivative of $\sin^{-1} x$ $(-1 \le x \le 1)$

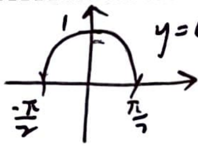
It is noted that $f(x) = \sin x$ is a 1–1 function for $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$, which is also known as the principal range for x.

We denote the inverse function $f^{-1}(x)$ by $\sin^{-1} x$.

Thus, if $y = \sin^{-1}x$ where $-1 \le x \le 1$, and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

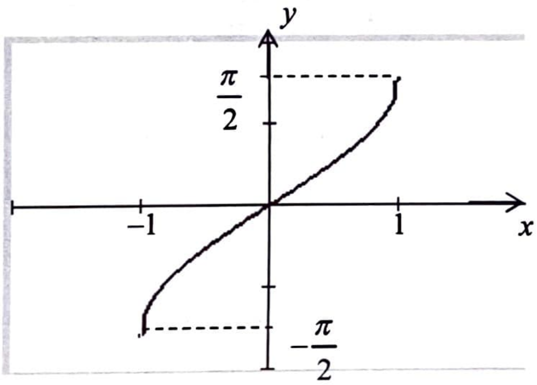
then $\sin y = x$

$y = \cos x$ $1 \ge \cos x \ge 0$ $-\frac{\pi}{2} < x < \frac{\pi}{2}$



Differentiating implicitly w.r.t x

$$\frac{dy}{dx}\cos y = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$



However, $\cos y = \pm\sqrt{1-\sin^2 y}$ and

since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, we have $\cos y \ge 0$

Thus, $\cos y = \sqrt{1-\sin^2 y}$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

Note that, from the graph of $y = \sin^{-1}x$ for $-1 \le x \le 1$, it is a strictly increasing function, hence $\frac{dy}{dx} > 0$.

Hence we have the following results:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1 \text{ (in MF 26)}$$

$$\frac{d}{dx}\left(\sin^{-1} f(x)\right) = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}, \quad -1 < f(x) < 1$$

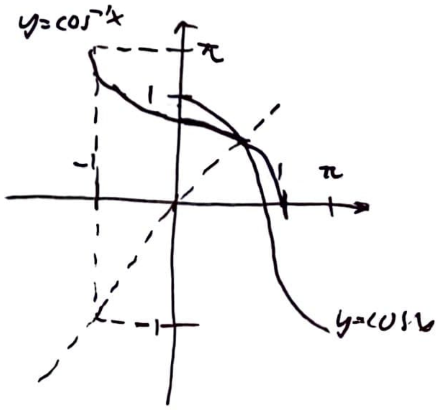
Self-Review 3

Without the use of tables or standard results, find $\frac{dy}{dx}$ for

(i) $y = \cos^{-1} x$, $(-1 \le x \le 1)$ **(ii)** $y = \tan^{-1} x$, $(x \in \mathbb{R})$ $\left[-\frac{1}{\sqrt{1-x^2}}, \frac{1}{1+x^2}\right]$

Solution

(i) It is also noted that $f(x) = \cos x$ is a 1–1 function where $0 \le x \le \pi$, which is known as the principal values for x.



We denote the inverse function $f^{-1}(x)$ by $\cos^{-1} x$.

Thus, if $y = \cos^{-1} x$ where $-1 \le x \le 1$, and $0 \le y \le \pi$

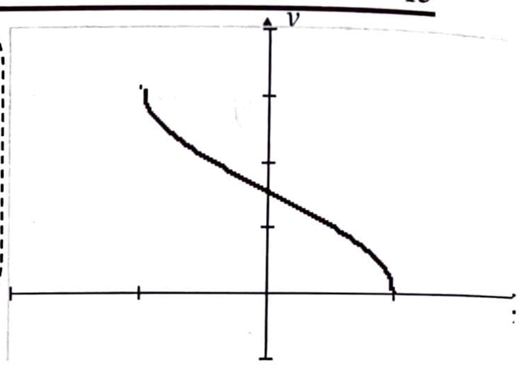
then $\cos y = x$

Differentiating implicitly w.r.t x,

$-\sin y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y}$

But $\sin y = \sqrt{1-\cos^2 y}$, as $\sin y \geq 0$ for $0 \leq y \leq \pi$

Thus, $\frac{dy}{dx} = -\frac{1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$



Note that, from the graph of $y = \cos^{-1}x$ for $-1 \leq x \leq 1$, it is a strictly decreasing function, hence $\frac{dy}{dx} < 0$

Thus, we have the following results:

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{(1-x^2)}}, \quad -1 < x < 1 \text{ (in MF 26)}$$

$$\frac{d}{dx}\left(\cos^{-1}f(x)\right) = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}, \quad -1 < f(x) < 1$$

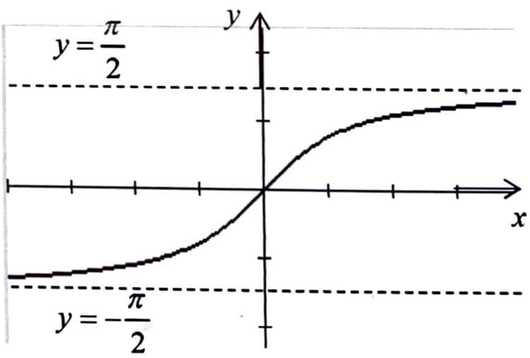
(ii) It is also noted that $f(x) = \tan x$ is a 1–1 function where $-\frac{\pi}{2} < x < \frac{\pi}{2}$, which is known as the principal values for x.

We denote the inverse function $f^{-1}(x)$ by $\tan^{-1}x$.

Thus if $y = \tan^{-1}x$ where $-\infty < x < \infty$, and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

then $\tan y = x \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$



Thus, we have the following results:

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \quad \text{(in MF 26)}$$

$$\frac{d}{dx}\left(\tan^{-1}f(x)\right) = \frac{f'(x)}{1+[f(x)]^2}$$

A Note about Notation:

a. $\sin^{-1}x \neq \frac{1}{\sin x}$, $\cos^{-1}x \neq \frac{1}{\cos x}$, $\tan^{-1}x \neq \frac{1}{\tan x}$

Inverse trigonometric functions $\neq$ Reciprocal of trigonometric functions.

b. Instead, the correct notations for the reciprocal of trigonometric functions are as follows:

$$(\sin x)^{-1} = \frac{1}{\sin x} = \operatorname{cosec} x, \quad (\cos x)^{-1} = \frac{1}{\cos x} = \sec x, \quad (\tan x)^{-1} = \frac{1}{\tan x} = \cot x$$

$\frac{d}{dx} \sin^{-1} f(x) = \frac{1}{\sqrt{1-[f(x)]^2}}$

$\frac{d}{dx} \cos^{-1} f(x) = -\frac{1}{\sqrt{1-[f(x)]^2}}$

$\frac{d}{dx} \tan^{-1} f(x) = \frac{1}{1+x^2}$

Example 10

Differentiate the following with respect to x:

(a) $\sin^{-1}(\sqrt{x})$ (b) $\cos^{-1}\left(\frac{x}{3}\right)$ (c) $\tan^{-1}(xe^x)$ (d) $x \sin^{-1}(x^2)$

(e) $\operatorname{cosec}^{-1} x$

Solution:

(a) $\frac{d}{dx}\left(\sin^{-1}\sqrt{x}\right) = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \frac{d}{dx}\sqrt{x}$ $= \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(1-x)}}$	Remember to differentiate '$\sqrt{x}$' w.r.t. x.
(b) $\frac{d}{dx}\cos^{-1}\left(\frac{x}{3}\right) = -\frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}} \times \left(\frac{1}{3}\right)$ $= -\frac{3}{\sqrt{3^2-x^2}} \times \frac{1}{3} = -\frac{1}{\sqrt{9-x^2}}$	Remember to differentiate '$\frac{x}{3}$' w.r.t. x.
(c) $\frac{d}{dx}\tan^{-1}(xe^x) = \frac{1}{1+(xe^x)^2} \times (e^x + xe^x) = \frac{(1+x)e^x}{1+x^2e^{2x}}$	Remember to differentiate 'xe^x' w.r.t. x.
(d) $\frac{d}{dx}\left(x\sin^{-1}(x^2)\right) = \sin^{-1}(x^2) + x\left(\frac{2x}{\sqrt{1-x^4}}\right)$ $= \sin^{-1}(x^2) + \frac{2x^2}{\sqrt{1-x^4}}$	
Let $y = \operatorname{cosec}^{-1} x \Rightarrow \operatorname{cosec} y = x$ $\Rightarrow \frac{1}{\sin y} = x \Rightarrow \sin y = \frac{1}{x} \Rightarrow y = \sin^{-1}\left(\frac{1}{x}\right)$ Thus, $\operatorname{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$	Note that : For all $x \in \mathbb{R}$, (a) $x^2 = \lvert x\rvert^2$ (b) $\sqrt{x^2} = \lvert x\rvert$. Reason for (b): When $x > 0, \sqrt{x^2} = x$ e.g. $\sqrt{2^2} = 2$. When $x < 0$, $\sqrt{x^2} = -x$.

$\frac{dy}{dx}=\frac{d}{dx}\sin^{-1}\left(\frac{1}{x}\right)=\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\frac{d}{dx}\left(\frac{1}{x}\right)=\sqrt{\frac{x^2}{x^2-1}}\left(\frac{-1}{x^2}\right)$ $=-\frac{\lvert x\rvert}{\sqrt{x^2-1}}\left(\frac{1}{\lvert x\rvert^2}\right)=-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	e.g. $\sqrt{(-3)^2}=-(-3)=3$. So when we are not sure whether x is positive or negative, we must state: $\sqrt{x^2}=\lvert x\rvert$ as $\sqrt{x^2}>0$.

Self-Review 4

$\cot^{-1}x = y$
$\cot y = x$

$\frac{1}{x} = \tan y$
$\tan^{-1}\frac{1}{x} = y$

① example 10
② implicit differentiation

Find the derivative of $\cot^{-1}x$.

$\frac{1}{\tan y} = x$

$\frac{dy}{dx}$ ~~tan~~ $\cot x = \frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)$ $\left[-\frac{1}{1+x^2}\right]$

$= (-1)\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(\frac{1}{x^2}\right) = \frac{-1}{x^2+1}$

$\frac{\cos^{-1}x}{\sin^{-1}x} = y$

$\frac{dy}{dx} = \frac{(\sin^{-1}x)\left(\frac{-1}{\sqrt{1-x^2}}\right)-(\cos^{-1}x)\left(\frac{1}{\sqrt{1-x^2}}\right)}{(\sin^{-1}x)^2} = \frac{\left(\frac{1}{\sqrt{1-x^2}}\right)(\sin^{-1}x-\cos^{-1}x)}{(\sin^{-1}x)^2}$

$\cot y = x$
$-\operatorname{cosec}^2 y\,\frac{dy}{dx} = 1$

7.9 Differentiation of Parametric equations

Recall from Chapter 3, Section 3.4, when the relationships between x and y are complicated, it may be easier to express x and y each in terms of a third variable (say t), called a ***parameter***. In this case the equation of the curve may be expressed via a pair of ***parametric equations***: $x = f(t)$ and $y = g(t)$ where t is a variable and f and g are functions of t.

For example, the expression $x = 2(y+3)^2 - 4$ can be replaced by the pair of parametric equations: $x = 2t^2 - 4$ and $y = t - 3$, where t is a parameter.

Since $x = 2t^2 - 4 \Rightarrow \frac{dx}{dt} = 4t$, and, $y = t - 3 \Rightarrow \frac{dy}{dt} = 1$, by the chain rule, we have $\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$, then we can find $\frac{dy}{dx}$ as follows:

$$\boxed{\text{If } \frac{dx}{dt} \neq 0, \text{ then } \frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.}$$

In this case, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{4t}$.

Is this easier?

Note: (a) $x = 2(y+3)^2 - 4 \Rightarrow 1 = 2\times 2(y+3)\frac{dy}{dx} - 0 \Rightarrow \frac{dy}{dx} = \frac{1}{4(y+3)} = \frac{1}{4t}$.

(b) The curve has a vertical tangent when $\frac{dx}{dt} = 0$, that is $t = 0$.

(c) $\dfrac{d^2y}{dx^2} \neq \dfrac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$. In fact, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \times \dfrac{dt}{dx} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

Example 11

Given that $x = (t^2+4)^{\frac{1}{2}}$ and $y = \dfrac{\ln t}{t}$, where t (>0) is a parameter, find $\dfrac{dy}{dx}$ in terms of t.

Solution:

$x = (t^2+4)^{\frac{1}{2}} \Rightarrow \dfrac{dx}{dt} = \dfrac{1}{2}(t^2+4)^{-\frac{1}{2}}(2t) = t(t^2+4)^{-\frac{1}{2}}$. Also, $y = \dfrac{\ln t}{t} \Rightarrow \dfrac{dy}{dt} = \dfrac{t\left(\frac{1}{t}\right) - \ln t}{t^2} = \dfrac{1-\ln t}{t^2}$. Hence, $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\frac{1-\ln t}{t^2}}{t(t^2+4)^{-\frac{1}{2}}} = \dfrac{1-\ln t}{t^3(t^2+4)^{-\frac{1}{2}}} = \dfrac{(1-\ln t)\sqrt{t^2+4}}{t^3}$.	Are you able to find $\dfrac{dy}{dx}$ directly?

Example 12

Given that $x = e^{3t}\cos 3t$, $y = e^{3t}\sin 3t$ where t is a parameter, show that $\dfrac{dy}{dx} = \tan(3t + \dfrac{\pi}{4})$.

