Tampines Meridian Junior College
2024 H2 Mathematics (9758)
Chapter 6B
3D Vector Geometry (Lines & Planes)
Learning Package

Resources

- ☐ Core Concept Notes
- ☐ Discussion Questions
- ☐ Extra Practice Questions

SLS Resources

- ☐ Recordings on Core Concepts
- ☐ Quick Concept Checks
- ☐ (Optional) Supplementary Geogebra Applets for exploration and visualisation

Reflection or Summary Page

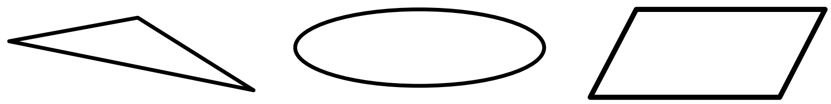
H2 Mathematics (9758)
Chapter 6B
3D Vector Geometry (Lines & Planes)
Core Concept Notes

Success Criteria:

Surface Learning	**Deep Learning**	**Transfer Learning**
☐ Interpret and find equations of planes in the form $ax+by+cz=d$ or $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0$ or $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$. ☐ Convert equations of planes from one form to another. ☐ Determine whether a line lies in a plane, is parallel to a plane, or intersects a plane. ☐ Find the angle between a line and a plane, and between two planes.	☐ Find the point of intersection of a line and a plane when it exists. ☐ Find foot of the perpendicular from a point to a plane ☐ Find perpendicular distance between a point and a plane, between a line and a plane, and between two planes. ☐ Find the line of intersection and the angle between two non-parallel planes.	☐ Find the reflection of a point in a plane ☐ Interpret given information in contextual question. ☐ Solve 3D vector geometry questions involving unknowns.

§1 <u>Equation of Planes</u>

A plane is a flat, two-dimensional surface that extends infinitely in all directions. Some examples of planes are:

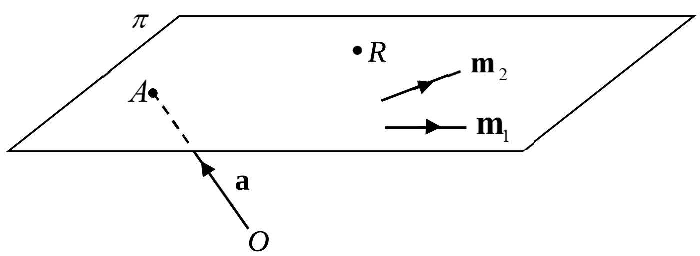


In the rest of this section, we will diagrammatise planes in the shape of a parallelogram.

1.1 <u>Vector Equation of a Plane</u>

Consider the plane π, which contains a given fixed point A, with position vector $\mathbf{a}$. Suppose that the plane π is parallel to the **two non-parallel** vectors $\mathbf{m_1}$ and $\mathbf{m_2}$.

Let R be a general point on π, and $\mathbf{r}$ be the position vector of R.



Now, $\mathbf{r} = \overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$

Given two non-parallel vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ on π, any vector on π may be expressed as a linear combination of those two vectors.
Hence, $\overrightarrow{AR}$ may be expressed as a linear combination of $\mathbf{m}_1$ and $\mathbf{m}_2$.
i.e $\overrightarrow{AR} = \lambda\mathbf{m}_1 + \mu\mathbf{m}_2$.
So, $\mathbf{r} = \mathbf{a} + \left(\lambda\mathbf{m}_1 + \mu\mathbf{m}_2\right)$, λ, μ are real parameters.

V**ector equation** of a plane π which contains the point A with position vector $\mathbf{a}$ and parallel to the vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ is given by:

$$\pi : \mathbf{r} = \mathbf{a} + \lambda\mathbf{m}_1 + \mu\mathbf{m}_2 \quad \text{where } \lambda, \mu \in \mathbb{R}$$

Position vector of **<u>a general point</u>** on the plane π

Position vector of a **<u>fixed point</u>** on the plane π

$\mathbf{m}_1$ and $\mathbf{m}_2$ are two non-parallel vectors that are parallel to the plane π

real parameters

Note:

(i) The position vector of any point P on π can be expressed as $\overrightarrow{OP} = \mathbf{a} + \lambda \mathbf{m}_1 + \mu \mathbf{m}_2$ for some $\lambda, \mu \in \mathbb{R}$.

(ii) If a plane π contains three non-collinear points *A*, *B* and *C*, then a **vector equation** of the plane can be given by :

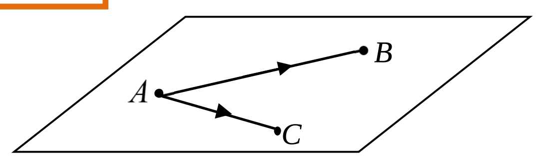
$$\mathbf{r} = \overrightarrow{OA} + \lambda \overrightarrow{AB} + \mu \overrightarrow{AC} \quad , \quad \lambda, \mu \in \mathbb{R}$$

The equation is not unique.
E.g. Another possible equation is $\mathbf{r} = \overrightarrow{OB} + \lambda \overrightarrow{AB} + \mu \overrightarrow{BC}$, $\lambda, \mu \in \mathbb{R}$.

Example 1

Find a vector equation of the plane that contains the points $A(2,1,4)$, $B(4,2,4)$ and $C(1,1,9)$.

Sketch a simple parallelogram to represent a plane and include points *A*, *B* and *C* on it to help with visualisation



Solution:
Find (any) two non-parallel vectors that are parallel to the plane:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$$

A vector equation of the plane is: $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$.

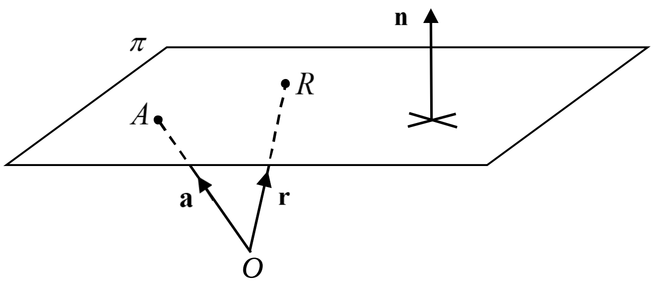
Note: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} x &= 2 + 2\lambda - \mu \\ y &= 1 + \lambda \\ z &= 4 \qquad + 5\mu \end{aligned}$

The equation of the plane in **parametric form** is $\begin{aligned} x &= 2 + 2\lambda - \mu \\ y &= 1 + \lambda \\ z &= 4 \qquad + 5\mu \end{aligned}$.

1.2 Equation of a Plane in Scalar Product Form

Consider a plane, π, which passes through a given fixed point A with position vector **a**, and is perpendicular to a given vector **n**.

Let R be a general point on π, and **r** be the position vector of R.



Since the vector **n** is perpendicular to the plane, π, **n** is perpendicular to any vector that lies on π.
Since points A and R lie on π, the vector $\overrightarrow{AR}$ lies on π.
Therefore, $\overrightarrow{AR}$ is perpendicular to **n**.

$$\Rightarrow \quad \overrightarrow{AR}\cdot\mathbf{n}=0$$
$$(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0$$
$$\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$$
$\therefore \mathbf{r}\cdot\mathbf{n}=D$, where

$D=\mathbf{a}\cdot\mathbf{n}$ is a scalar constant.

This is known as the equation of the plane π in **scalar product form.**

Equation of the plane π in **scalar product form** which contains the point A with position vector **a** and perpendicular to the vector **n** is given by:

$$\pi : \mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$$

Position vector of **<u>a general point</u>** on the plane π.	**<u>Normal vector</u>** (a vector perpendicular to the plane π).	Position vector of **<u>a fixed point</u>** on the plane π.

Example 2

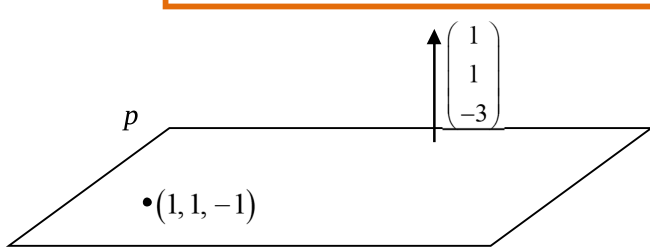
Find an equation of plane p in the form $\mathbf{r}\bullet\mathbf{n}=D$ given that plane p passes through the point $(1,1,-1)$ and is perpendicular to the vector $\mathbf{i}+\mathbf{j}-3\mathbf{k}$.

Solution:
Equation of the plane p:

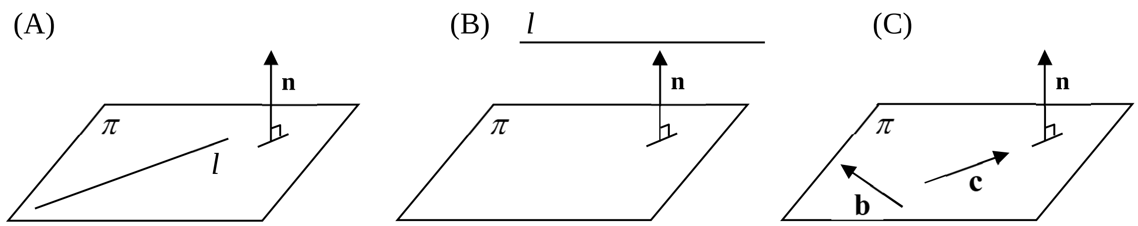
$$\mathbf{r}\bullet\begin{pmatrix}1\\1\\-3\end{pmatrix}=\begin{pmatrix}1\\1\\-1\end{pmatrix}\bullet\begin{pmatrix}1\\1\\-3\end{pmatrix}=5$$

i.e. $\mathbf{r}\bullet\begin{pmatrix}1\\1\\-3\end{pmatrix}=5$.