Solution:

$x = e^{3t}\cos 3t$ $\Rightarrow \dfrac{dx}{dt} = e^{3t}(-3\sin 3t) + 3e^{3t}\cos 3t = 3e^{3t}(\cos 3t - \sin 3t)$ Also, $y = e^{3t}\sin 3t$ $\Rightarrow \dfrac{dy}{dt} = e^{3t}(3\cos 3t) + 3e^{3t}\sin 3t = 3e^{3t}(\cos 3t + \sin 3t)$ $\therefore \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{3e^{3t}(\cos 3t + \sin 3t)}{3e^{3t}(\cos 3t - \sin 3t)} = \dfrac{\sin 3t + \cos 3t}{\cos 3t - \sin 3t}$ ……(1) Since $\sin 3t + \cos 3t = \sqrt{2}\sin(3t + \dfrac{\pi}{4})$, and $\cos 3t - \sin 3t = \sqrt{2}\cos(3t + \dfrac{\pi}{4})$.	Recall that : $a\sin\alpha + b\cos\alpha = R\sin(\alpha+\theta)$ where $R = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\dfrac{b}{a}$, and also

Then $\dfrac{dy}{dx}=\dfrac{\sqrt{2}\sin(3t+\frac{\pi}{4})}{\sqrt{2}\cos(3t+\frac{\pi}{4})}=\tan(3t+\dfrac{\pi}{4})$. **Alternatively,** From (1), $\dfrac{dy}{dx}=\dfrac{\cos 3t+\sin 3t}{\cos 3t-\sin 3t}$ $=\dfrac{\frac{\cos 3t+\sin 3t}{\cos 3t}}{\frac{\cos 3t-\sin 3t}{\cos 3t}}$ $=\dfrac{\frac{\cos 3t}{\cos 3t}+\frac{\sin 3t}{\cos 3t}}{\frac{\cos 3t}{\cos 3t}-\frac{\sin 3t}{\cos 3t}}$ $=\dfrac{1+\tan 3t}{1-\tan 3t}$ $=\dfrac{\tan\frac{\pi}{4}+\tan 3t}{1-\tan\frac{\pi}{4}\tan 3t}=\tan(3t+\dfrac{\pi}{4})$	$a\cos\alpha-b\sin\alpha=R\cos(\alpha+\theta)$ where $R=\sqrt{a^2+b^2}$ and $\theta=\tan^{-1}\dfrac{b}{a}$ Here : $R=\sqrt{1^2+1^2}=\sqrt{2}$ and $\theta=\tan^{-1}1=\dfrac{\pi}{4}$ $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

Self-Review 5

$5(\cos 2t)^3$

//

Given that $x=5\cos^3 2t$ and $y=5\sin^3 2t$, where t is a parameter, find $\dfrac{dy}{dx}$ in terms of t.

[$-\tan 2t$]

$\frac{dx}{dt}=3(5)\cos^2 2t(-\sin 2t)(2)$

$=-30\cos^2(2t)\sin(2t)$

$\frac{dy}{dt}=(3)(5)\sin^2 2t(\cos 2t)(2)$

$=30\sin^2(2t)\cos(2t)$

$\frac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{30\sin^2(2t)\cos(2t)}{30\sin(2t)(\cos 2t)}$

$=-\dfrac{\sin(2t)}{\cos(2t)}$

$=-\tan 2t$

Chapter 8: Applications of Differentiation

At the end of this chapter, students should be able to:

(a) interpret the graphs of $\mathrm{f}'(x)>0$, $\mathrm{f}'(x)=0$, $\mathrm{f}'(x)<0$, $\mathrm{f}''(x)>0$ and $\mathrm{f}''(x)<0$;

(b) relate the graph of $y=\mathrm{f}'(x)$ to the graph of $y=\mathrm{f}(x)$;

(c) determine the nature of stationary points (local maximum, minimum and points of inflexion) analytically using first derivative test or second derivative test;

(d) find the local maximum and minimum points using GC;

(e) find equations of tangents and normal to curves defined implicitly or parametrically;

(f) apply and solve local maxima and minima problems;

(g) apply and solve problems with connected rates of change.

8 The Gradient Function

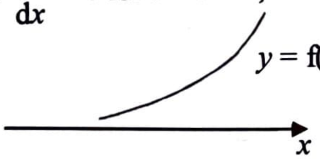
Let $y=\mathrm{f}(x)$, then $y=\mathrm{f}'(x)$ where $\mathrm{f}'(x)=\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is called the **gradient function** of $y=\mathrm{f}(x)$. $y=\mathrm{f}'(x)$ is the function whose value at x is the slope or gradient of the tangent line to the graph of $y=\mathrm{f}(x)$ at x.

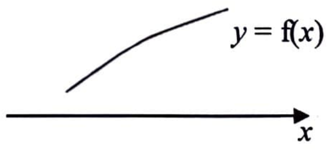
8.1 Interpretation of $\mathrm{f}'(x)>0$ and $\mathrm{f}'(x)<0$

Algebraically, a function f is said to be **strictly increasing** on a domain D if for all $x_1, x_2 \in D$, $x_1<x_2 \Rightarrow \mathrm{f}(x_1)<\mathrm{f}(x_2)$. Similarly, a function f is said to be **strictly decreasing** on a domain D if for all $x_1, x_2 \in D$, $x_1<x_2 \Rightarrow \mathrm{f}(x_1)>\mathrm{f}(x_2)$.

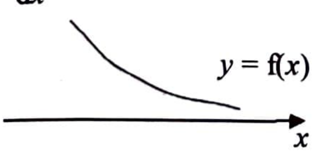
Consider a function given by $y = \mathrm{f}(x)$.

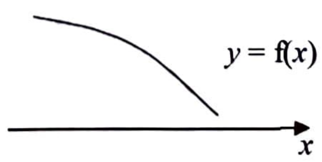
If $\dfrac{\mathrm{d}y}{\mathrm{d}x}>0$ for $a<x<b$, then $y=\mathrm{f}(x)$ is **strictly increasing** for $a<x<b$.





If $\dfrac{\mathrm{d}y}{\mathrm{d}x}<0$ for $a<x<b$, then $y=\mathrm{f}(x)$ is **strictly decreasing** for $a<x<b$.





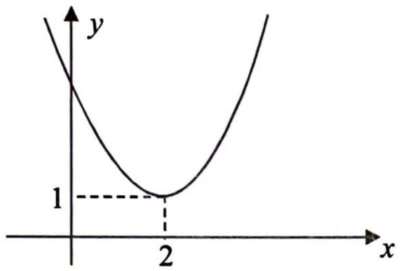
Note: This is true even if $y = f'(x)$ is not defined. For example $y = x^{\frac{1}{3}}$ is an increasing function for all real values of x but the derivative is not defined on $x = 0$.

Example 1

Using a sketch, find the range of values of x for which the graph of $y = x^2 - 4x + 5$ is

(i) strictly increasing, (ii) strictly decreasing.

Solution:



Using the sketch

(i) The graph is strictly increasing for $\boxed{}$

(ii) The graph is strictly decreasing for $\boxed{}$

Q: What is the value of the gradient at $x = 2$?

A:

Example 2

Given that $f : x \mapsto \ln x,\ x \in \mathbb{R},\ x > 0$, prove that f is a strictly increasing function. Hence show that $0 < x_1 < x_2 \Rightarrow \ln x_1 < \ln x_2$.

Solution:

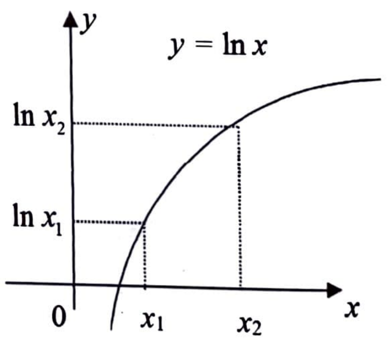
Let $y = \ln x$, then $\dfrac{dy}{dx} = \dfrac{1}{x}$.

As $x > 0$, $\dfrac{dy}{dx} > 0$.

Hence f is a strictly increasing function.

$0 < x_1 < x_2 \Rightarrow \ f(x_1) < f(x_2)$

$\Rightarrow \ \ln x_1 < \ln x_2$

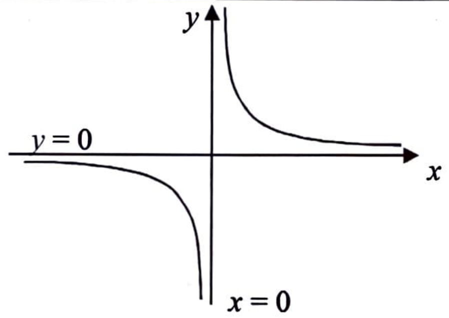


The graph verifies the results for $0 < x_1 < x_2 \Rightarrow \ln x_1 < \ln x_2$

Example 3

Sketch the graph of $y = \dfrac{1}{x}$ for $x \in \mathbb{R},\ x \neq 0$. Explain why $f(x) = \dfrac{1}{x}$ is not strictly decreasing. Given that f is strictly decreasing for $x < a$ or $x > a$, $a \in \mathbb{R}$, state the value of a.

Solution:



Consider $x_1 = -1$, $x_2 = 1$, then $f(x_1) = -1$ and $f(x_2) = 1$. Since $x_1 < x_2$ but $f(x_1) < f(x_2)$, f is not strictly decreasing.

From the graph, f is strictly decreasing for $x < 0$ or $x > 0$. $\therefore a = 0$.

Question: If f is strictly increasing (or decreasing) on a domain *D*, prove that f is a 1–1 function on *D*. Illustrate your result with an example.
[Recall that a function f is 1-1 on *D* if for all $x_1, x_2 \in D$, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$]

8.2 Stationary Points

A point (*a*, f(*a*)) on the graph of $y = f(x)$ is called a **stationary point** if $\frac{dy}{dx} = 0$ at that point. In other words, $f'(a) = 0$. The value of f(*a*) is called the **stationary value** at this stationary point (*a*, f(*a*)).

8.2.1 Nature of Stationary Points

(a) Turning Points

Turning points are either **maximum** points or **minimum** points.

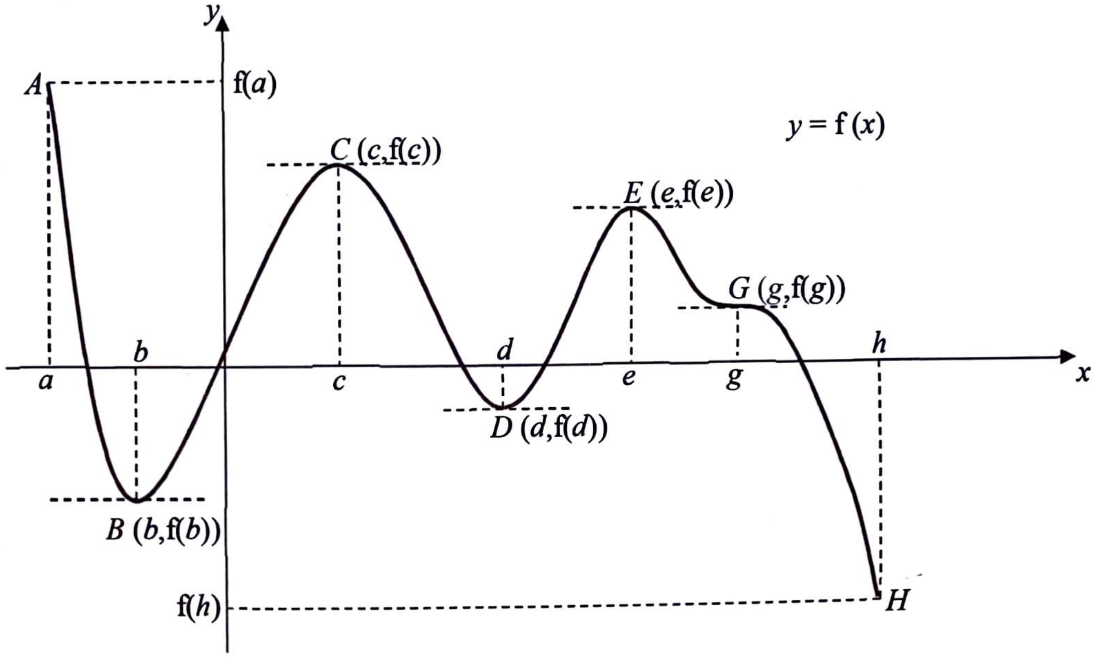
A **maximum** point on a curve is a stationary point at which the *y*-coordinate is greater than the *y*-coordinates of any other points on the curve in the neighbourhood of the point.

A **minimum** point on a curve is a stationary point at which the *y*-coordinate is smaller than the *y*-coordinates of any other points on the curve in the neighbourhood of the point.

(b) Stationary Points of Inflexion

A **stationary point of inflexion** is a point with zero gradient and the curvature of the curve changes in the neighbourhood of the stationary point.

Graphical representations



(a) Stationary points: B, C, D, E and G.

(b) Turning points: B, C, D and E.

(c) Minimum points: B and D. Maximum points: C and E.

f(b) and f(d) are called the minimum values of $y = \mathrm{f}(x)$; f(c) and f(e) are the maximum values of $y = \mathrm{f}(x)$.

(d) G is called a stationary point of inflexion of $y = \mathrm{f}(x)$.

(e) The greatest value of $y = \mathrm{f}(x)$ occurs at

(f) The least value of $y = \mathrm{f}(x)$ occurs at

(g) $y = \mathrm{f}(x)$ is strictly increasing when

(h) $y = \mathrm{f}(x)$ is strictly decreasing when

Example 4 (Using GC to sketch $y = \mathrm{f}'(x)$)

Given that $\mathrm{f}(x) = x^3 + 3x^2 - x - 3$, sketch the graph $y = \mathrm{f}(x)$. On the same axes, sketch the graph of $y = \mathrm{f}'(x)$.

State the set of values of x for which the graph is

(a) strictly increasing,

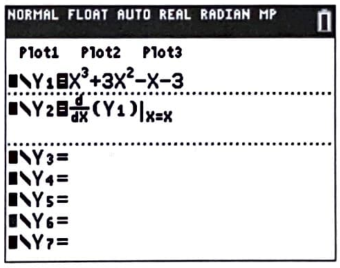
(b) strictly decreasing,

(c) stationary.

Solution:

Using a graphic calculator to sketch the graphs of $y = x^3 + 3x^2 - x - 3$ and $y = f'(x)$.

Step 1: Press [stat plot f1 / y=] and enter the expression **Y_1=X³+3X²–X–3**

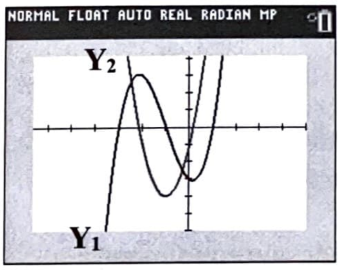


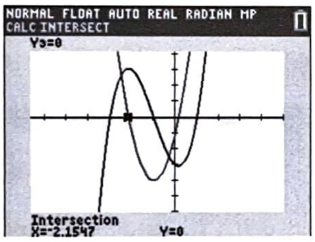
Step 2: At **Y_2**, press [a-lock / alpha] [tbl set f2 / window].

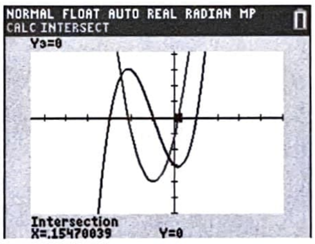
Select **3:nDeriv(** and press [entry solve / enter].

Enter the expression **Y_2=** $\frac{d}{dx}(Y_1)\big|_{X=X}$

Step 3: Press [table f5 / graph]. Press [tbl set f2 / window] to change the window settings to your preference.







(a) Graph is strictly increasing when $\frac{dy}{dx} > 0$ That is, $Y_2 > 0$.

Therefore, the set of values required is $\{x \in \mathbb{R} : x < -2.15 \text{ or } x > 0.155\}$.

(b) Graph is strictly decreasing when $\frac{dy}{dx} < 0$ That is, $Y_2 < 0$. Therefore, the set of values required is $\{x \in \mathbb{R} : -2.15 < x < 0.155\}$.

(c) Stationary points occur at $Y_2 = 0$ or $\frac{dy}{dx} = 0$, i.e. $x = -2.15$ and $x = 0.155$

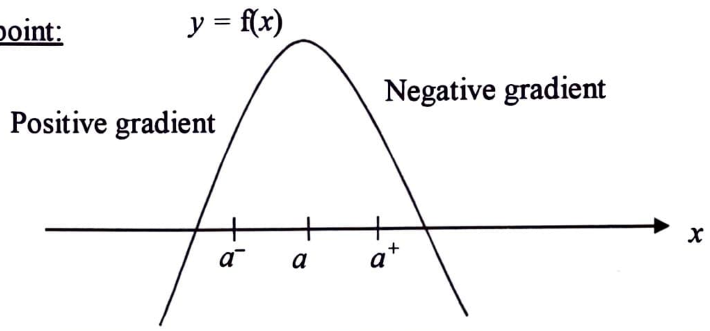
8.2.2 Investigating the Nature of Stationary Points

There are various ways to determine whether a stationary point is a maximum, a minimum or a stationary point of inflexion. Below are two methods.

Method 1: The First Derivative Test

This method involves constructing a table and finding the sign of $f'(x)$ in the neighbourhood of the stationary point. Let $f(x)$ be a function and $(a, f(a))$ a stationary point of $y = f(x)$. Let a^- and a^+ denote respectively arbitrary numbers which are slightly less than and larger than a respectively.

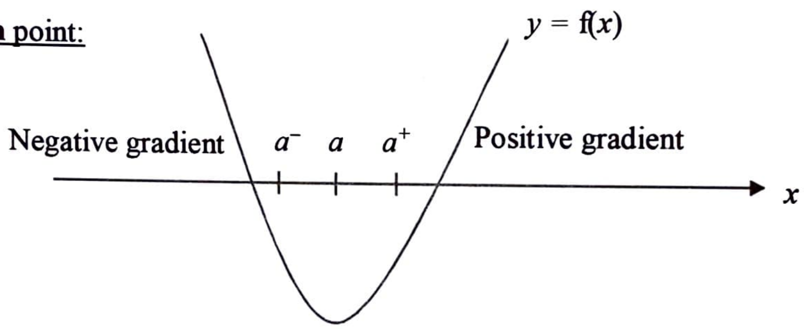
<u>Test for **maximum** point:</u>



x	a^-	a	a^+
$f'(x)$	> 0	0	< 0
slope	/	—	\

From the table, we conclude that the graph of $y = f(x)$ has a maximum point at $(a, f(a))$.

<u>Test for **minimum** point:</u>

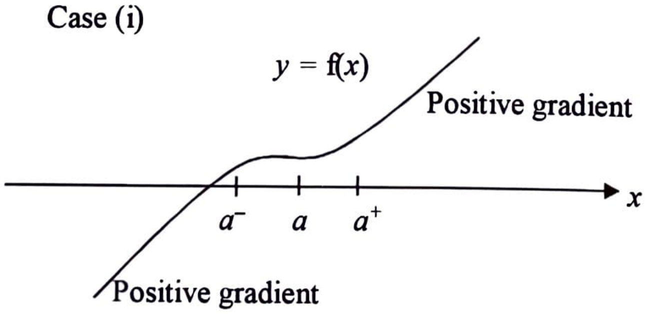


x	a^-	a	a^+
$f'(x)$	< 0	0	> 0
slope	\	—	/

From the table, we conclude that the graph of $y = f'(x)$ has a minimum point at $(a\ ,\ f(a))$.

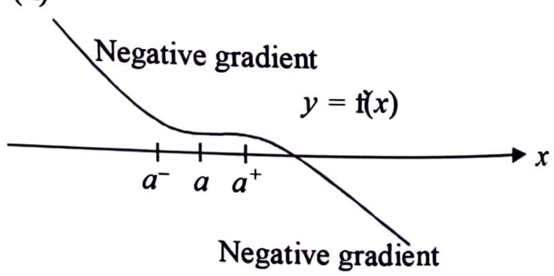
<u>Test for stationary point of inflexion:</u>

Case (i)



x	a^-	a	a^+
$f'(x)$	> 0	0	> 0
slope	/	—	/

Case (ii)



x	a^-	a	a^+
$f'(x)$	< 0	0	< 0
slope	\	—	\

From the two tables, we conclude that the graph of $y = f(x)$ has a stationary point of inflexion at $(a, f(a))$.

Note:

(a) In case (i), $f'(x)$ decreases from a positive value to 0 as x increases from a^- to a, and then, $f'(x)$ increases from 0 as x increases from a to a^+. $f'(x)$ has the minimum value 0 when $x = a$.

(b) In case (ii), $f'(x)$ increases from a negative value to 0 as x increases from a^- to a, and then, $f'(x)$ decreases from 0 as x increases from a to a^+. $f'(x)$ has the maximum value 0 when $x = a$.

Method 2: The Second Derivative Test (Given $f'(a) = 0$)

Test for **maximum** point: The graph $y = f(x)$ has a maximum point at $(a, f(a))$ if $f''(a) < 0$

Test for **minimum** point: The graph $y = f(x)$ has a minimum point at $(a, f(a))$ if $f''(a) > 0$

Note: When $f''(a) = 0$, we have no conclusion as to whether the point is maximum or minimum so we have to perform the 1st derivative test.

Example 5(a) (MC J91/I/16a)

Find the exact coordinates of the turning points on the curve $y = x^4 e^{-x}$ and determine the nature of each turning point.

Solution:

Do not use GC if the question requires the **exact** coordinates.

$y = x^4 e^{-x} \Rightarrow \dfrac{dy}{dx} = (4x^3)e^{-x} + x^4(-e^{-x}) = x^3 e^{-x}(4 - x)$

At turning points, $\dfrac{dy}{dx} = 0$

$\Rightarrow x^3 e^{-x}(4 - x) = 0 \Rightarrow x = 0 \text{ or } x = 4$

When $x = 0 \Rightarrow y = 0$.

When $x = 4 \Rightarrow y = 256e^{-4}$.

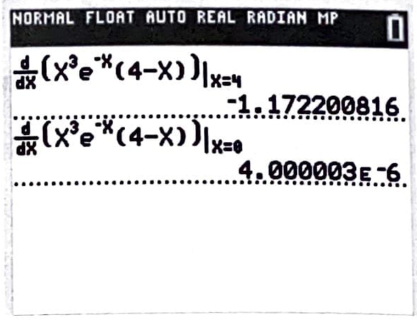
Using first derivative test:

x			
$\dfrac{dy}{dx}$			
slope			

$\therefore$(0, 0) is a $\boxed{\qquad\qquad}$

x			
$\dfrac{dy}{dx}$			
slope			

$\therefore$(4, $256e^{-4}$) is a $\boxed{\qquad\qquad}$

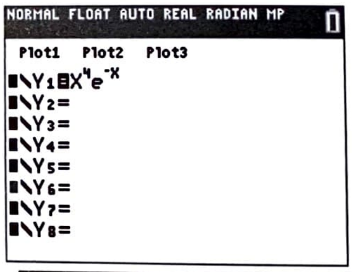
Alternatively, using second derivative test: $\frac{d^2y}{dx^2} = 12x^2e^{-x} - 4x^3e^{-x} - 4x^3e^{-x} + x^4e^{-x}$ $= x^2e^{-x}(12 - 8x + x^2) = x^2e^{-x}(x-2)(x-6)$ In this case, only the first derivative test can be used.	You may use GC for this part instead of the algebraic approach: 

Example 5(b) (MC J91/I/16a) (Independent Reading)

Find the coordinates of the turning points on the curve $y = x^4e^{-x}$ and state the nature of each turning point.

Solution: Graphic Calculator Approach

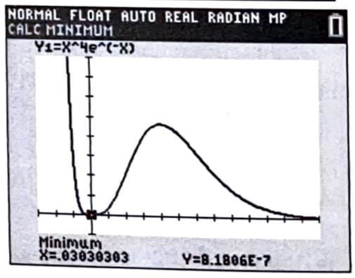
Step1: Press [stat plot f1 / y=] to display the Function Display Screen. Then key in the equation.



Step 2: Press [table f5 / graph] to display the graph of the equation.

Step 3: Press [2nd] [calc f4 / trace] to activate the Calculate Display Screen.

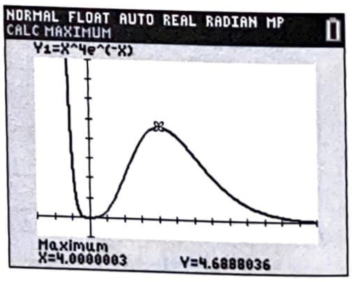
Step 4: Press [L3 θ / 3] for finding minimum turning point or [L4 T / 4] for finding maximum turning point.



Step 5: For minimum turning point, scroll to the left of the minimum point, press [entry solve / enter], then scroll to the right of the minimum point and press [entry solve / enter], then [entry solve / enter].

Minimum turning point is at (0, 0).

Step 6: Repeat the process, maximum point is at (4, 4.69).



Example 6 (MC N93/I/16a)

Find by differentiation the x-coordinate of the stationary point of the curve $y = x^2 - k^2\ln\left(\frac{x}{a}\right)$, where k and a are positive constants, and determine the nature of the stationary point.

Solution:

$y = x^2 - k^2 \ln(\frac{x}{a}) = x^2 - k^2(\ln x - \ln a)$ $\Rightarrow \frac{dy}{dx} = 2x - \frac{k^2}{x}$ $= \frac{2x^2 - k^2}{x}$ $= \boxed{\frac{(\sqrt{2}x - k)(\sqrt{2}x + k)}{x}}$	What if we did not use $\ln(\frac{x}{a}) = \ln x - \ln a$?
When $\frac{dy}{dx} = 0$, we have $\boxed{\frac{(\sqrt{2}x - k)(\sqrt{2}x + k)}{x} = 0}$ Thus, $\boxed{x = \pm\frac{k}{\sqrt{2}}}$ As $x > 0$, therefore, $\boxed{x = \frac{k}{\sqrt{x}}}$	**Q:** Why $x > 0$? **A:** $\ln(\frac{x}{a})$ is well defined only if $\frac{x}{a} > 0$. As $a > 0$, then $x > 0$.
Method 1 (First Derivative Test): Observe that $\sqrt{2}x + k > 0$ for positive values of x and k. When $x = \left(\frac{k}{\sqrt{2}}\right)^-$, $x < \frac{k}{\sqrt{2}} \Rightarrow \sqrt{2}x < k \Rightarrow \sqrt{2}x - k < 0$, so $\frac{dy}{dx} = \boxed{\frac{(\sqrt{2}x - k)(\sqrt{2}x + k)}{x} < 0}$ When $x = \left(\frac{k}{\sqrt{2}}\right)^+$, $x > \frac{k}{\sqrt{2}} \Rightarrow \sqrt{2}x > k \Rightarrow \sqrt{2}x - k > 0$, so $\frac{dy}{dx} = \boxed{\frac{(\sqrt{2}x - k)(\sqrt{2}x + k)}{x} > 0}$	
(table below)	Note that in this example, method 2 is easier.

x	$\left(\frac{k}{\sqrt{2}}\right)^-$	$\left(\frac{k}{\sqrt{2}}\right)$	$\left(\frac{k}{\sqrt{2}}\right)^+$
$\frac{dy}{dx}$	<0	0	>0
slope	\	—	/

Hence, when $x = \frac{k}{\sqrt{2}}$, it is a minimum point.