Sketch a simple parallelogram to represent a plane and include the given point and the normal vector to help with visualisation



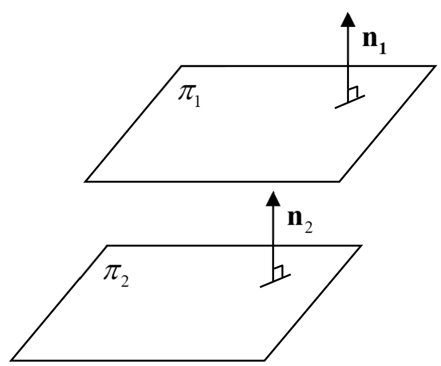
1.2.1 <u>Normal Vector to a Plane</u>



(i) A normal to a plane is perpendicular to any line in the plane (A) or any line parallel to the plane (B). If a vector is perpendicular to any two non-parallel vectors, say **b** and **c** in the plane (C), then this vector must be a normal vector, **n**, to the plane.

(ii) Normal to a plane is not unique. Thus, vector equation of a plane is **not** unique. If **n** is a normal to a plane, then 2**n**, $-\frac{2}{3}\mathbf{n}$ are also normal to the plane.

(iii) The planes π_1 and π_2 are parallel $\Leftrightarrow$ their normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel.



(iv) From the equation of a plane in **scalar product form,** if we divide both sides of the equation of the plane by the magnitude of the normal vector **n**, we will get the following:

$$\mathbf{r}\bullet\mathbf{n} = \mathbf{a}\bullet\mathbf{n}$$

$$\mathbf{r}\bullet\frac{\mathbf{n}}{|\mathbf{n}|} = \mathbf{a}\bullet\frac{\mathbf{n}}{|\mathbf{n}|}$$

$$\mathbf{r}\bullet\hat{\mathbf{n}} = \mathbf{a}\bullet\hat{\mathbf{n}}$$

$$\mathbf{r}\bullet\hat{\mathbf{n}} = D'$$

where **(a)** $\hat{\mathbf{n}} = \dfrac{\mathbf{n}}{|\mathbf{n}|}$ is a unit vector normal to the plane,

(b) $D' = \mathbf{a}\bullet\dfrac{\mathbf{n}}{|\mathbf{n}|}$ is a constant,

Note:

Constant $|D'| = \left|\mathbf{a}\bullet\dfrac{\mathbf{n}}{|\mathbf{n}|}\right|$ is the **perpendicular distance from the origin, *O*, to the plane**.

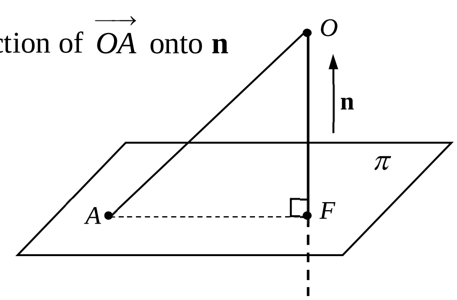
Distance from origin O to the plane π

$= OF$

$=$ length of projection of $\overrightarrow{OA}$ onto **n**

$= \left|\overrightarrow{OA}\bullet\hat{\mathbf{n}}\right|$

$= \left|\mathbf{a}\bullet\dfrac{\mathbf{n}}{|\mathbf{n}|}\right| = |D'|$



E.g. Given equation of plane

$$\mathbf{r}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix} = 3 \Rightarrow \mathbf{r}\bullet\frac{\begin{pmatrix}1\\1\\1\end{pmatrix}}{\sqrt{3}} = \frac{3}{\sqrt{3}}$$

$\left|\dfrac{3}{\sqrt{3}}\right|$ is the perpendicular distance from origin to plane.

(v) $\mathbf{r}\bullet\mathbf{n} = 0 \Leftrightarrow$ the plane passes through the origin.

(vi) If the point P lies on π, then the position vector of P will satisfy $\overrightarrow{OP}\bullet\mathbf{n} = \mathbf{a}\bullet\mathbf{n}$.

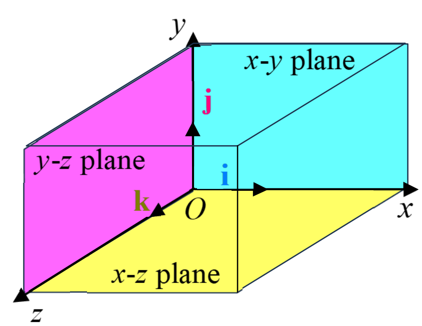
(vii) In scalar product form, equations of

(a) *x-y* plane : $\mathbf{r}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix} = 0$

Any vector parallel to *z*-axis is perpendicular to the *x-y* plane. Choose the simplest vector $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

(b) *y-z* plane : $\mathbf{r}\bullet\begin{pmatrix}1\\0\\0\end{pmatrix} = 0$

(c) *x-z* plane : $\mathbf{r}\bullet\begin{pmatrix}0\\1\\0\end{pmatrix} = 0$



Example 3

A plane passes through the point $A(4,-2,-2)$ and is perpendicular to the vector $\mathbf{i}+\mathbf{j}-3\mathbf{k}$. Write down an equation of the plane in scalar product form. Hence find the shortest distance from the origin to the plane.

Do the points $B(2,0,-2)$ and $C(1,2,0)$ lie in the plane?

Use previously found part(s) or result(s)

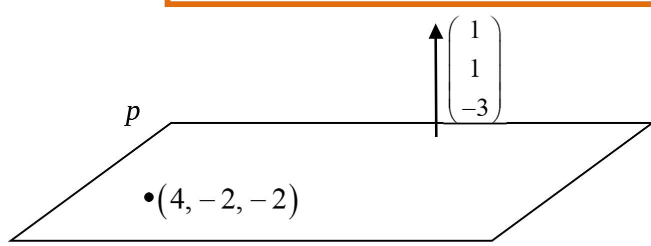
Solution:

Sketch a simple parallelogram to represent a plane and include the given point and the normal vector to help with visualisation.

An equation of the plane in scalar product form is:

$$\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}\Rightarrow\mathbf{r}\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}=\begin{pmatrix}4\\-2\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}=8$$

i.e $\mathbf{r}\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}=8$



Shortest distance from Origin to Plane

Recall formula

$$=\frac{|8|}{\left|\begin{pmatrix}1\\1\\-3\end{pmatrix}\right|}=\frac{8}{\sqrt{11}}$$

Check if B lies in the plane:

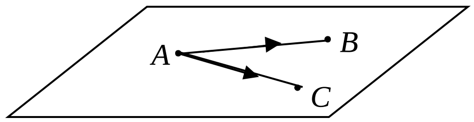
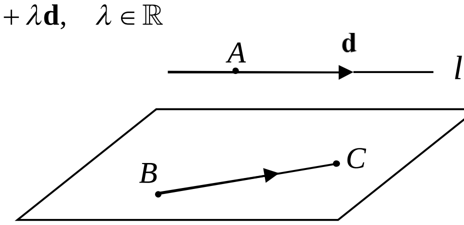
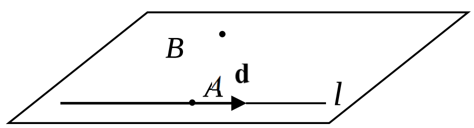
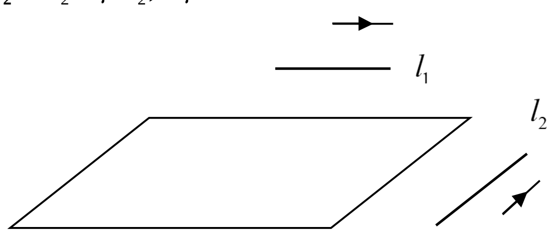
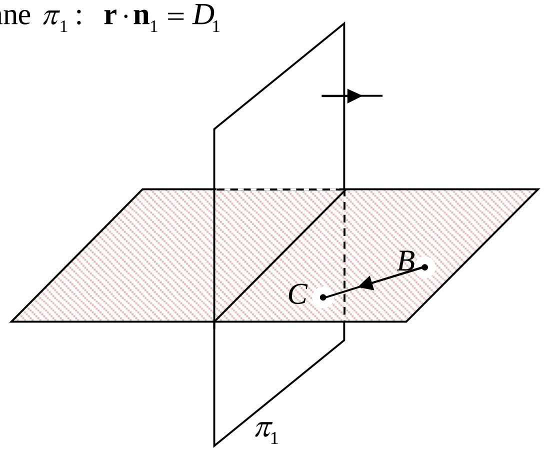
$$\begin{pmatrix}2\\0\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}=2+0+6=8 \qquad \therefore\ B \text{ lies in the plane.}$$

Check if C lies in the plane:

$$\begin{pmatrix}1\\2\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}=1+2+0=3\neq 8 \qquad \therefore\ C \text{ does not lie in the plane.}$$

Check if the position vector satisfy equation of plane LHS=RHS

1.2.2 <u>Some Examples for the Determination of Normal n to a Plane</u>

	Information given	**Normal**
1	Plane containing 3 given points A, B and C 	$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$
2	Plane containing 2 given points B and C and parallel to line $l : \mathbf{r} = \mathbf{a} + \lambda \mathbf{d}, \quad \lambda \in \mathbb{R}$ 	$\mathbf{n} = \overrightarrow{BC} \times \mathbf{d}$
3	Plane containing the line $l : \mathbf{r} = \mathbf{a} + \lambda \mathbf{d}, \quad \lambda \in \mathbb{R}$ and the point B (not on l). 	$\mathbf{n} = \overrightarrow{AB} \times \mathbf{d}$
4	Plane parallel to 2 lines (non-parallel) $l_1 : \mathbf{r_1} = \mathbf{a}_1 + \lambda \mathbf{d_1}, \quad \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r_2} = \mathbf{a}_2 + \mu \mathbf{d}_2, \quad \mu \in \mathbb{R}$ 	$\mathbf{n} = \mathbf{d_1} \times \mathbf{d_2}$
5	Plane containing 2 given points B and C and perpendicular to a given plane $\pi_1 : \ \mathbf{r} \cdot \mathbf{n}_1 = D_1$ 	$\mathbf{n} = \mathbf{n}_1 \times \overrightarrow{BC}$

Example 4

Find an equation of the plane in scalar product form that contains the lines,

$$l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ where } \lambda, \mu \in \mathbb{R} .$$

Solution:

A normal vector to the plane is $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}$

Remember to check that the normal vector is correct. Can you recall how?