Method 2 (Second Derivative Test):

From (1), we have $\frac{d^2y}{dx^2} = 2 + \frac{k^2}{x^2}$

At $x = \frac{k}{\sqrt{2}}$, $\frac{d^2y}{dx^2} = \boxed{}$

Hence, minimum point occurs at $x = \frac{k}{\sqrt{2}}$.

Self-Review 1

Find the exact coordinates of the stationary point on the curve $y=\dfrac{\ln x}{x}$, $x>0$ and determine its nature.

Sketch the curve. $\left[\left(e,\dfrac{1}{e}\right)\text{, maximum point}\right]$

8.2.3 Stationary Points of curves defined by Parametric Equations

Given $x = f(t)$ and $y = g(t)$ are the parametric equations of a curve where t is the parameter, then we have,

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ if } \frac{dx}{dt}\neq 0.$$

Example 7

Find the stationary points on the curve $x=t^2$, $y=t+\dfrac{1}{t}$ where $t\in\mathbb{R}$, $t\neq 0$.

Solution:

$\dfrac{dx}{dt}=2t$, $\dfrac{dy}{dt}=1-\dfrac{1}{t^2}=\dfrac{t^2-1}{t^2}$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{t^2-1}{t^2}}{2t}=\frac{t^2-1}{2t^3}$$

At stationary points, $\dfrac{dy}{dx}=0$, i.e. $\dfrac{t^2-1}{2t^3}=0 \Rightarrow t^2-1=0 \Rightarrow t=1$ or -1

When $t=-1$, we have $x=(-1)^2=1$, $y=-1+\dfrac{1}{-1}=-2$.

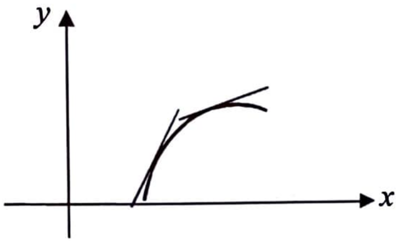
When $t=1$, we have $x=1^2=1$, $y=1+\dfrac{1}{1}=2$.

So $(1,-2)$ and $(1,2)$ are the stationary points.

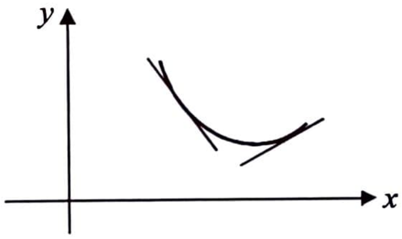
8.3 Graphical Interpretation of $f''(x)>0$ and $f''(x)<0$

The second derivative, $\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ or $f''(x)$ measures the rate of change of $f'(x)$, as x changes.

That is, it measures the rate of change of the gradient of $y = f(x)$ with respect to x.



As x increases, the gradient of the tangent on the curve decreases ie $f''(x) < 0$



As x increases, the gradient of the tangent on the curve increases ie $f''(x) > 0$

8.4 Concavity

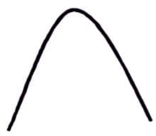
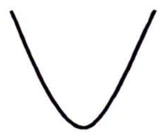
The graph of $y = f(x)$ is **concave downward** on the interval (a, b) if $f'(x)$ is **decreasing** on (a, b). $\cap$

Geometrically, the graph lies below the tangent lines in (a, b).

The graph of $y = f(x)$ is **concave upward** on the interval (a, b) if $f'(x)$ is **increasing** on (a, b). $\cup$

Geometrically, the graph lies above its tangent lines in (a, b).

Using the second derivative test, the graph is **concave downward** if $f''(x) < 0$ and **concave upward** if $f''(x) > 0$.

$f'(a) = 0$	
$(a, f(a))$ is a maximum point	$(a, f(a))$ is a minimum point
	
Notice that $f'(x)$ decreases as x increases from a^- to a^+.	Notice that $f'(x)$ increases as x increases from a^- to a^+.
$f''(a) < 0$	$f''(a) > 0$

Example 8 (Independent Reading)

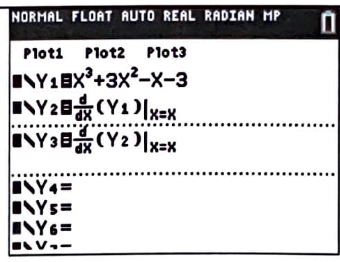
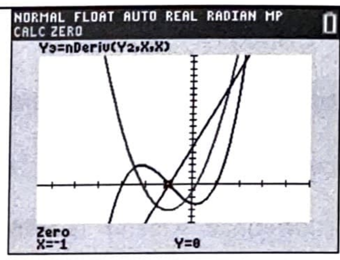
Refer to the graphs of $y = f(x)$ and $y = f'(x)$, where $f(x) = x^3 + 3x^2 - x - 3$, in Example 4.

State the set of values of x for which the graph is

(d) concave upwards,

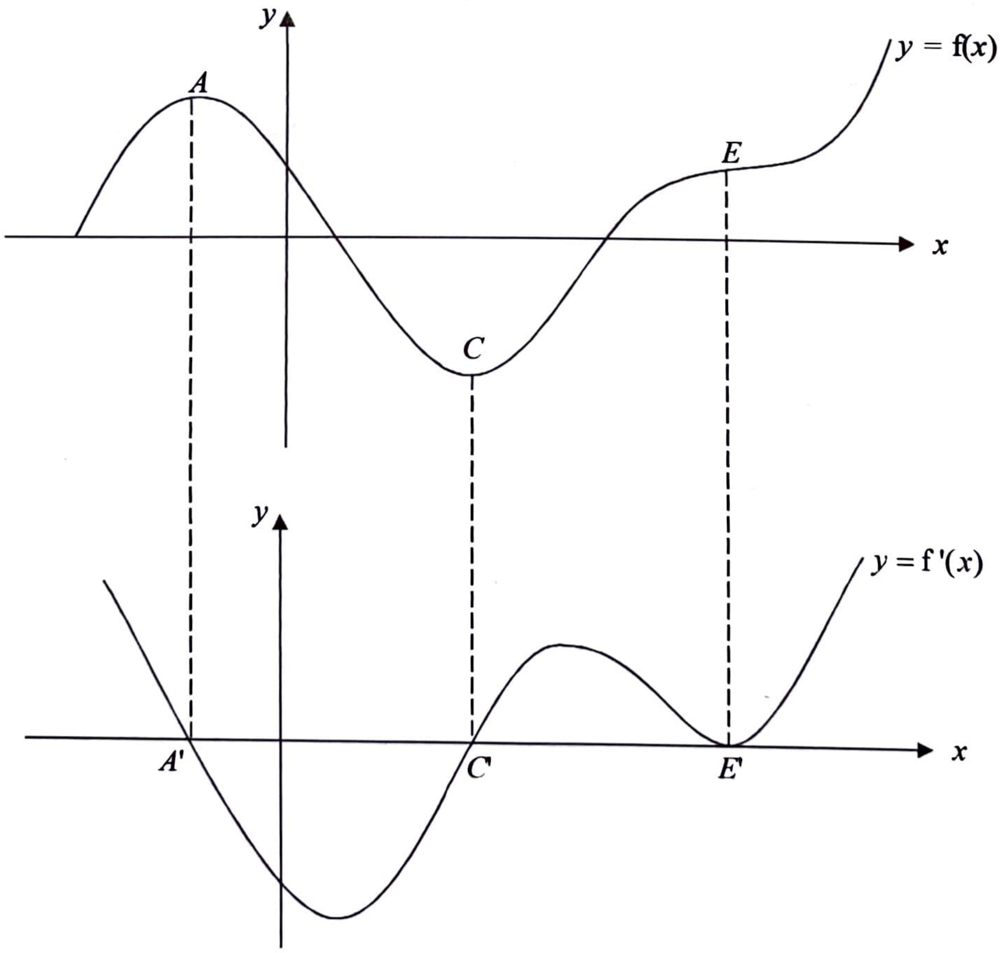
(e) concave downwards.

Solution:

(d) Graph is concave upwards when $\frac{d^2y}{dx^2} > 0$, i.e. the values of x where $\frac{dy}{dx}$ or $\mathbf{Y_2}$ is strictly increasing. Use the graphic calculator to determine the minimum point of the $\frac{dy}{dx}$ or $\mathbf{Y_2}$ graph or plot the $\frac{d^2y}{dx^2}$ or $\mathbf{Y_3}$ graph and find its roots. Therefore, the set of values required is $\{x \in \mathbb{R} : x > -1\}$.	
(e) Graph is concave downwards when $\frac{d^2y}{dx^2} < 0$ or $\mathbf{Y_3} < 0$, i.e. the values of x where $\frac{dy}{dx}$ is strictly decreasing. Therefore, the set of values required is $\{x \in \mathbb{R} : x < -1\}$.	

8.5 Relationship between the graphs of $y = f(x)$ and $y = f'(x)$

The diagram below shows the graph of $y = f(x)$ with stationary points at points A, C and E, and the corresponding graph of the gradient function $\frac{dy}{dx}$ or $y = f'(x)$.



Procedure to obtain the graph of $y = f'(x)$ from the graph of $y = f(x)$:

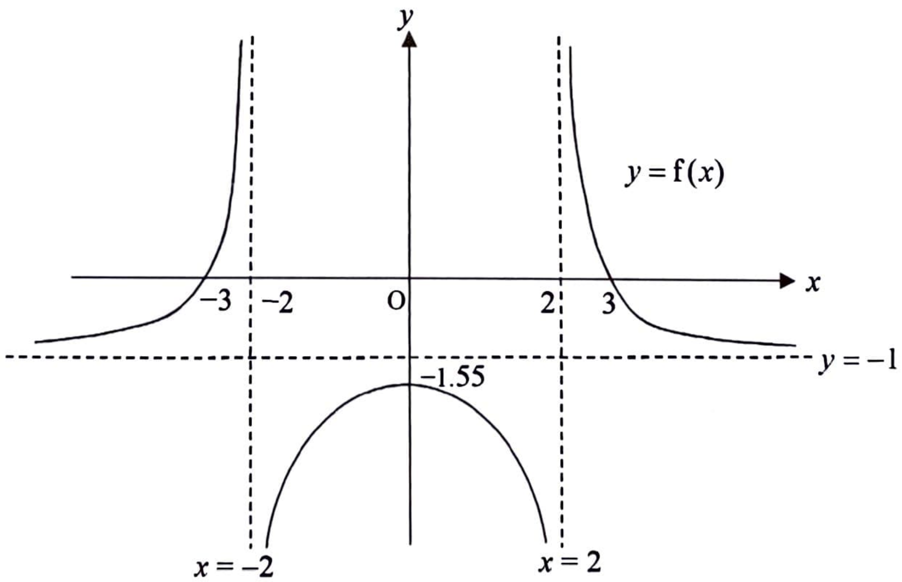
Step	$y = f(x)$ ⇨	$y = f'(x)$	Remarks
1	Stationary points at $x = h$	x-intercept at $x = h$	
	Maximum point at (h, k)		Graph of $y = f'(x)$ changes from positive to negative
	Minimum point at (h, k)		Graph of $y = f'(x)$ changes from negative to positive
2	Vertical asymptote at $x = a$	Vertical asymptote at $x = a$	
3	Horizontal asymptote (if any) $y = a$	Horizontal asymptote $y = 0$	Graph of $y = f'(x)$ has horizontal asymptote at $y = k$, where k = gradient of asymptote in graph of $y = f(x)$
	Oblique asymptote (if any) $y = ax + b$	Horizontal asymptote $y = a$	

In particular, the gradient function for the graph of $y = f'(x)$ is the second derivative $\frac{d^2y}{dx^2}$ $(= f''(x))$.

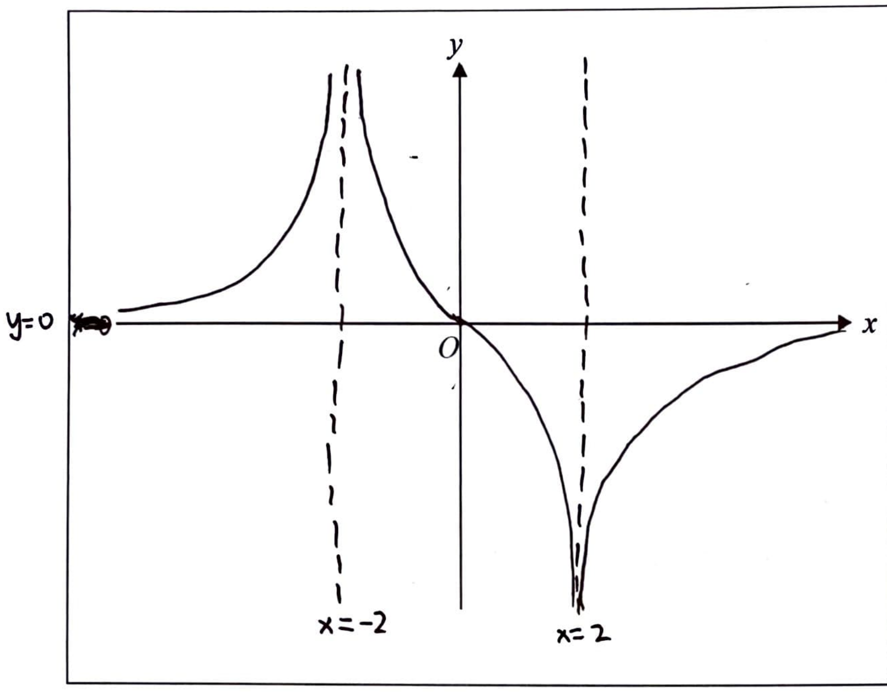
In other words, $\frac{d^2y}{dx^2}$ $(= f''(x))$ is the gradient of the graph of $y = f'(x)$.

Example 9

The diagram below shows the graph of $y = f(x)$. Sketch the graph of $y = f'(x)$ on a separate diagram.



Solution:



Note:

In sketching the graph of $y = f'(x)$, bear in mind that the y-coordinates of the graph of $y = f'(x)$ are the gradients of the tangents to the graph of $y = f(x)$ at the points corresponding to x.

In other words, the point $(x, f(x))$ in the graph of $y = f(x)$ will become the point $(x, f'(x))$ in the graph of $y = f'(x)$

Example 10

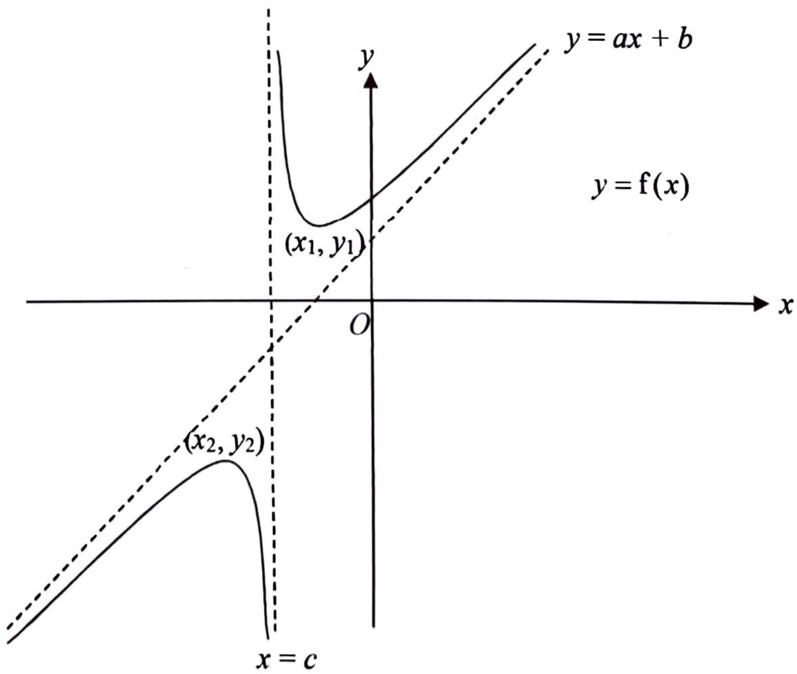
The diagram shows the graph of $y = f(x)$. State the values of x for which

(i) $f'(x) < 0$,

(ii) the graph of $y = f(x)$ is stationary,

(iii) the graph of $y = f(x)$ is strictly increasing.

Sketch the graph of $y = f'(x)$ on a separate diagram.



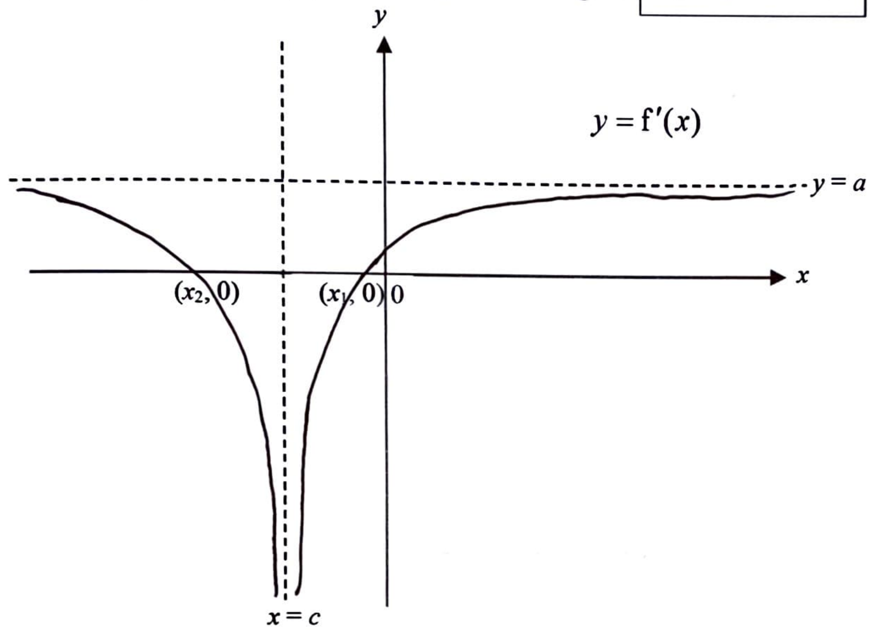
Solution:

From the graph of $y=\mathrm{f}(x)$, we deduce that

(i) $\mathrm{f}'(x)<0$ for $\boxed{x_2<x<c \text{ or } c<x<x_1}$

(ii) the graph of $y=\mathrm{f}(x)$ is stationary at $x=$ $\boxed{x_1 \text{ or } x_2}$

(iii) the graph of $y=\mathrm{f}(x)$ is strictly increasing for $\boxed{x<x_2 \text{ or } x>x_1}$



Note:

For the graph of $y=\mathrm{f}'(x)$, observe that as $x\to\infty$, $y=\mathrm{f}'(x)\to a$.

This is because as $x\to\infty$, $\mathrm{f}(x)\to ax+b$ so that $\mathrm{f}'(x)\to a$.

The same applies to the situation when $x\to-\infty$.

Self-Review 2

Without the use of GC, sketch the graph of $y=\dfrac{x}{1+x}$ $(x\in\mathbb{R}, x\neq-1)$ and hence deduce the graph of $y=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x}{1+x}\right)$. What is the set of values of x for which the graph of $y=\dfrac{x}{1+x}$ $(x\in\mathbb{R}, x\neq-1)$ is concave downward?

$[(-1,\infty)]$

8.6 Tangents and Normals for $y = \mathrm{f}(x)$

The gradient of the tangent to a curve with equation $y = \mathrm{f}(x)$ at any arbitrary point (x_1, y_1) is $\dfrac{\mathrm{d}y}{\mathrm{d}x}\Big|_{x=x_1}$ or $\mathrm{f}'(x_1)$.

Let $m = \mathrm{f}'(x_1)$. Then the equation of the tangent to the curve $y = \mathrm{f}(x)$ at the point (x_1, y_1) is

$$\boxed{\frac{y-y_1}{x-x_1}=m \quad \text{or} \quad y-y_1=m(x-x_1).}$$

It also follows that the gradient of the normal to the curve at (x_1, y_1) is $-\dfrac{1}{m}$ or $-\dfrac{1}{\mathrm{f}'(x_1)}$.

Thus the equation of the normal to the curve $y = \mathrm{f}(x)$ at the point (x_1, y_1) will be

$$\boxed{\frac{y-y_1}{x-x_1}=-\frac{1}{m} \quad \text{or} \quad y-y_1=-\frac{1}{m}(x-x_1)}$$

Example 11 (2015 Promo/RI(JC)/1/4) (Independent Reading)

A curve C has equation $xy^2 = 1$ for $x \neq 0$, $y \neq 0$.

(i) Find $\frac{dy}{dx}$ in terms of x and y.

(ii) Show that the equation of the normal to C at the point $P\left(\frac{1}{p^2}, p\right)$, where p is a non-zero constant, is $p^5 y = 2p^2 x - 2 + p^6$.

(iii) Hence, write down the equation of the normal at the point $Q\ (1, -1)$ and use an algebraic method to determine if this normal meets C again.

Solution:

(i) Differentiate $xy^2 = 1$ implicitly with respect to x, $y^2 + (x)\left(2y\frac{dy}{dx}\right) = 0$ Rearranging, $\frac{dy}{dx} = -\frac{y}{2x}$	We do not need to make y the subject of the equation before differentiating.
(ii) At $P\left(\frac{1}{p^2}, p\right)$, $\frac{dy}{dx} = -\frac{p}{2\left(\frac{1}{p^2}\right)} = -\frac{p^3}{2}$ Gradient of normal at P is $\frac{2}{p^3}$ Equation of normal at P is $y - p = \frac{2}{p^3}\left(x - \frac{1}{p^2}\right)$ $y = \frac{2}{p^3}x - \frac{2}{p^5} + p$ $p^5 y = 2p^2 x - 2 + p^6$	
(iii) At $Q\ (1, -1)$, $p = -1$ Equation of normal at Q is $-y = 2x - 2 + 1$ $y = -2x + 1$ When the normal meets the curve, $x(-2x+1)^2 = 1$ $4x^3 - 4x^2 + x - 1 = 0$ $(x-1)(4x^2+1) = 0$ The only real solution is $x = 1$. When $x = 1$, $y = -1$.	The normal meets the curve at the point(s) of intersection.

Since the normal meets the curve only at the point $Q\,(1, -1)$, the normal does not meet the curve again.	

Example 12

The equation of a curve is $3x^2 + 8xy + y^2 = -13$. Find the equations of the two tangents which are parallel to the y-axis.

Solution:

$3x^2 + 8xy + y^2 = -13$

Differentiate implicitly with respect to x,

$6x + 8x\frac{dy}{dx} + 8y \cancel{=} 2y\frac{dy}{dx} = 0 \quad \frac{dy}{dx} = \frac{3x+4y}{4x+y}$

Since the tangent is parallel to the y-axis, $\frac{dy}{dx}$ is undefined.

Thus, denominator of $\frac{dy}{dx} = \frac{3x+4y}{4x+y}$ is zero,

$4x + y = 0 \Rightarrow y = -4x$

Substitute $y = -4x$ into $3x^2 + 8xy + y^2 = -13$ yields

$3x^2 + 8x(-4x) + (-4x)^2 = -13$

$\Rightarrow 3x^2 - 3x^2 + 16x^2 = -13$

$x^2 = 1$

$x = 1$ or $x = -1$

(DO NOT USE $y - y_1 = m(x_1 - x_1)$ as m is undefined

So, the equation of the tangents are $x = 1$ and $x = -1$.

Alternatively, you can think of the tangent being parallel to the y-axis as the normal being parallel to the x-axis. As the gradient of the normal is

$-\frac{1}{\frac{dy}{dx}} = \frac{4x+y}{3x+4y}$,

then $\frac{4x+y}{3x+4y} = 0 \Rightarrow 4x + y = 0$

as before.

Example 13 (MC J93 / I / 16 modified) (Independent Reading)

The equation of a closed curve is $(x+y)^2 + 2(x-y)^2 = 24$. Show that $\frac{dy}{dx} = \frac{3x-y}{x-3y}$.

Find the coordinates of all the points on the curve at which the tangent is parallel to either the x-axis or the y-axis. State the equations of the tangents for each of the points.

Solution:

$(x+y)^2 + 2(x-y)^2 = 24$

Differentiate implicitly with respect to x,

Extension:
Find the coordinates of the point(s) on the curve at which the tangent makes an angle of 45° with the positive x-axis.

$$2(x+y)(1+\frac{dy}{dx})+2(2)(x-y)(1-\frac{dy}{dx})=0$$

$$2(x+y)+2(x+y)\frac{dy}{dx}+4(x-y)-4(x-y)\frac{dy}{dx}=0$$

$$(2x+2y-4x+4y)\frac{dy}{dx}=-2(x+y)-4(x-y)$$

$$(-2x+6y)\frac{dy}{dx}=-6x+2y \quad \Rightarrow \quad \frac{dy}{dx}=\frac{3x-y}{x-3y}$$

For tangent parallel to the x-axis, $\frac{dy}{dx}=\frac{3x-y}{x-3y}=0$.

Thus, $3x-y=0 \Rightarrow y=3x$.

Substitute $y=3x$ into $(x+y)^2+2(x-y)^2=24$ gives

$(4x)^2+2(-2x)^2=24$

$x=1$ or -1

$y=3x=3$ or -3

The points at which the tangents are parallel to the x-axis are $(1,3)$ and $(-1,-3)$.

The equations of the tangents parallel to x-axis are $y=3$ at the point $(1,3)$ and $y=-3$ at the point $(-1,-3)$.

When the tangent is parallel to y-axis, we must have $x-3y=0$ (see Example 12)

Substitute $x=3y$ into $(x+y)^2+2(x-y)^2=24$ we have,

$(4y)^2+2(2y)^2=24$

$y=1$ or -1

$x=3$ or -3

The points at which the tangents are parallel to the y-axis are $(3,1)$ and $(-3,-1)$.

Hence, the equations of the tangents parallel to y-axis are $x=3$ at the point $(3, 1)$ and $x=-3$ at the point $(-3, -1)$.

Ans:

When $\frac{dy}{dx}=1$,

$3x-y=x-3y$

$x=-y$

$(-y+y)^2+2(-y-y)^2=24$

$y=\pm\sqrt{3}$

The coordinates are

$(\sqrt{3},-\sqrt{3})$ or $(-\sqrt{3},\sqrt{3})$

8.6.1 Tangents and Normals for Parametric Equations

Example 14

A curve is defined parametrically by $x=at^2$, $y=2at$, where a is a positive constant.