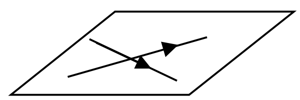
Check if $\mathbf{n} \cdot \mathbf{d} = 0$

$\begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = 2 - 2 = 0$ $\quad$ $\begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 2 - 2 = 0$

An equation of the plane in scalar product form is:

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} \Rightarrow \mathbf{r} \cdot \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} = -4$$

i.e $\mathbf{r} \cdot \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} = -4$ or $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 4$.



Sketch a simple parallelogram to represent a plane and include relevant details.

Example 5

Find an equation of the plane in scalar product form that contains the points $A(2, -1, 1)$ and the line l with equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} - 2\mathbf{k}),\ \lambda \in \mathbb{R}$.

[Solution]

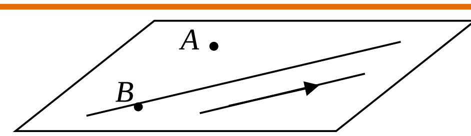
Let point B on line l be $(2,1,1)$.

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix} = 2\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

An equation of the plane in scalar product form is:

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} \Rightarrow \mathbf{r} \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = -3$$

Sketch a simple parallelogram to represent a plane and include relevant details.



i.e $\mathbf{r} \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = -3$

1.3 <u>Equation of a Plane in Cartesian Form</u>

Consider the scalar product form of a plane $\pi : \mathbf{r}\cdot\mathbf{n} = D$, where $D = \mathbf{a}\cdot\mathbf{n}$ and $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Since $\mathbf{r}$ is the position vector of a general point on π, let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$\mathbf{r}\cdot\mathbf{n} = D \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}\cdot\begin{pmatrix} a \\ b \\ c \end{pmatrix} = D \Leftrightarrow ax + by + cz = D.$$

E.g.

$$\mathbf{r}\bullet\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 5 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}\bullet\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 5$$
$$\Rightarrow x + 2y + 3z = 5$$

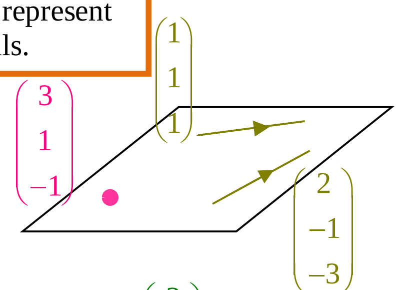
$$ax + by + cz = D$$

This is known as the **Cartesian equation** of the plane π.

Example 6

Find the equation of the plane passing through the point $(3,1,-1)$ and parallel to the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$ in **(a)** vector form **(b)** scalar product form. Find also the Cartesian equation of the plane.

Sketch a simple parallelogram to represent a plane and include relevant details.



Solution:

(a) Vector equation, $\pi : \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}$

(b) A normal vector to the plane is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\times\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix} = -\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$. Choose $\mathbf{n} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$

Equation in scalar product form, $\pi : \mathbf{r}\cdot\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = -2 \Rightarrow \mathbf{r}\cdot\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = -2$

$$\mathbf{r}\cdot\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = -2 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}\cdot\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = -2$$

Cartesian equation of π , $2x-5y+3z=-2$

1.4 Convert Equation of Plane from One Form to Another Form

Vector Equation form | Scalar Product form | Cartesian form

$\pi:\ \mathbf{r}=\mathbf{a}+\lambda\mathbf{u}+\mu\mathbf{v},\ \ \lambda,\mu\in\mathbb{R} \longrightarrow \pi:\ \mathbf{r}\bullet\mathbf{n}=\mathbf{a}\bullet\mathbf{n}$ where $\mathbf{n}=\mathbf{u}\times\mathbf{v} \longleftrightarrow ax+by+cz=d$

Step 1: Find the normal vector
$\mathbf{n}=\mathbf{u}\times\mathbf{v}$

Step 2: Since $\mathbf{a}$ is the position vector of a fixed point on the plane,
$\mathbf{r}\bullet\mathbf{n}=\mathbf{a}\bullet\mathbf{n}$

Let $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$. Given that $\mathbf{n}=\begin{pmatrix}a\\b\\c\end{pmatrix}$,

$\mathbf{r}\bullet\mathbf{n}=\begin{pmatrix}x\\y\\z\end{pmatrix}\bullet\begin{pmatrix}a\\b\\c\end{pmatrix}=ax+by+cz$

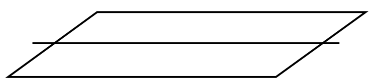
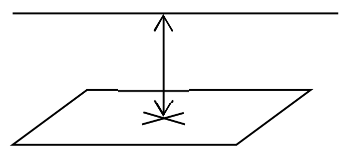
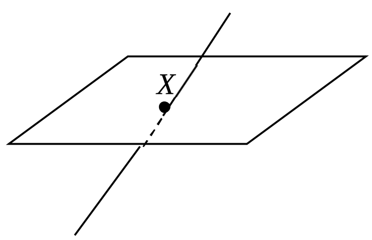
§2 Relationship Between a Line and a Plane

Activity

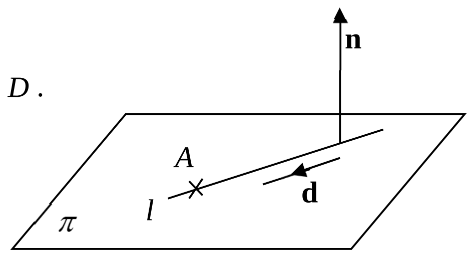
Step 1- Take out 1 pen (representing a line) and a paper (representing a plane)

Step 2- Using the pen and the paper, determine the possible relationships that a line and a plane can have in a 3-dimensional space.

Consider the line $l:\mathbf{r}=\mathbf{a}+\lambda\mathbf{d},\ \lambda\in\mathbb{R}$ and the plane $\pi:\mathbf{r}\cdot\mathbf{n}=D$.

l and π	**intersect**	**do not intersect**
parallel	 l is contained in the plane π	
not parallel	 l and π intersect at a point X	Impossible

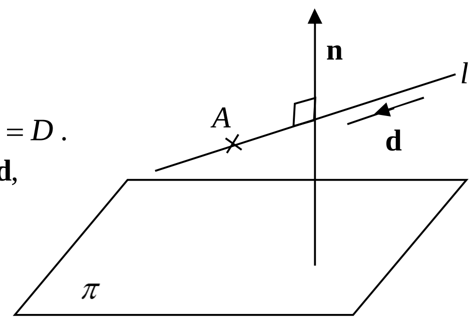
2.1 Line l parallel to Plane π and lies in π



Consider the line $l:\mathbf{r}=\mathbf{a}+\lambda\mathbf{d},\ \lambda\in\mathbb{R}$ and the plane $\pi:\mathbf{r}\cdot\mathbf{n}=D$.
l lies in π if its direction vector, **d**, is parallel to the plane (and therefore perpendicular to **n**), and if any one point on l (eg. A) lies in π.

l lies in π if:
(i) l is parallel to π i.e. $\mathbf{n}\cdot\mathbf{d}=0$ **and**
(ii) the point with position vector **a** on l is also in π i.e. $\mathbf{a}\cdot\mathbf{n}=D$.

2.2 Line l parallel to Plane π but does NOT lie in π



Consider the line $l:\mathbf{r}=\mathbf{a}+\lambda\mathbf{d},\ \lambda\in\mathbb{R}$ and the plane $\pi:\mathbf{r}\cdot\mathbf{n}=D$.
l is parallel to, but does not lie in π if its direction vector, **d**, is parallel to the plane (and therefore perpendicular to **n**), and if **<u>no</u>** point on l lies in π.

l is parallel to π but does NOT lie in π if:
(i) l is parallel to π i.e. $\mathbf{n}\cdot\mathbf{d}=0$ and
(ii) the point with position vector **a** on l is not in π, i.e. $\mathbf{a}\cdot\mathbf{n}\neq D$.

Example 7

Given a plane $\pi : \mathbf{r} \bullet \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 4$, determine whether the following lines:

- intersect the plane, or
- are parallel to but do not lie in the plane, or
- lie in the plane.

(i) $l_1 : \mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$ **(ii)** $l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $\mu \in \mathbb{R}$ parameters.