Find $\frac{dy}{dx}$ in terms of t.

(i) State the equation of the tangent at $t=0$.

(ii) Show that the equation of the normal at the point P (ap^2 , $2ap$) where $p \neq 0$ is $y + xp - ap^3 - 2ap = 0$.

(iii) The normal to the curve at the point P meets the curve again at the point Q. Find the value of the parameter at Q in terms of p.

(iv) The normal cuts the y-axis at the point R. Find, in term of a and p, the area of the triangle ROQ, where O is the origin.

Solution:

$x = at^2 \Rightarrow \dfrac{dx}{dt} = \boxed{2at}$ and $y = 2at \Rightarrow \dfrac{dy}{dt} = \boxed{2a}$

Thus $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = \dfrac{2a}{2at} = \dfrac{1}{t}$.

(i) At $t = 0$, $x = y = 0$ and gradient of tangent is undefined.

Hence tangent is parallel to y-axis.

Equation of tangent is $\boxed{x = 0}$

(ii) At the point P $(ap^2, 2ap)$, $t = p$, then $\dfrac{dy}{dx} = \dfrac{1}{p}$

$\Rightarrow$ Gradient of the normal at the point P with parameter p is $-p$

The equation of the normal at the point P is

$$y - 2ap = -px + ap^3$$
$$y + px - ap^3 - 2ap = 0$$

(iii) C_N : $y + px - ap^3 - 2ap = 0$

C: $x = at^2$, $y = 2at$

Equating: $2at + p(at^2) - ap^3 - 2ap = 0$

$\Rightarrow \quad pt^2 + 2t - p(2 + p^2) = 0$

$\Rightarrow \quad (t - p)\left[pt + (2 + p^2)\right] = 0$

$$t = p \quad \text{or} \quad t = -\frac{2+p^2}{p}$$

As the parameter at the point P is p, thus the parameter at Q should be $-\dfrac{2 + p^2}{p}$.

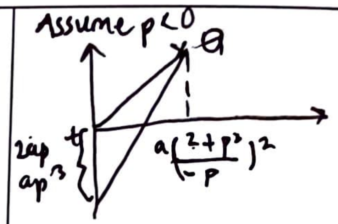
"The normal meets the curve **again** at point Q" implies there are 2 **points of intersections** between the normal and the curve.

(iv) At R: $x = 0$ for $y + px - 2ap - ap^3 = 0$

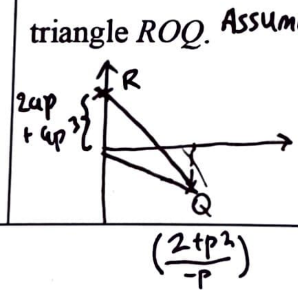
$y = 2ap + ap^3$.

Thus, the coordinates of the point R is $(0,\ 2ap+ap^3)$.

At Q: coordinates is $\left(a\left(-\dfrac{2+p^2}{p}\right)^2,\ 2a\left(-\dfrac{2+p^2}{p}\right)\right)$.



Obtain a sketch of the triangle ROQ. Assume $p>0$



Area of the triangle $ROQ = \dfrac{1}{2}\times$ base$\times$height

$$= \frac{1}{2}\times(2ap+ap^3)\times a\left(-\frac{2+p^2}{p}\right)^2 = \frac{1}{2}ap(2+p^2)\,a\frac{(2+p^2)^2}{p^2}$$

$$= \frac{a^2(2+p^2)^3}{2p}$$

Example 15 (2012 Prelim/RVHS/1/8 modified)

The parametric equations of a curve are $x = a\sin t,\quad y = b\cos t$ where a and b are positive constants.

(i) Find $\dfrac{dy}{dx}$ in terms of t. Express your answer as a single trigonometric function.

(ii) Show that the equation of the tangent to the curve at the point $P(a\sin\alpha,\ b\cos\alpha)$ is of the form $\dfrac{x\sin\alpha}{a}+\dfrac{y\cos\alpha}{b}=1$.

(iii) The tangent at the point $Q\ (a\sin\theta,\ b\cos\theta)$ intersects the x-axis and the y-axis at the points A and B respectively. Find a cartesian equation of the locus of the mid-point of AB as θ varies.
(Note: A locus is a set of points which satisfies a certain condition. Please refer to Annex for a short note about locus.)

Solution:

(i) $x = a\sin t \Rightarrow \dfrac{dx}{dt} = a\cos t$, and $y = b\cos t \Rightarrow \dfrac{dy}{dt} = -b\sin t$

$$\frac{dy}{dx} = \frac{dy}{dt}\div\frac{dx}{dt} = -\frac{b\sin t}{a\cos t} = -\frac{b}{a}\tan t$$

(ii) At point $P\ (a\sin\alpha,\ b\cos\alpha)$, $t=\alpha$, then $\dfrac{dy}{dx} = -\dfrac{b}{a}\tan\alpha$

Equation of tangent at the point P:

$$y - b\cos\alpha = -\frac{b}{a}\tan\alpha\ (x - a\sin\alpha)$$

$$\Rightarrow y - b\cos\alpha = -\frac{b\sin\alpha}{a\cos\alpha}(x - a\sin\alpha)$$

$$\Rightarrow \quad ay\cos\alpha - ab\cos^2\alpha = -bx\sin\alpha + ab\sin^2\alpha$$

$$\Rightarrow \quad bx\sin\alpha + ay\cos\alpha = ab(\sin^2\alpha + \cos^2\alpha)$$

$\Rightarrow \quad bx\sin\alpha + ay\cos\alpha = ab$

Dividing both sides by ab, $\quad \dfrac{bx\sin\alpha}{ab} + \dfrac{ay\cos\alpha}{ab} = \dfrac{ab}{ab}$

Hence, $\dfrac{x\sin\alpha}{a} + \dfrac{y\cos\alpha}{b} = 1$ (shown)

Recall: Midpoint of (x_1, y_1) and (x_2, y_2) is given by $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$.

(iii) At A: $y = 0 \Rightarrow x = \dfrac{a}{\sin\theta}$. Coordinates of A : $\left(\dfrac{a}{\sin\theta}, 0\right)$

At B: $x = 0 \Rightarrow y = \dfrac{b}{\cos\theta}$. Coordinates of B : $\left(0, \dfrac{b}{\cos\theta}\right)$

Midpoint of AB is $\boxed{\left(\dfrac{a}{2\sin\theta}, \dfrac{b}{2\cos\theta}\right)}$

As θ varies, the mid-point of AB changes such that the coordinates is

$\boxed{x = \dfrac{a}{2\sin\theta} \text{ and } y = \dfrac{b}{2\cos\theta}}$

Then $\sin\theta = \dfrac{a}{2x}$ and $\cos\theta = \dfrac{b}{2y}$

Since $\sin^2\theta + \cos^2\theta = 1$, then $\left(\dfrac{a}{2x}\right)^2 + \left(\dfrac{b}{2y}\right)^2 = 1$

$\boxed{a^2y^2 + b^2x^2 = 4x^2y^2 \text{ where } x \neq 0, y \neq 0}$

Self-Review 3

(a) The tangent at the point P on the curve with equation $y = x^2 + 1$ passes through the origin. Find the possible coordinates of P. [(1,2), (–1,2)]

(b) A curve C is defined by the parametric equations $x = t^3$ and $y = 2t^2$. Show that the equation of the normal to C at the point corresponding to the parameter t is $4y + 3tx - 8t^2 - 3t^4 = 0$.

(i) Find the equation of the tangent to C at the point where C crosses the x-axis. [$x = 0$]

(ii) Prove that the normal to C at the point where $t = -1$ does not meet C again.

8.7 Applications of Differentiation to Rates of Change

If $y = f(x)$, then $\dfrac{dy}{dx}$ is the rate of change of y with respect to x. Very often we are interested to find out how one variable, say y, changes with respect to a variable t when another variable, say x, changes with respect to t.

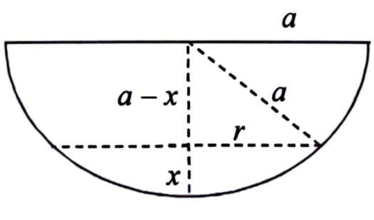
To solve problems of this nature, we either use implicit differentiation or apply the chain rule which says $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$.

Example 16

A hemispherical bowl of radius a cm is initially filled with water. The water runs through a small hole at the bottom of the bowl. It is given that, at the instant when the depth of the water is $\frac{1}{2}a$, the water level is decreasing at a rate of $\frac{a}{27}$ cm/s.

Find, in terms of a, the rate at which the top surface area of the water is decreasing at the instant when the depth of the water is $\frac{1}{2}a$.

Solution:

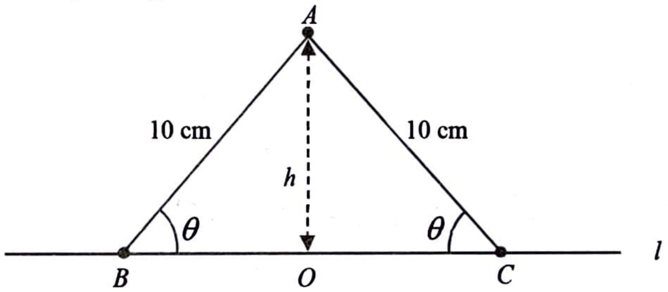
	Question says " ...	**Interpretation in context**
STEP 1: Define variables	*Find the rate at which the* ***top surface area*** *of the water is decreasing at the instant when the* ***depth of the water*** *is* $\frac{1}{2}a$.	Let A cm^2 represents the area of the top surface area of the water and x cm represents the depth of water. You need to find $\frac{dA}{dt}$ when $x = \frac{1}{2}a$.
STEP 2: Translate given info to notations	*It is given that, when the depth of the water is* $\frac{1}{2}a$, *the water level is decreasing at a rate of* $\frac{a}{27}$ *cm/s.*	When $x = \frac{1}{2}a$, $\frac{dx}{dt} = -\frac{a}{27}$ cm/s.
STEP 3: Construct an equation that links the identified variables		You need to obtain an equation that connects A and x. Obviously, when the depth of water is x cm, $A = \pi r^2$ where r is the radius of the top surface area of the water. So, you need an equation that connects r and x ! Sketch diagrams to help in visualizing. From the diagram, note that: $r^2 = a^2 - (a-x)^2 = 2ax - x^2$ Also, $A = \pi r^2$ So $A = \pi(2ax - x^2)$.
STEP 4: Perform implicit differentiation with respect to t or apply chain rule	*At the instant when* $x = \frac{1}{2}a$, *find, in terms of a, the rate at which the top surface area of the water is decreasing.*	Differentiate $A = \pi(2ax - x^2)$ implicitly w.r.t. t, $\frac{dA}{dt} = \pi(2a\frac{dx}{dt} - 2x\frac{dx}{dt}) = \pi(2a - 2x)\frac{dx}{dt}$. (alternatively, $\frac{dA}{dt} = \frac{dA}{dx}\frac{dx}{dt} = \pi(2a - 2x)\frac{dx}{dt}$)
STEP 5: Solve for the required rate		When $x = \frac{1}{2}a$, $\frac{dx}{dt} = -\frac{a}{27}$ cm/s. i.e.: $\frac{dA}{dt} = \pi(2a - 2\cdot\frac{a}{2})(-\frac{a}{27}) = -\frac{\pi a^2}{27}$ cm^2/s

of change at the specific instant of time		
STEP 6: Write conclusion		When $x=\frac{1}{2}a$, the top surface area of the water is **decreasing** at the rate of $\frac{\pi a^2}{27}$ cm^2/s

Example 17 (NJC/ Prelim 2006 / I / 5 modified)

The diagram below shows an isosceles triangle ABC with fixed lengths AB and AC of 10 cm each and $\angle ACB = \angle ABC = \theta$ radians.

A is a variable point which is at a height h cm directly above the point O while B and C are variable points which move horizontally along the line l.



Given that A descends vertically towards the point O such that the area of triangle ABC is decreasing at a constant rate of 0.7 cm^{2}s^{-1}, determine, at the instant when A is 6 cm above O,

(i) the rate of change of θ,

(ii) the rate of descent of A,

(iii) the rate at which C is moving away from O.

Solution:

Let the area of the triangle ABC be P cm^2 and length OC be x cm. Considering the right-angled triangle AOC, we have $x=10\cos\theta$ ---- (1) $h=10\sin\theta$ ---- (2) $P=\frac{1}{2}\text{Base} \times \text{height} = \frac{1}{2}(2x)h = xh = 100\sin\theta\cos\theta$ $P=50\sin 2\theta$ ....(3) (i) To find the rate of change of θ, i.e. $\frac{d\theta}{dt}$. Differentiate (3) implicitly, w.r.t to t,	When $h=6$ cm, find (i) $\frac{d\theta}{dt}$, (ii) $\frac{dh}{dt}$ and (iii) $\frac{dx}{dt}$. You are given that $\frac{dP}{dt}=-0.7$, to find $\frac{d\theta}{dt}$, you need to find an equation connecting P and θ.