[Solution]

(i) $\mathbf{n} \bullet \mathbf{d} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 2 - 2 = 0$ $\therefore$ l_1 is parallel to π

Since $\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = -4 - 1 - 1 = -6 \neq 4$, point $(-2, 1, -1)$ is on l_1 but does not lie in π.

$\therefore$ l_1 is parallel to π but does not lie in π.

(ii) $\mathbf{n} \bullet \mathbf{d} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = -2 + 2 = 0$ $\therefore$ l_2 is parallel to π

$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 4 - 1 + 1 = 4$, point $(2, 1, 1)$ is on l_2 and lies in π.

$\therefore$ l_2 is parallel to π and lies in π.

Alternatively:

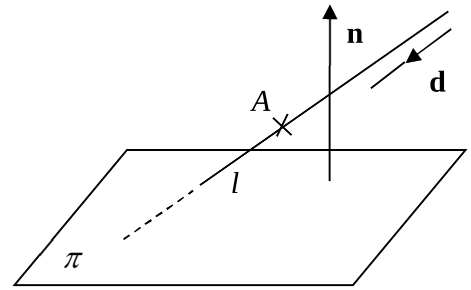
$\begin{pmatrix} 2-\mu \\ 1 \\ 1+2\mu \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 4 - 2\mu - 1 + 1 + 2\mu = 4$, therefore l_2 is parallel to π and lies in π.

2.3 Line l is not parallel to the plane and therefore intersects the plane

Point of Intersection between a line l and a plane π

Consider the line, $l: \mathbf{r} = \mathbf{a} + \lambda\mathbf{d},\ \lambda \in \mathbb{R}$ — (1)

and the plane, $\pi:\ \mathbf{r}\cdot\mathbf{n} = D$ — (2)



If l and π are not parallel, then they must intersect at a point.

The coordinates of the intersection point can be found by:

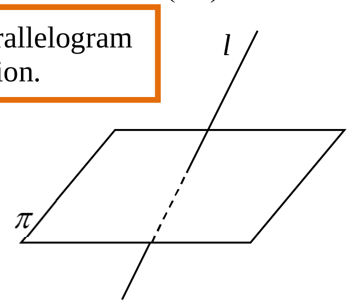
Step 1: Substitute (1) in (2) and solve for λ.

Step 2: Substitute the value of λ obtained into (1) to obtain the intersection point.

Example 8

Find the coordinates of the point of intersection between the plane $\pi : \mathbf{r}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix} = 2$ and the line $l:\mathbf{r} = \begin{pmatrix}2\\1\\4\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\0\end{pmatrix},\ \lambda \in \mathbb{R}$.

Sketch a line and a parallelogram to help with visualisation.



Solution:

$l:\mathbf{r} = \begin{pmatrix}2\\1\\4\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\0\end{pmatrix},\ \lambda \in \mathbb{R}$ — (1) $\qquad \pi : \mathbf{r}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix} = 2$ — (2)

Substitute (1) in (2): $\begin{pmatrix}2\\1+\lambda\\4\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix} = 2 \quad \Rightarrow \lambda = 1$

Substitute equation of line into equation of plane

Substitute $\lambda = 1$ into (1): $\mathbf{r} = \begin{pmatrix}2\\1\\4\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\2\\4\end{pmatrix}$

Therefore the coordinates of the point of intersection are $(2, 2, 4)$.

Example 9

The equations of a line l and a plane π are

$$l:\mathbf{r}=\begin{pmatrix}1\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix},\ \lambda\in\mathbb{R} \qquad \text{and} \qquad \pi:\mathbf{r}\cdot\begin{pmatrix}a\\-1\\1\end{pmatrix}=b\,.$$

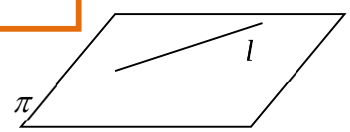
respectively, where a and b are constants.

(i) Given that l lies in π, find the values of a and b.

(ii) Given instead that l is parallel to π and does not lie in π, what can be said about the values of a and b?

Sketch a line and a parallelogram to help with visualisation.

Solution:



(i) Since l lies in π,

$\mathbf{n}\cdot\mathbf{d}=0$

$$\begin{pmatrix}a\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\0\end{pmatrix}=0 \Rightarrow \quad a-2=0 \quad \Rightarrow \quad a=2$$

$$\begin{pmatrix}1\\0\\4\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=b \Rightarrow \quad b=6$$

the point with position vector on l is also in π i.e. $\mathbf{a}\cdot\mathbf{n}=D$.

(ii) Given instead that l does not lie in π, $a=2$, $\quad b\neq 6$

§3 Angle Between a Line and a Plane

Recall from Vectors 1 that the angle θ between two **vectors** $\mathbf{a}$ and $\mathbf{b}$ is found by

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

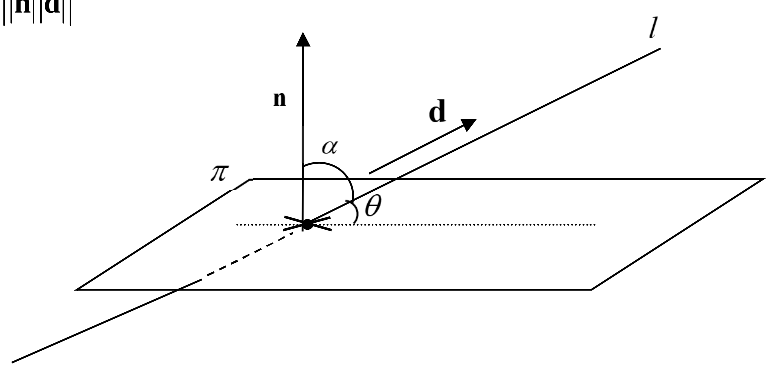
Consider a line l and a plane π given by

$$l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}, \lambda \in \mathbb{R},$$
$$\pi : \mathbf{r}\cdot\mathbf{n} = D.$$

Recall by convention, we want to find the **acute** angle θ between l and π.

To find the **acute** angle between l and π , we first consider the **acute** angle α between the normal vector $\mathbf{n}$ and the direction vector $\mathbf{d}$ using the scalar product formula

$$\cos\alpha = \left|\frac{\mathbf{n}\bullet\mathbf{d}}{|\mathbf{n}||\mathbf{d}|}\right|$$



Since

$$\alpha + \theta = 90^\circ$$
$$\alpha = 90^\circ - \theta$$

Therefore, we have

$$\cos\alpha = \left|\frac{\mathbf{n}\bullet\mathbf{d}}{|\mathbf{n}||\mathbf{d}|}\right|$$
$$\cos\left(90^\circ - \theta\right) = \left|\frac{\mathbf{n}\bullet\mathbf{d}}{|\mathbf{n}||\mathbf{d}|}\right|$$
$$\sin\theta = \left|\frac{\mathbf{n}\bullet\mathbf{d}}{|\mathbf{n}||\mathbf{d}|}\right|$$

In conclusion, the **acute** angle θ between line l and plane π can be found using the formula

$$\boxed{\sin\theta = \left|\frac{\mathbf{n}\bullet\mathbf{d}}{|\mathbf{n}||\mathbf{d}|}\right|}$$

Example 10

Find the acute angle between the line l and the plane π, where

$$l:\ \mathbf{r}=\begin{pmatrix}1\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\4\\-1\end{pmatrix},\ \lambda\in\mathbb{R},\quad \text{and}\quad \pi:\ 2x-y+z=0.$$

Solution:

$$\pi: 2x-y+z=0 \Rightarrow \mathbf{r}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=0$$

Convert to scalar product form first

Let θ be the acute angle between l and π.

$$\sin\theta=\left|\frac{\mathbf{n}\cdot\mathbf{d}}{|\mathbf{n}||\mathbf{d}|}\right| \Rightarrow \sin\theta=\frac{\left|\begin{pmatrix}2\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}2\\4\\-1\end{pmatrix}\right|}{\sqrt{6}\sqrt{21}}=\frac{1}{\sqrt{126}}\quad \therefore \theta=5.1^\circ$$

§4 Foot of the Perpendicular from Point to Plane

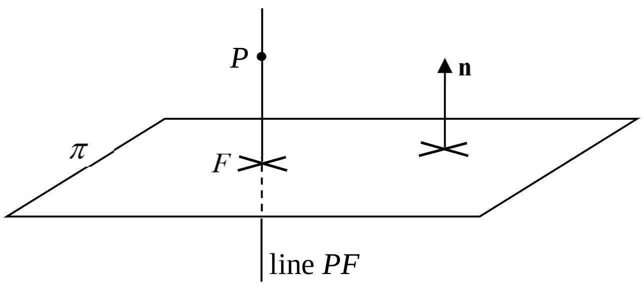
Example 11

Find the foot of the perpendicular from the point $P(1,-3,3)$ to the plane $\pi:\mathbf{r}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}=6$.

Solution:

Let F be the foot of the perpendicular from the point P to π .

Sketch a diagram with relevant details to help with visualisation.



What do you observe from the diagram?

F is the point of intersection of the line PF and the plane π .

Key idea: to solve equations of l_{PF} and π simultaneously

Step 1: Find l_{PF}

Line PF: $\mathbf{r}=\begin{pmatrix}1\\-3\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix},\ \lambda\in\mathbb{R}$ ----- (1)

$\pi:\mathbf{r}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}=6$ -----(2)

Find intersection of line PF and plane π :

Step 2: Substitute l_{PF} into π

Substituting (1) in (2) ,

$$\begin{pmatrix}1+\lambda\\-3+\lambda\\3\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}=6 \quad\Rightarrow 1+\lambda-3+\lambda=6 \quad\Rightarrow \lambda=4$$

Step 3: Solve for λ

Step 4: Substitute λ into l_{PF} to find foot of perpendicular

Thus the position vector of the foot of the perpendicular from P to π is:

$$\overrightarrow{OF}=\begin{pmatrix}1\\-3\\3\end{pmatrix}+4\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}5\\1\\3\end{pmatrix}$$

§5 Perpendicular Distance from a Point to a Plane

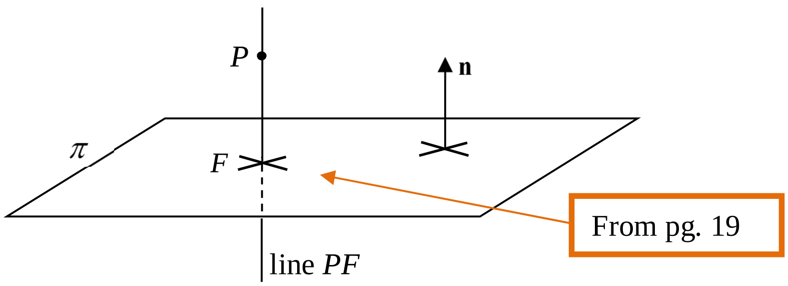
Example 11 (Continued) – where the foot of perpendicular is found (“Hence” method)

Find the foot of the perpendicular from the point $P(1,-3,3)$ to the plane $\pi : \mathbf{r}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}=6$.

Hence find the shortest distance from P to the plane π.

Solution:

Let F be the foot of the perpendicular from the point P to π .



Since we have already found the position vector of the foot of the perpendicular from P to π to be $\overrightarrow{OF}=\begin{pmatrix}5\\1\\3\end{pmatrix}$, the shortest distance from P to the plane π is none other than $\left|\overrightarrow{PF}\right|$.

$$\overrightarrow{PF}=\overrightarrow{OF}-\overrightarrow{OP}=\begin{pmatrix}5\\1\\3\end{pmatrix}-\begin{pmatrix}1\\-3\\3\end{pmatrix}=\begin{pmatrix}4\\4\\0\end{pmatrix}$$

Thus the shortest distance from P to the plane π $=\left|\overrightarrow{PF}\right| = \sqrt{4^2+4^2+0^2}$

$$=\sqrt{32} = 4\sqrt{2} \text{ units}$$

Example 12 (Using the concept of length of projection – "Otherwise" method)

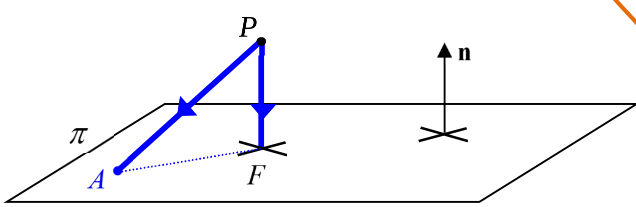
Find the shortest distance from $P(2,3,-4)$ to the plane π with equation $2x+y-2z+9=0$.

Solution:

Aim: To find PF without find $\overrightarrow{OF}$

Convert to scalar product form first

$$2x+y-2z+9=0 \Rightarrow \mathbf{r}\bullet\begin{pmatrix}2\\1\\-2\end{pmatrix}=-9 \qquad \text{----------} \ (1)$$



Choose an easy random vector that satisfies the equation

First, observe that point $A(0,-9,0)$ lies in π, since $\begin{pmatrix}0\\-9\\0\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-2\end{pmatrix}=-9$.

Next, observe that finding the shortest distance from P to the plane π which is $\left|\overrightarrow{PF}\right|$ is same as finding the length of projection of $\overrightarrow{PA}$ onto $\mathbf{n}$.

Let $\overrightarrow{OA}=\begin{pmatrix}0\\-9\\0\end{pmatrix}$. $\overrightarrow{PA}=\overrightarrow{OA}-\overrightarrow{OP}=\begin{pmatrix}0\\-9\\0\end{pmatrix}-\begin{pmatrix}2\\3\\-4\end{pmatrix}=\begin{pmatrix}-2\\-12\\4\end{pmatrix}$

Shortest distance from P to the plane π

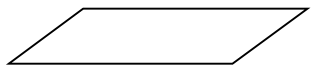
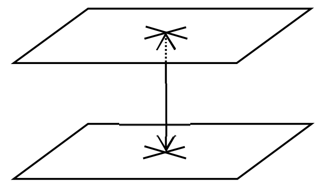
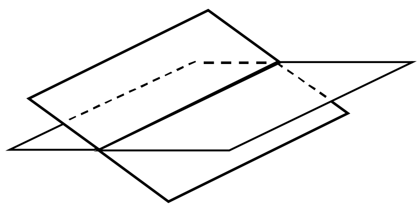
= length of projection of $\overrightarrow{PA}$ onto $\mathbf{n}$

$=\left|\overrightarrow{PA}\bullet\hat{\mathbf{n}}\right|$ Recall

$$=\frac{\left|\begin{pmatrix}-2\\-12\\4\end{pmatrix}\bullet\begin{pmatrix}2\\1\\-2\end{pmatrix}\right|}{\left|\begin{pmatrix}2\\1\\-2\end{pmatrix}\right|}=\frac{|-4-12-8|}{\sqrt{9}}=\frac{24}{3}=8 \text{ units}$$

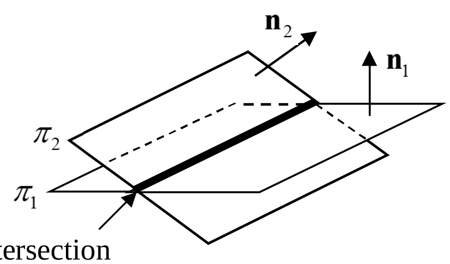
§6 Relationship Between Two Planes

Consider the planes $\pi_1 : \mathbf{r}\cdot\mathbf{n}_1 = D_1$ and $\pi_2 : \mathbf{r}\cdot\mathbf{n}_2 = D_2$. Again, there are 3 possibilities.

π_1 and π_2	**intersect**	**do not intersect**
Parallel (i.e. $n_1 = \lambda n_2$ $\lambda \in \mathbb{R}$)	 π_1 and π_2 are in fact the same plane	
not parallel (i.e. $n_1 \neq \lambda n_2$ $\lambda \in \mathbb{R}$)	 π_1 and π_2 intersect in a line l	Impossible

6.1 Non-Parallel Planes: Line of Intersection

Two distinct planes are either parallel or intersecting. If the two planes are non-parallel, they **always** intersect in a line.



Example 13 **[numbers in vectors: use GC]**

Find the line of intersection of the planes $\pi_1 : \mathbf{r}\cdot(\mathbf{i}-5\mathbf{j}-3\mathbf{k}) = 0$ and $\pi_2 : \mathbf{r}\cdot(\mathbf{i}-\mathbf{j}-3\mathbf{k}) = 8$.

Solution:

It is good to observe that since $\begin{pmatrix}1\\-5\\-3\end{pmatrix} \neq \lambda\begin{pmatrix}1\\-1\\-3\end{pmatrix}, \lambda\in\mathbb{R}$, π_1 and π_2 are not parallel,

therefore they are intersecting.

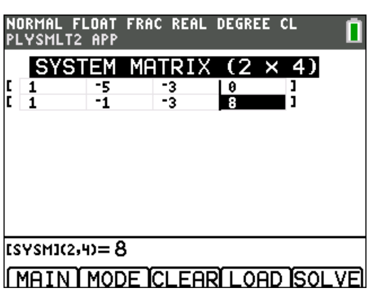
Step 1: Convert the equations to Cartesian form

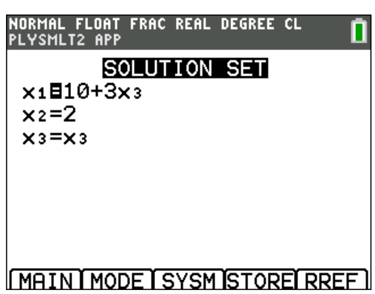
$$\pi_1 : \quad \mathbf{r}\cdot(\mathbf{i}-5\mathbf{j}-3\mathbf{k}) = 0 \quad \Rightarrow \quad \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\-5\\-3\end{pmatrix} = 0 \quad \Rightarrow \quad x-5y-3z=0$$

$$\pi_2 : \quad \mathbf{r}\cdot(\mathbf{i}-\mathbf{j}-3\mathbf{k}) = 8 \quad \Rightarrow \quad \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-3\end{pmatrix} = 8 \quad \Rightarrow \quad x-y-3z=8$$

Step 2: Use GC to solve simultaneous equations

Using GC (Polysmlt2) (refer to steps on next page)





$$\Rightarrow \quad \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}10 + 3z\\2\\z\end{pmatrix}$$

Step 3: Let $z=\lambda$

Let $z=\lambda$, $\lambda\in\mathbb{R}$: $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}10+3\lambda\\2\\\lambda\end{pmatrix}=\begin{pmatrix}10\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}3\\0\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$.

$\therefore$ the planes intersect in the line $\mathbf{r}=\begin{pmatrix}10\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}3\\0\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$.

Important Note:
Given the two planes $\pi_1:\mathbf{r}.\mathbf{n}_1=D_1$ and $\pi_2:\mathbf{r}.\mathbf{n}_2=D_2$
The line of intersection of the planes π_1 and π_2 **has direction vector parallel to** $\mathbf{n}_1\times\mathbf{n}_2$.