$\frac{dP}{dt} = 50(2\cos 2\theta)\frac{d\theta}{dt} = 100\cos 2\theta \frac{d\theta}{dt}$ As P is decreasing at constant rate 0.7 cm²s⁻¹, then $\frac{dP}{dt} = -0.7$ $\Rightarrow\ 100\cos 2\theta \frac{d\theta}{dt} = -0.7 \quad \cdots\cdots(4)$	Here we have used the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$ in establishing (3).
When $h=6$, (2) $\Rightarrow \sin\theta = \frac{h}{10} = \frac{6}{10} = \frac{3}{5}$. (4) $\Rightarrow\ 100(1-2\sin^2\theta)\frac{d\theta}{dt} = -0.7$ $\Rightarrow\ 100\left[1-2\left(\frac{3}{5}\right)^2\right]\frac{d\theta}{dt} = -0.7$ $\Rightarrow\ \frac{d\theta}{dt} = -0.025 \text{ rad s}^{-1}$. Hence rate of change of θ is -0.025 rad s⁻¹.	Notice how we have cleverly used the trigonometric identity $\cos 2\theta = 1-2\sin^2\theta$ here to facilitate the computation by simply substituting $\sin\theta = \frac{3}{5}$ thus removing the need to find angle θ.
(ii) Differentiating (2) with respect to t, $\frac{dh}{dt} = \boxed{10\cos\theta\left(\frac{d\theta}{dt}\right) = 10\left(\frac{4}{5}\right)(-0.025) = -0.2 \text{ cm s}^{-1}}$ So, when $h=6$, rate of descent of A is 0.2 cms⁻¹.	For part (ii), rate of descent of A, means $\frac{dh}{dt}$.
(iii) Differentiating (1) w.r.t. t, we have $\frac{dx}{dt} = \boxed{-10\sin\theta\frac{d\theta}{dt} = -10\left(\frac{3}{5}\right)(-0.025) = 0.15 \text{ cm s}^{-1}}$ So, when $h=6$, the rate at which C moves away from O is 0.15 cms⁻¹.	For part (iii), the rate at which C moves away from O means $\frac{dx}{dt}$.

Self-Review 4

The volume V of a sphere of radius r is $V = \frac{4}{3}\pi r^3$ and its surface area A is $A = 4\pi r^2$.

The radius r of a sphere is increasing at a rate such that when r = 8 m, its volume V is increasing at a rate 10 m³s⁻¹. The surface area of the sphere is denoted by A. Find

(i) the corresponding exact rate of increase of r at this instant, [$\frac{5}{128\pi}$ ms⁻¹]

(ii) the corresponding rate of increase of A at this instant. [2.5 m²s⁻¹]

8.7.1: General guide on how to solve rates of change problems

STEP 1:	Know what the question wants, for example, $\frac{dA}{dt}$.

STEP 2:	Write down what you are given, for example, x and $\frac{dx}{dt}$.
STEP 3:	Form an equation relating the variables, for example, A and x. If there are **more than 2** variables in the equation, for example, $A = xy$, **form another equation** to relate the two variables x and y, for example, $y = \frac{1}{2}x$. Substitute this equation into $A = xy$ to **reduce** the number of **variables** to **only 2**. The equation becomes $A = \frac{1}{2}x^2$. If the variables are not defined in the question, you have to define them.
STEP 4:	**Differentiate** the equation **implicitly** with respect to t, substituting the known values of x and $\frac{dx}{dt}$ to get the desired result.
STEP 5:	Write **conclusion** statement.

8.8 Derivative as a Rate of change

Suppose that a particle is moving along a straight line so that we know its position s, relative to a fixed point O on the line, as a function of time t: $s = f(t)$.

That is, the function $f(t)$ tells us its position from point O at any time t. This is the **displacement** of the particle at time t.

The rate at which a particle's displacement changes is called the **velocity** of the particle. Then, if a particle's displacement at time t is $s = f(t)$, then the particle's velocity **(instantaneous velocity)** at time t is the derivative of displacement with respect to time, that is, $v(t) = \frac{ds}{dt}$.

Similarly, the rate at which a particle's velocity changes is called the **acceleration** of the particle.

The acceleration measures how quickly the particle picks up or loses speed.

If a particle's position at time t is $s = f(t)$, then the particle's acceleration at time t is the derivative of velocity with respect to time , that is, $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

Scalar	Vector
Distance	Displacement
Speed	Velocity
-	Acceleration

Example 18

A weather balloon is released at time $t = 0$ and allowed to rise. Suppose its height above the ground at time t seconds is given by $H(t) = \frac{1}{3}t^3 - 5t^2 + 23t$ m.

(i) Find the velocity of the balloon t seconds after its release.

(ii) Find when the balloon is rising and when (if ever) it is falling.

(iii) Because of unforeseen problems, the balloon bursts after 11 seconds of flight. How fast is it rising at the moment it bursts?

Solution:

(i) The velocity, $V(t)$, of the balloon t seconds after its release $V(t) = H'(t) = t^2 - 10t + 23$ m s^{-1} (ii) The balloon is rising when $V(t) > 0$, thus $t^2 - 10t + 23 > 0 \Rightarrow (t-5)^2 - 2 > 0 \Rightarrow t < 5 - \sqrt{2}$ or $t > 5 + \sqrt{2}$ The balloon is falling when $V(t) < 0$, $t^2 - 10t + 23 < 0 \Rightarrow 5 - \sqrt{2} < t < 5 + \sqrt{2}$ Since $t \geq 0$, the balloon is rising from the start till the time is $(5-\sqrt{2})$ sec, then begin falling until the time is $(5+\sqrt{2})$ sec, and after this moment, it will be continue rising forever. (iii) When $t = 11$ sec, $V(11) = 11^2 - 10(11) + 23 = 34$ ms^{-1}. The balloon is rising at the ~~speed~~ velocity of 34 ms^{-1}	**Note:** The starting time is $t = 0$.

Example 19

A particle moves in a straight line. Its displacement s m from a fixed point on the line is given by $s = t^2 - 4t - 5$, at a time t after the start, where $t \geq 0$. Find

(i) where the particle starts and its initial velocity,

(ii) when and where it comes to instantaneous rest,

(iii) when it passes through the fixed point,

(iv) its acceleration,

(v) the total distance travelled by the particle when it passes the fixed point.

Solution

(i) $v = \dfrac{ds}{dt} = 2t - 4$ When $t = 0$, $s = -5$, $v = -4$ So the particle starts at -5 m from the fixed point with an initial velocity -4 ms^{-1}. (ii) When $v = 0$, $t = 2$, $s = -9$ The particle comes to rest 2 seconds after it started at a distance of 9 m from the fixed point. (iii) When $s = 0$, $t^2 - 4t - 5 = 0$	

$t = 5$ or $t = -1$ (NA) The particle passes through the fixed point 5 seconds after it started. (iv) $a = \dfrac{dv}{dt} = 2$ The acceleration of the particle is 2 m/s^2. (v) Total distance travelled $= (-5-(-9)) + 9 = 13$ m. The total distance travelled is 13 m.	Sketch the path of travel of the particle.

8.8.1 Application of the relationship between the graphs $y = f(x)$ and $y = f'(x)$

(i) Displacement-Time graph and Velocity-Time graph

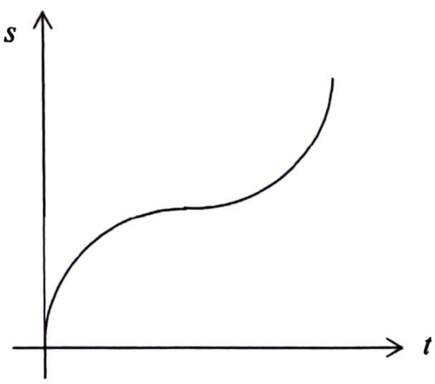
Suppose that a particle is moving in a straight line, and $s(t)$ is the distance of the particle from a fixed point O at time t, then the displacement-time graph is the graph of the function $y = s(t)$, and the velocity-time graph is the graph of the function $y = s'(t)$.

(ii) Velocity-Time graph and Acceleration-Time graph

Similarly, if $v(t)$ is the velocity of the particle at time t, then the velocity-time graph is the graph of the function $y = v(t)$, and the acceleration-time graph is the graph of the function $y = v'(t)$.

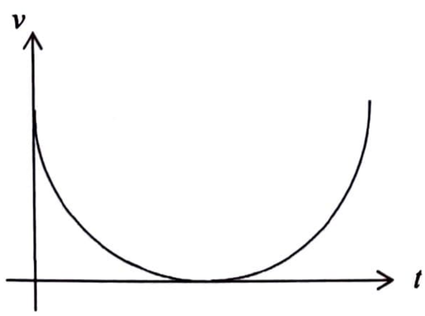
Example 20

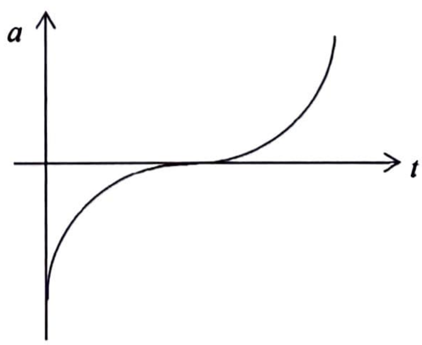
A particle moves along a straight line and O is a fixed point on that line. The displacement s m of the particle from O at time t seconds is given by:



Draw the velocity-time graph and the acceleration-time graph for the particle.

Solution



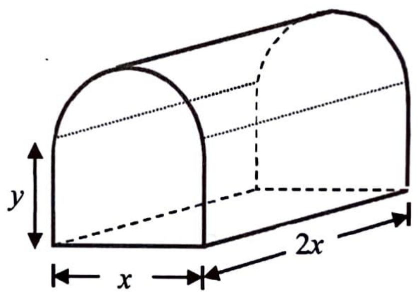


8.9 Applications of Differentiation to Optimisation Problems (Maxima or Minima Problems)

In many real-life problems, we are interested to optimize, that is, maximize or minimize certain quantities under certain constraints. For example, given some constraints, we may wish to maximize the volume of a geometrical solid or minimize the cost of building a fence round a garden. Problems of this nature are called maximum and minimum problems and differentiation is an extremely useful tool to solve this class of problems.

Example 21 (DHS Prelim 2009 / I / 11 modified)

A mould in the shape of a letter-box as shown in the diagram below, is to be made from a metal sheet. The lower portion of the cross-section of the mould is in the shape of a rectangle with a length of x cm and a height of y cm. The upper portion is in the shape of a semi-circle with diameter x cm. When completed, the mould has a length of $2x$ cm and the total area of the metal sheet used, excluding the base of the mould, is $800\pi\ \text{cm}^2$.



Considering the total area of the metal sheet used, show that $5\pi x^2 + 24xy = 3200\pi$.

Deduce that the volume, V cm^3 of the mould (neglecting thickness of the metal) is given by $V = \frac{1}{6}\pi x\left(1600 - x^2\right)$.

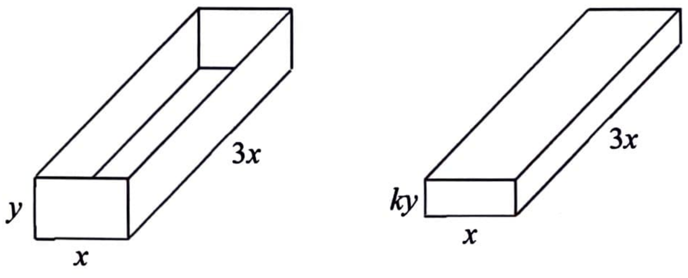
If x and y may vary, use differentiation to find the values of x and y for which V has its maximum value and give this maximum value in the form $k\pi$ where k is a number rounded off to the nearest integer.

Solution:

Area of the metal sheet $= 2\left[xy + (2x)y + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2\right] + \frac{\pi x}{2}(2x) = 800\pi$ (Given) $\Rightarrow 2\left[3xy + \frac{1}{8}\pi x^2\right] + \pi x^2 = 800\pi$ $\Rightarrow 5\pi x^2 + 24xy = 3200\pi$	Step 1: Obtain equation of constraint. Hint: Total area used have to be $800\pi\ \text{cm}^2$.
$\Rightarrow y = \dfrac{3200\pi - 5\pi x^2}{24x}$	Write y in terms of x.
$V =$ base area $\times$ height	

$V=\left[xy+\frac{1}{2}\pi\left(\frac{x}{2}\right)^2\right](2x)=2x^2y+\frac{1}{4}\pi x^3$	Step 2: Obtain equation connecting variable to be optimized, V, and x and y.
$V=2x^2\left(\frac{3200\pi-5\pi x^2}{24x}\right)+\frac{1}{4}\pi x^3=\frac{x}{12}\left(3200\pi-5\pi x^2\right)+\frac{1}{4}\pi x^3$ $=\frac{800}{3}\pi x-\frac{5\pi x^3}{12}+\frac{1}{4}\pi x^3=\frac{800}{3}\pi x-\frac{\pi x^3}{6}$ $V=\frac{1}{6}\pi x\left(1600-x^2\right)$ (Shown)	Step 3: Substitute $y=\frac{3200\pi-5\pi x^2}{24x}$ into equation for V to obtain an equation connecting V and the variable x only.
Since $V=\frac{1}{6}\pi x(1600-x^2)=\frac{1}{6}\pi(1600x-x^3)$ $\Rightarrow \frac{dV}{dx}=\frac{1}{6}\pi(1600-3x^2)$	Step 4: Differentiate V w.r.t x
For stationary V, $\frac{dV}{dx}=0$. Then $3x^2=1600$. $\Rightarrow x=23.094$, -23.094 (rejected, as $x>0$) $\Rightarrow y=\frac{3200\pi-5\pi x^2}{24x}=\frac{3200\pi-5\pi(23.094)^2}{24(23.094)}=3.0230$ So the values of x and y are 23.1 and 3.02cm respectively.	Step 5: Set $\frac{dV}{dx}=0$. Then, find the value of x.
$\frac{d^2V}{dx^2}=\frac{1}{6}\pi(-6x)=-\pi x<0 \Rightarrow V$ is maximum.	Step 6: Determine nature of stationary value of V.
Max $V=\frac{1}{6}\pi x\left(1600-x^2\right)=\frac{1}{6}\pi\left(23.094\right)\left(1600-23.094^2\right)$ $V\approx 4106\pi$ cm^3.	Step 7: Obtain optimum value.