Direction vector of the line of intersection can be found by
$\mathbf{d}=\mathbf{n}_1\times\mathbf{n}_2$

Example 14 [unknowns in vectors]

Find the line of intersection of the planes $\pi_1:\mathbf{r}\cdot\begin{pmatrix}1\\2\\\alpha\end{pmatrix}=0$ and $\pi_2:\mathbf{r}\cdot\begin{pmatrix}4\\5\\0\end{pmatrix}=0$ in terms of the constant α.

Unknown: cannot use GC

Notice that RHS=0:
it means that $\begin{pmatrix}0\\0\\0\end{pmatrix}$ lies on both planes

Solution:

$$\mathbf{n}_1\times\mathbf{n}_2=\begin{pmatrix}1\\2\\\alpha\end{pmatrix}\times\begin{pmatrix}4\\5\\0\end{pmatrix}=\begin{pmatrix}-5\alpha\\4\alpha\\-3\end{pmatrix}$$

Since the origin $(0,0,0)$ is a common point on the planes π_1 and π_2, the line of intersection of the planes π_1 and π_2 is $\mathbf{r}=\begin{pmatrix}0\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}-5\alpha\\4\alpha\\-3\end{pmatrix}$, $\lambda\in\mathbb{R}$.

Note: If a common point between the 2 planes cannot be easily observed, simply equate one of the coordinates (either x or y or z) in both equations to 0, and solve the resulting pair of simultaneous equations to obtain a common point between the 2 planes.

6.2 <u>Non-Parallel Planes: Angle Between Two Planes</u>

Recall from ***Vectors 1*** that the angle θ between two **vectors** $\mathbf{a}$ and $\mathbf{b}$ is found by

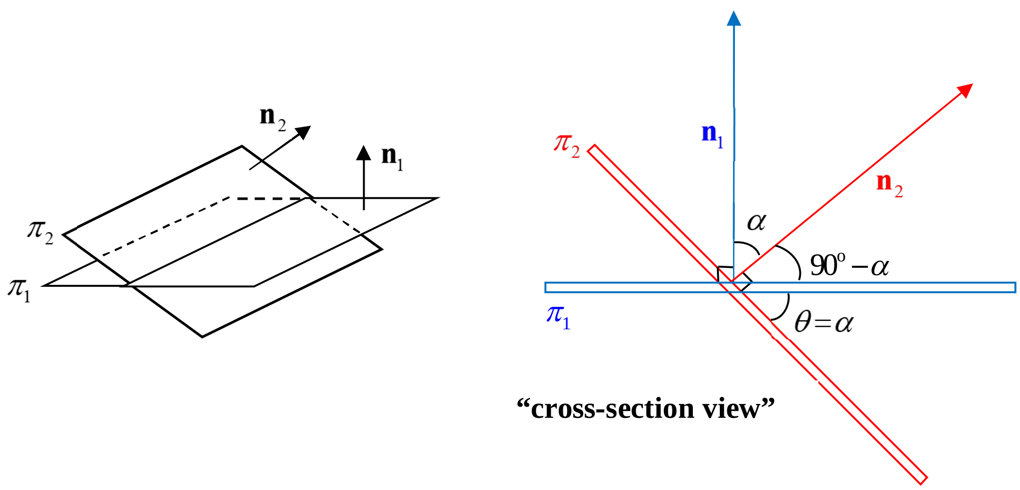
$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Consider the two planes $\pi_1 : \mathbf{r}\cdot\mathbf{n}_1 = D_1$ and $\pi_2 : \mathbf{r}\cdot\mathbf{n}_2 = D_2$.

Recall by convention, we want to find the **acute** angle θ between π_1 and π_2.

To find the **acute** angle between π_1 and π_2 , we first consider the **acute** angle α between the two normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ using the scalar product formula

$$\cos\alpha = \left|\frac{\mathbf{n}_1 \bullet \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right|$$



"cross-section view"

From the "cross-section view", observe that **the acute angle between the planes** π_1 and π_2 **is also the acute angle between their normal vectors** $\mathbf{n}_1$ and $\mathbf{n}_2$**, i.e.** $\theta = \alpha$ **.**

In conclusion, the **acute** angle θ between planes π_1 and π_2 can be found using the formula

$$\boxed{\cos\theta = \left|\frac{\mathbf{n}_1 \bullet \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right|}$$

Special Case: Two planes π_1 and π_2 **are perpendicular** if and only if $\mathbf{n}_1 \bullet \mathbf{n}_2 = 0$.

Example 15

Find the acute angle between the planes $\pi_1 : x + y + z = 3$ and $\pi_2 : 2x - 2y + z = 1$.

Convert to scalar product form first

Solution:

$$\pi_1 : \mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3 \text{ and } \pi_2 : \mathbf{r} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 1$$

Let θ be the acute angle between π_1 and π_2.

$$\cos\theta = \left| \frac{\mathbf{n_1} \bullet \mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|} \right| \Rightarrow \cos\theta = \frac{\left| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right|}{\sqrt{3}\sqrt{9}} = \frac{1}{3\sqrt{3}}$$

$\therefore \theta = 78.9°$

6.3 <u>Parallel Planes: Distance Between Two Planes</u>

Example 16

The plane p_1 has equation $\mathbf{r}\bullet\begin{pmatrix}7\\2\\-3\end{pmatrix}=-4$. The plane p_2 passes through $(2,4,5)$ and is parallel to p_1. Find the equation of p_2 and determine the distance between the two planes.

Solution:

The equation of p_2 is $\mathbf{r}\bullet\begin{pmatrix}7\\2\\-3\end{pmatrix}=\begin{pmatrix}2\\4\\5\end{pmatrix}\bullet\begin{pmatrix}7\\2\\-3\end{pmatrix}=14+8-15=7$ i.e. $\mathbf{r}\bullet\begin{pmatrix}7\\2\\-3\end{pmatrix}=7$

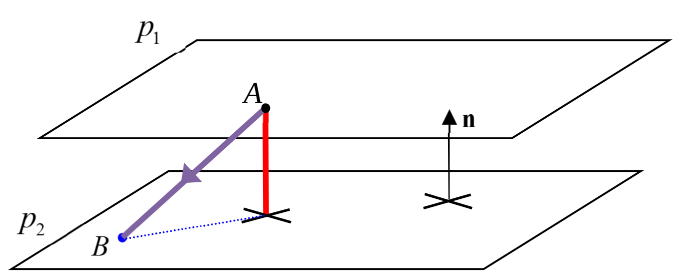
Method 1: (Using the concept of length of projection)

First, observe that the point $A(0,-2,0)$ lies in p_1 and $B(1,0,0)$ lies in p_2.

Step 1: Find a point on each plane to form vector $\overrightarrow{AB}$

Next, observe that finding the distance between the two planes is the same as finding the length of projection of $\overrightarrow{AB}$ onto **n**.

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}1\\0\\0\end{pmatrix}-\begin{pmatrix}0\\-2\\0\end{pmatrix}=\begin{pmatrix}1\\2\\0\end{pmatrix}$



Distance between p_1 and p_2

= length of projection of $\overrightarrow{AB}$ onto **n**

$=\left|\overrightarrow{AB}\bullet\hat{\mathbf{n}}\right|$

Step 2: Find length of projection of $\overrightarrow{AB}$ onto **n**

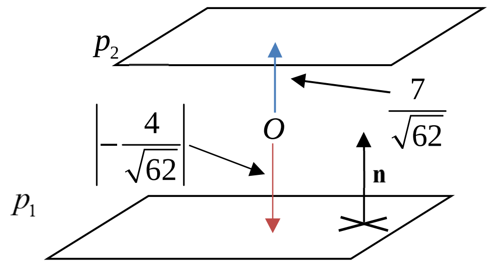
$$=\frac{\left|\begin{pmatrix}1\\2\\0\end{pmatrix}\bullet\begin{pmatrix}7\\2\\-3\end{pmatrix}\right|}{\left|\begin{pmatrix}7\\2\\-3\end{pmatrix}\right|}=\frac{|7+4|}{\sqrt{62}}=\frac{11}{\sqrt{62}}\text{ units}$$

Method 2: (Using distance from origin to plane p_1 and p_2)

To find the distance between p_1 and p_2, we first convert the equations into the form $\mathbf{r}\cdot\hat{\mathbf{n}} = D'$.

$$\mathbf{r}\bullet\begin{pmatrix}7\\2\\-3\end{pmatrix} = -4 \;\Rightarrow\; \mathbf{r}\bullet\frac{1}{\sqrt{62}}\begin{pmatrix}7\\2\\-3\end{pmatrix} = -\frac{4}{\sqrt{62}}$$

$$\mathbf{r}\bullet\begin{pmatrix}7\\2\\-3\end{pmatrix} = 7 \;\Rightarrow\; \mathbf{r}\bullet\frac{1}{\sqrt{62}}\begin{pmatrix}7\\2\\-3\end{pmatrix} = \frac{7}{\sqrt{62}}$$



$\therefore$ Distance between the two planes $= \dfrac{7}{\sqrt{62}} + \left|-\dfrac{4}{\sqrt{62}}\right| = \dfrac{11}{\sqrt{62}}$ units.

In general, given 2 parallel planes $\pi_1 : \mathbf{r}\cdot\mathbf{n}_1 = \mathrm{D}_1$ and $\pi_2 : \mathbf{r}\cdot\mathbf{n}_2 = \mathrm{D}_2$, the distance between them can be found using the formula

$$\boxed{\text{Distance between two parallel planes} = \left|\frac{\mathrm{D}_1}{|\mathbf{n}|} - \frac{\mathrm{D}_2}{|\mathbf{n}|}\right|}$$

H2 Mathematics (9758)
Chapter 6B
3D Vector Geometry (Lines & Planes)
Discussion Questions

Level 1

1 Find the equation of the following planes in parametric form, scalar product form and Cartesian form.

(a) The plane passing through the points $A(0, 1, 1)$, $B(1, -3, 2)$ and $C(1, 0, 1)$.

(b) The plane containing the lines $x = 3, \dfrac{y+1}{2} = z - 3$ and $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \mu \in \mathbb{R}$.

(c) The plane that includes the line $\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$ and the point with position vector $2\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$.

In each case, find the coordinates of the point of intersection of the plane and the line $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}, \alpha \in \mathbb{R}$.

2 Find the coordinates of the point where the line $\mathbf{r} = \mathbf{i} + \lambda(\mathbf{i} - \mathbf{k})$ intersects the plane with equation $2x - 3y + z = 1$.

3 Find the equation of the line of intersection between the two planes with equations $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 10$ and $\mathbf{r} \cdot \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} = 6$ respectively.

4 Find the acute angle between

(a) the line $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$, and the plane $x - 3y + z = 2$;

(b) the planes $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = 6$ and $x + 2y - 5z = 8$.

Level 2

5 Find the position vector of *F*, the foot of the perpendicular from *B*(2, 3, –4) to the plane *p*, whose equation is $2x+y-2z+9=0$.

6 Find the perpendicular distance from point $A(4,5,6)$ to the plane $\mathbf{r}\bullet\mathbf{k}=2$.

7 The plane π contains the point $P(2,2,4)$ and the line $l_1:\ \mathbf{r}=\begin{pmatrix}1\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$.

(i) Find the vector equation of the plane π in scalar product form.

(ii) Find the position vector of the foot of the perpendicular from the point $S(1,0,-4)$ to π.

(iii) Verify that $S(1,0,-4)$ lies on line $l_2:\mathbf{r}=\begin{pmatrix}1\\-1\\1\end{pmatrix}+\mu\begin{pmatrix}0\\1\\-5\end{pmatrix}$, $\mu\in\mathbb{R}$.

(iv) Find the equation of the image obtained by reflecting the line l_2 in the plane π.

8 **2011(9740)/I/11**

The plane *p* passes through the points with coordinates $(4,-1,-3)$, $(-2,-5,2)$ and $(4,-3,-2)$.

(i) Find the Cartesian equation of *p*. [4]

The line l_1 has equation $\dfrac{x-1}{2}=\dfrac{y-2}{-4}=\dfrac{z+3}{1}$ and the line l_2 has equation $\dfrac{x+2}{1}=\dfrac{y-1}{5}=\dfrac{z-3}{k}$, where *k* is a constant. It is given that l_1 and l_2 intersect.

(ii) Find the value of *k*. [4]

(iii) Show that l_1 lies in *p* and find the coordinates of the point at which l_2 intersects *p*. [4]

(iv) Find the acute angle between l_2 and *p*. [3]

9 2013(9740)/II/4 (modified)

The planes p_1 and p_2 have equations $\mathbf{r} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 1$ and $\mathbf{r} \bullet \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix} = -1$ respectively, and meet in the line *l*.

(i) Find the shortest distance from the origin to plane p_2. [2]

(ii) Find the acute angle between p_1 and p_2. [3]

(iii) Find a vector equation for *l*. [4]

(iv) The point $A(4, 3, c)$ is equidistant from the planes p_1 and p_2. Calculate the two possible values of *c*. [6]

10 2016(9740)/I/11

The plane *p* has equation $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} a \\ 4 \\ -2 \end{pmatrix}$, and the line *l* has equation $\mathbf{r} = \begin{pmatrix} a-1 \\ a \\ a+1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$, where *a* is a constant and λ, μ and *t* are parameters.

(i) In the case where $a = 0$,

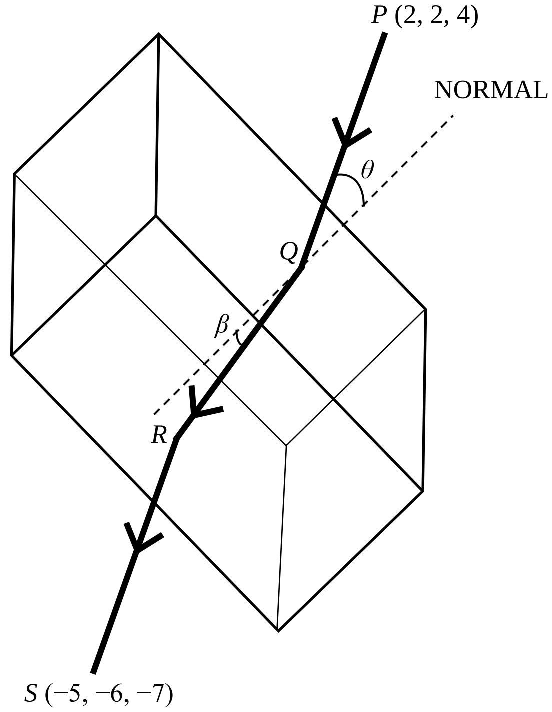
(a) show that *l* is perpendicular to *p* and find the values of λ, μ and *t* which give the coordinates of the point at which *l* and *p* intersect, [5]

(b) find the cartesian equations of the planes such that the perpendicular distance from each plane to *p* is 12. [5]

(ii) Find the value of *a* such that *l* and *p* do not meet in a unique point. [3]

Level 3

11 2019(9758)/I/12



A ray of light passes from air into a material made into a rectangular prism. The ray of light is sent in direction $\begin{pmatrix} -2 \\ -3 \\ -6 \end{pmatrix}$ from a light source at the point P with coordinates (2, 2, 4). The prism is placed so that the ray of light passes through the prism, entering at the point Q and emerging at the point R and is picked up by a sensor at point S with coordinates (−5, −6, −7). The acute angle between PQ and the normal to the top of the prism at Q is θ and the acute angle between QR and the same normal is β (see diagram).

It is given that the top of the prism is a part of the plane $x+y+z=1$, and that the base of the prism is a part of the plane $x+y+z=-9$. It is also given that the ray of light along PQ is parallel to the ray of light along RS so that P, Q, R and S lie in the same plane.

(i) Find the exact coordinates of Q and R. [5]

(ii) Find the values of $\cos\theta$ and $\cos\beta$. [3]

(iii) Find the thickness of the prism measured in the direction of the normal at Q. [3]

Snell's law states that $\sin\theta = k\sin\beta$, where k is a constant called the refractive index.

(iv) Find k for the material of this prism. [1]

(v) What can be said about the value of k for a material for which $\beta > \theta$? [1]

12 2017(9758)/I/6

(i) Interpret geometrically the vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and t is a parameter. [2]

(ii) Interpret geometrically the vector equation $\mathbf{r} \cdot \mathbf{n} = d$, where $\mathbf{n}$ is a constant unit vector and d is a constant scalar, stating what d represents. [3]

(iii) Given that $\mathbf{b} \cdot \mathbf{n} \neq 0$, solve the equations $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{r} \cdot \mathbf{n} = d$ to find $\mathbf{r}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ and d. Interpret the solution geometrically. [3]

13 2008/HCI/I/12(b) (modified)

Referring to the origin *O*, two planes Π_1 and Π_2 are given by

$$\Pi_1 : \ \mathbf{r} \bullet \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = 13 \text{ and } \Pi_2 : \ \mathbf{r} \bullet \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} = -8 .$$

Find an equation of the plane which is the image of Π_2 when Π_2 is reflected in Π_1. [9]

Answer Key
1(a) $x = \lambda + \mu, y = 1 - 4\lambda - \mu, z = 1 + \lambda$, where $\lambda, \mu \in \mathbb{R}$; $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 4$; $x + y + 3z = 4$; $\left(\frac{7}{5}, -4, \frac{11}{5}\right)$
1(b) $x = 3 + 2\mu, y = -1 + 2\lambda + 4\mu, z = 3 + \lambda + 3\mu$, where $\lambda, \mu \in \mathbb{R}$; $\mathbf{r}.\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = -4$; $x + y - 2z = -4$; $\left(\frac{11}{10}, -\frac{5}{2}, \frac{13}{10}\right)$
1(c) $x = 1 + 2\lambda - \mu, y = -2 + 3\lambda - 7\mu, z = -4 - 6\lambda + 2\mu$, where $\lambda, \mu \in \mathbb{R}$; $\mathbf{r} \bullet \begin{pmatrix} 36 \\ -2 \\ 11 \end{pmatrix} = -4$; $36x - 2y + 11z = -4$; $\left(\frac{24}{79}, \frac{117}{79}, -\frac{86}{79}\right)$
2 (0, 0, 1)
3 $\mathbf{r} = \begin{pmatrix} 22/13 \\ 32/13 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 11 \\ 13 \end{pmatrix}, \lambda \in \mathbb{R}$
4 (a) 29.3° **(b)** 35.5°