Example 22 (H2 Math 2010/I /9)



Box Lid

A company requires a box made of cardboard of negligible thickness to hold 300 cm^3 of powder when full. The length of the box is $3x$ cm, the width is x cm and the height is y cm. The lid has depth ky cm, where $0 < k \le 1$ (see diagram).

(i) Use differentiation to find, in terms of k, the value of x which gives a minimum total external surface area of the box and the lid.

(ii) Find also the ratio of the height to the width, $\dfrac{y}{x}$, in this case, simplifying your answer.

(iii) Find the values between which $\dfrac{y}{x}$ must lie.

(iv) Find the value of k for which the box has square ends.

Solution:

(i) Find the value of x which gives a minimum total external surface area of the box and the lid. Volume of box = $3x^2y = 300 \;\Rightarrow\; y = \dfrac{100}{x^2}$. External surface area of box $= (3x)(x) + 2(3x)(y) + 2(x)(y)$ $= 3x^2 + 8xy$ External surface of lid $= (3x)(x) + 2(ky)(3x) + 2(ky)(x)$ $= 3x^2 + 8kxy$ Total external surface area of box and lid, denoted by A, is given by: $A = 3x^2 + 8xy + 3x^2 + 8kxy = 6x^2 + 8xy(k+1)$ Substitute $y = \dfrac{100}{x^2}$ into the above, we have	External surface of the box consists of the two long sides ($3x \times y$), the two ends ($x \times y$) and the base ($3x \times x$). Can you identify the external surface for the lid as well?

$$A = 6x^2 + 8x\left(\frac{100}{x^2}\right)(k+1) = 6x^2 + \frac{800(k+1)}{x}$$

When A is stationary, $\frac{dA}{dx} = 0$. Thus

As $\frac{d^2A}{dx^2} = 12 + \frac{1600(k+1)}{x^3} > 0$ since $x > 0$, and $k > 0$.

Hence A is a minimum when $x = \left[\frac{200(k+1)}{3}\right]^{\frac{1}{3}}$.

(ii) Find the ratio $\frac{y}{x}$ in terms of k.

$$y = \frac{100}{x^2} \Rightarrow \frac{y}{x} = \frac{100}{x^3} = 100 \div \frac{200(k+1)}{3} = \frac{3}{2(k+1)}$$

(iii) Find the values between which $\frac{y}{x}\left(=\frac{3}{2(k+1)}\right)$ must lie.

$$0 < k \le 1 \Rightarrow 1 < k+1 \le 2 \Rightarrow 2 < 2(k+1) \le 4$$

$$\Rightarrow \frac{1}{2} > \frac{1}{2(k+1)} \ge \frac{1}{4} \Rightarrow \frac{3}{2} > \frac{3}{2(k+1)} \ge \frac{3}{4}$$

Hence $\frac{3}{2} > \frac{y}{x} \ge \frac{3}{4}$.

(iv) Find the value of k when the box has square ends.

If the box has square ends, then Thus ,

Therefore

It is mandatory to do a first or second derivative test to ensure that A is minimum even if only one value of x is obtained for $\frac{dA}{dx} = 0$.

$x > 0$ since x represents the length of one side of the box.

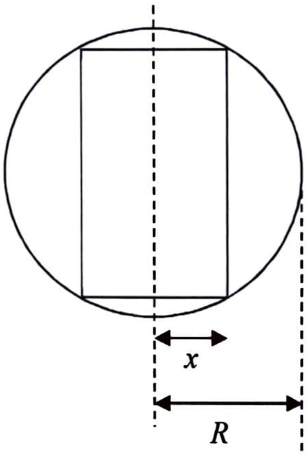
We have used the fact that: when $a, b > 0$ then $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$.

Q: Is it still true that: when $a, b < 0$, then $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$?

What about the case when $a < 0$, $b > 0$?

Example 23 (SRJC Prelim 2008 / I / 6)

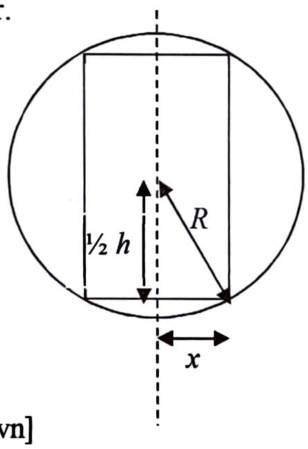
The diagram below shows the cross-section of a cylinder of radius x that is inscribed in a sphere of fixed internal radius R. Show that $A^2 = 16\pi^2 x^2 (R^2 - x^2)$, where A is the curved surface area of the cylinder. Prove that, as x varies, the maximum value of A is obtained when the height of the cylinder is equal to its diameter.



Solution:

Let h be the height of the cylinder.

By Pythagoras' theorem,



$$R^2 = x^2 + \left(\frac{1}{2}h\right)^2$$

$\Rightarrow\ h^2 = 4R^2 - 4x^2$

$A = 2\pi x h$

$\Rightarrow\ A^2 = 4\pi^2 x^2 h^2$

$A^2 = 4\pi^2 x^2 (4R^2 - 4x^2)$ [shown]

$\Rightarrow\ A^2 = 16\pi^2 R^2 x^2 - 16\pi^2 x^4$ ------ (1)

Differentiate (1) w.r.t x,

$$2A\frac{\mathrm{d}A}{\mathrm{d}x} = 32\pi^2 R^2 x - 64\pi^2 x^3$$

Since $A \neq 0$, $\dfrac{\mathrm{d}A}{\mathrm{d}x} = \dfrac{32\pi^2 R^2 x - 64\pi^2 x^3}{2A}$

$$= \frac{16\pi^2 x(R^2 - 2x^2)}{A} = \frac{16\pi^2 x(R - \sqrt{2}x)\left(R + \sqrt{2}x\right)}{A}$$

------ (2)

At stationary point, $\dfrac{\mathrm{d}A}{\mathrm{d}x} = 0$

$$16\pi^2 x(R - \sqrt{2}x)\left(R + \sqrt{2}x\right) = 0$$

It is very common that the diagram of the question always provide an equation connecting two variables.

Here, $R^2 = x^2 + \left(\dfrac{1}{2}h\right)^2$

connects

x and h.

Notice that it is not necessary to write A explicitly in terms of x (by taking square root) before differentiating as A^2 can be differentiated *implicitly* to give a neater form. This is a useful skill to employ in many situations.

$x = 0$ (reject) , $\dfrac{R}{\sqrt{2}}, -\dfrac{R}{\sqrt{2}}$ (reject)

Therefore $x = \dfrac{R}{\sqrt{2}}$

Observe that $R + \sqrt{2}x > 0$ since $R > 0$ and $x > 0$.

When $x = \left(\dfrac{R}{\sqrt{2}}\right)^-$, $x < \dfrac{R}{\sqrt{2}} \Rightarrow R - \sqrt{2}x > 0$, so

$\dfrac{dA}{dx} = \boxed{}$

When $x = \left(\dfrac{R}{\sqrt{2}}\right)^+$, $x > \dfrac{R}{\sqrt{2}} \Rightarrow R - \sqrt{2}x < 0$, so

$\dfrac{dA}{dx} = \boxed{}$

x	$\left(\dfrac{R}{\sqrt{2}}\right)^-$	$\dfrac{R}{\sqrt{2}}$	$\left(\dfrac{R}{\sqrt{2}}\right)^+$
$\dfrac{dA}{dx}$	> 0	0	< 0
Slope			

Hence, when $x = \dfrac{R}{\sqrt{2}}$, A is maximum.

$x = \dfrac{R}{\sqrt{2}} \Rightarrow 2x^2 = R^2$.

$\therefore\ h = \sqrt{4R^2 - 4x^2} = \sqrt{4(2x^2) - 4x^2}$

$= \sqrt{4x^2} = 2x =$ diameter of cylinder

Hence, the maximum value of A is obtained when the height h of the cylinder is equal to its diameter $2x$.

Note:
From (2) we have
$\dfrac{dA}{dx} = \dfrac{16\pi^2 x(R - \sqrt{2}x)(R + \sqrt{2}x)}{A}$

8.9.1: General steps involved in solving optimisation problems

Steps	Explanation and Illustration
1. Obtain the equation of constraint and write one variable in terms of the other.	This is usually an equation connecting two variables, say x and y. For example, the equation of constraint could be $x + y = 1$. Writing y in terms of x gives $y = 1 - x$. Note: If the variables are not defined in the question, you will need to define them first before proceeding.

2. Obtain an equation connecting the variable to be optimized, say A and the variables x and y. Write A in terms of x and y.	For example, this equation could look like $A = xy$. Notice that A is expressed in terms of two variables x and y.
3. Substitute the variable in terms of the other in step 1 into the equation in step 2.	This substitution reduces the number of variables by one. For example, substitute $y = 1 - x$ into $A = xy$ gives $A = x(1-x)$ so that the quantity A is in terms of only one variable, x.
4. Differentiate the variable to be optimized with respect to the single variable.	Differentiate A with respect to x in the equation $A = x(1-x)$ to obtain $\frac{dA}{dx} = 1 - 2x$.
5. Set the derivative to zero and solve the equation .	Set $\frac{dA}{dx} = 0 \Rightarrow 1 - 2x = 0 \Rightarrow x = \frac{1}{2}$.
6. Determine the nature of the stationary value of the optimized variable using either the first or second derivative test, whichever is easier. Note: This step is mandatory unless the question specifically says that this is unnecessary.	Construct a table for the first derivative test **or** find the sign of $\left.\frac{d^2A}{d^2x}\right\|_{x=\frac{1}{2}}$ for the second derivative test. For example, $\left.\frac{d^2A}{d^2x}\right\|_{x=\frac{1}{2}} = -2 < 0$. So A is a maximum when $x = \frac{1}{2}$. Note: In this example, the second derivative test is easier.
7. Obtain the optimum value of the variable to be optimized.	In this example, the maximum value of A is $\frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$.

Self-Review 5

(a) A piece of wire of length l units is cut into two pieces. One piece is bent to form an equilateral triangle while the other piece is bent to form a square. Prove that, when the combined area of the two figures attains a minimum, the length of each side of the equilateral triangle is approximately $0.188l$ units. (You need to verify that the value obtained is a minimum)

(b) Find the coordinates of the points on the curve $y = x^2 + 2x - 1$ that is closest to the point $(-1, 2)$.

[$(0.871, 1.50)$ and $(-2.871, 1.50)$]

Annex: A short note about locus

A **locus** is a path formed by a moving point which moves according to certain conditions.

Example 1

The locus of a point P moves so that it is always a same distance r from a fixed point C (h, k). What shape is it? Find its Cartesian equation.

Solution:

Obviously, it is a circle. The centre is the point C and the radius is r.

To find the Cartesian equation of the locus of a moving point, we always let the coordinates of the moving point P be (x, y), then we have

$$CP = r \quad \Rightarrow \quad \sqrt{(x-h)^2+(y-k)^2} = r$$

$$\Rightarrow \quad (x-h)^2+(y-k)^2 = r^2$$

Example 2

The locus of a point which moves so that it is an equal distance from two points, A (a, b) and B (c, d). What shape is it? Find its Cartesian equation.

Solution:

It is a straight line. In fact, it is the perpendicular bisector of the line joining A and B. To find the Cartesian equation of the locus, we let the coordinates of the moving point P be (x, y), then,

$$PA = PB \quad \Rightarrow \quad \sqrt{(x-a)^2+(y-b)^2} = \sqrt{(x-c)^2+(y-d)^2}$$

$$\Rightarrow \quad (x-a)^2+(y-b)^2 = (x-c)^2+(y-d)^2$$

$$\Rightarrow \quad [(x-a)-(x-c)][(x-a)+(x-c)] = [(y-d)-(y-b)][(y-d)+(y-b)]$$

$$\Rightarrow \quad (c-a)(2x-a-c) = (b-d)(2y-b-d)$$

$$\Rightarrow \quad 2y-b-d = \frac{c-a}{b-d}(2x-a-c) = \frac{2(c-a)x}{b-d} - \frac{(c-a)(a+c)}{b-d}$$

$$\Rightarrow \quad y = \frac{c-a}{b-d}x - \frac{a+c}{2}\left(\frac{c-a}{b-d}\right) + \frac{b+d}{2}$$

$$\Rightarrow \quad y = \frac{c-a}{b-d}x - \frac{c^2-a^2}{2(b-d)} + \frac{(b+d)(b-d)}{2(b-d)}$$

$$\Rightarrow \quad y = \frac{c-a}{b-d}x - \frac{c^2-a^2-b^2+d^2}{2(b-d)} \quad \cdots\cdots(1)$$

It is an equation for a straight line.

Now, we want to prove that this locus is the perpendicular bisector of the line AB.

Proof:

Gradient of AB is $\dfrac{d-b}{c-a}$.

Gradient of the locus $= -\dfrac{c-a}{d-b}$.

Since $\dfrac{d-b}{c-a} \times \left[-\dfrac{c-a}{d-b}\right] = -1$

The locus is perpendicular to the line joining A and B.

Hence the locus is a perpendicular bisector of the line joining A and B.

In general, to find the Cartesian equation of the locus of a moving point P that satisfies a certain rule or condition, we let the co-ordinates of the point P be (x, y), and form an expression connecting the variables x and y based on the given rule or condition.