5 $\frac{1}{3}\begin{pmatrix}-10\\1\\4\end{pmatrix}$
6 perpendicular distance from *A* to plane is 4
7(i) $\mathbf{r}\bullet\begin{pmatrix}3\\-2\\1\end{pmatrix}=6$ **(ii)** $\frac{1}{2}\begin{pmatrix}5\\-2\\-7\end{pmatrix}$ **(iii)** $S(1,0,4)$ **(iv)** $\mathbf{r}=\begin{pmatrix}1\\-1\\1\end{pmatrix}+\alpha\begin{pmatrix}3\\-1\\-4\end{pmatrix}, \alpha\in\mathbb{R}$
8(i) $x+y+2z=-3$ **(ii)** $k=-7$ **(iii)** $(-1,6,-4)$ **(iv)** 22.2°
9(i) $\frac{1}{7}$ **(ii)** 40.4° **(iii)** $\mathbf{r}=\begin{pmatrix}-\frac{1}{6}\\-\frac{2}{3}\\0\end{pmatrix}+\lambda\begin{pmatrix}\frac{7}{6}\\\frac{5}{3}\\1\end{pmatrix},\lambda\in\mathbb{R}$ **(iv)** -49 and $\frac{35}{13}$
10(ia) $\lambda=-\frac{8}{9},\ \mu=\frac{19}{18}$ and $t=-\frac{5}{9}$ **(ib)** $-2x+y+2z=35$ and $-2x+y+2z=-37$ **(ii)** $\frac{9}{2}$
11(i) $Q\left(\frac{8}{11},\frac{1}{11},\frac{2}{11}\right),\ R\left(-\frac{37}{11},-\frac{39}{11},-\frac{23}{11}\right)$ **(ii)** $\frac{11}{21}\sqrt{3}$ **,** $\frac{11}{255}\sqrt{510}$ **(iii)** $\frac{10}{3}\sqrt{3}$ **(iv)** 1.86 **(v)** $0<k<1$
12(iii) $\mathbf{r}=\mathbf{a}+\left(\frac{d-\mathbf{a.n}}{\mathbf{b.n}}\right)\mathbf{b}$
13 $\mathbf{r}=\begin{pmatrix}-3\\-1\\6\end{pmatrix}+\lambda\begin{pmatrix}18\\-7\\1\end{pmatrix}+\mu\begin{pmatrix}29\\-10\\-3\end{pmatrix}$ where λ and $\mu\in\mathbb{R}$

H2 Mathematics (9758)
Chapter 6 3D Vector Geometry
Extra Practice Questions

1 Show that the line $l: \mathbf{r} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$, $\lambda \in \mathbb{R}$, is parallel to the plane $\pi: x + 5y + z = 5$.

Find the shortest distance between the line l and the plane π.

2 **2007/IJC/I/5**

The position vectors of the points A, B, and C are given by $\mathbf{i}+\mathbf{j}+\mathbf{k}$, $4\mathbf{i}+3\mathbf{j}+2\mathbf{k}$ and $-7\mathbf{i}-2\mathbf{j}-\mathbf{k}$ respectively.

(i) Prove that the points A, B and C are not collinear. [2]

(ii) Find a vector which is perpendicular to the plane ABC. [2]

(iii) Deduce the exact length of projection of $\overrightarrow{PQ}$ on the plane ABC, given that $\overrightarrow{OP} = 2\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$ and $\overrightarrow{OQ} = 4\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$. [3]

3 **2007/PJC/I/6**

The line l_1 has equation $\mathbf{r} = \begin{pmatrix} p \\ 3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$. The line l_2 passes through points A and B with position vectors $3\mathbf{j}+3\mathbf{k}$ and $q\mathbf{i}+3\mathbf{j}+5\mathbf{k}$ respectively, where p and q are constants.

(i) If $q = 1$ and $p = 4$, find the position vector of the point C on l_1 such that A is the foot of perpendicular from C to l_2. [4]

(ii) If the acute angle between l_1 and l_2 is $60°$, find the possible value(s) of q. [3]

4 **2008/JJC/I/3**

The lines l_1 and l_2 have equations

$$l_1: \ \mathbf{r} = (\mathbf{i}+3\mathbf{j}+2\mathbf{k}) + \lambda(\mathbf{i}-\mathbf{j}+7\mathbf{k}),$$

$$l_2: \ x = 1, \ \frac{y-3}{-4} = \frac{z-2}{3},$$

where λ is a real parameter. The point P lies on the line l_1 with position vector $a\mathbf{i}+\mathbf{j}+16\mathbf{k}$. The point Q lies on the line l_2 such that PQ is perpendicular to the line l_2.

(i) Prove that a = 3. [1]

(ii) Find the position vector of the point Q. [3]

(iii) Find, in degrees, the acute angle between the lines l_1 and l_2. [2]

5 2008/HCI/I/12b

Referring to the origin *O*, two planes Π_1 and Π_2 are given by

$$\Pi_1 : \mathbf{r} \bullet \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = 13 \qquad \text{and} \qquad \Pi_2 : \mathbf{r} \bullet \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} = -8 .$$

(i) Given that a point $A(1,7,-10)$ lies on Π_2, show that the perpendicular distance from *A* to Π_1 is $2\sqrt{21}$. [2]

(ii) Hence or otherwise find $\overrightarrow{OB}$ where *B* is the image of *A* when reflected in the plane Π_1. [2]

(iii) Write down the Cartesian equations of both Π_1 and Π_2. [1]
Find a vector equation of the line of intersection of Π_1 and Π_2. [1]

(iv) Find a vector equation of the plane which is the image of Π_2 when Π_2 is reflected in Π_1. [3]

6 2010 DHS Prelim/P2/Q4

The planes Π_1 and Π_2 are defined by

$$\Pi_1 : \mathbf{r} \bullet \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = 10, \qquad \Pi_2 : \mathrm{r} \bullet \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = 8.$$

(i) Find the acute angle between the two planes. [3]

(ii) Obtain a vector equation of l_1, the line of intersection of the two planes. [4]

The Cartesian equation of another line, l_2, is given by $\dfrac{x-2}{6} = \dfrac{7-z}{3}, y = m,$ where *m* is a real constant.

(iii) If the plane Π_1 and line l_2 intersect at the point (6, *m*, 5), find the value of *m*. [2]

(iv) Show that the lines l_1 and l_2 are perpendicular for all values of *m*. [2]

7 2009/CJC/I/11 (Modified)

The planes p_1 and p_2, which meet in the line l, have vector equations

$$\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

$$\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

respectively, where $\lambda_1, \lambda_2, \mu_1$ and μ_2 are real constants.

(i) Show that l is parallel to the vector $5\mathbf{i} + 6\mathbf{j} + \mathbf{k}$. [3]

(ii) Calculate the acute angle between p_1 and p_2. [2]

(iii) Find, in exact form, the perpendicular distance from the point with coordinates (4, 2, 2) to p_2. [2]

The plane p_3 has equation $ax - 2y + 2z = b$, where $a, b \in \mathbb{R}$.

(iv) Given that $a = 2$, find the values of b, such that the distance between the planes p_1 and p_3 is $\frac{1}{\sqrt{3}}$ units. [3]

8 2009/MJC/I/9

The equations of two planes Π_1 and Π_2,

$$x + 3y + az = 8,$$
$$3x + y + bz = 0.$$

respectively, where a, b, p and q are constants.

(i) If the point $(\alpha, \beta, 0)$ lies on both Π_1 and Π_2, find the values of α and β. [2]

(ii) Given that the line l_1 lies on both Π_1 and Π_2, find a vector equation of l in terms of a and b. [2]

The line l_2 has equation $\mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$, where μ is a parameter.

(iii) Given that l_1 and l_2 intersect at a point and $a + b = 0$, find the values of a and b. [4]

9 2014 SRJC P2 Q1

The line l with Cartesian equation $\frac{4-x}{4}=\frac{z}{3}$, $y=1$ contains the point B with position vector $\mathbf{j}+3\mathbf{k}$.

A point A, not lying on l, has position vector $2\mathbf{i}+\left(1+\sqrt{5}\right)\mathbf{j}-\mathbf{k}$.

(i) Given that $\mathbf{c}$ denotes a unit vector parallel to l, find $\left|\overrightarrow{AB}\bullet\mathbf{c}\right|$ and give a geometrical interpretation of this quantity. [3]

(ii) Hence find the shortest distance from A to l. [2]

The foot of perpendicular from A to l is denoted by F and the foot of perpendicular from F to AB is denoted by G.

(iii) Write down the ratio between the area of ΔAGF and area of ΔBGF. [1]

(iv) Hence, deduce the ratio $AG:GB$ and find the position vector of G. [2]

Answer Key

No	**Year**	**JC/CI**	**Answers**
1			$\frac{10}{9}\sqrt{3}$
2	**2007**	**IJC**	(ii) $\begin{pmatrix}1\\2\\-7\end{pmatrix}$ (iii) $\frac{\sqrt{14}}{2}$
3	**2007**	**PJC**	(i) $\overrightarrow{OC}=-8\mathbf{i}-9\mathbf{j}+7\mathbf{k}$ (ii) $q=\pm 2$
4	**2008**	**JJC**	(ii) $\overrightarrow{OQ}=\begin{pmatrix}1\\-5\\8\end{pmatrix}$ (iii) 45.6°
5	**2008**	**HCI**	(bii) $\overrightarrow{OB}=\begin{pmatrix}-3\\-1\\6\end{pmatrix}$; (biii) Π_1 : $x + 2y - 4z = 13$; Π_2 : $x + 3y + 3z = -8$; $l:\mathbf{r}=\begin{pmatrix}55\\-21\\0\end{pmatrix}+\lambda\begin{pmatrix}18\\-7\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$ (biv) $\mathbf{r}=\begin{pmatrix}-3\\-1\\6\end{pmatrix}+\alpha\begin{pmatrix}18\\-7\\1\end{pmatrix}+\beta\begin{pmatrix}29\\-10\\-3\end{pmatrix}$ where α and $\beta\in\mathbb{R}$

6	**2010**	**DHS**	(i) 9.3° (ii) $\mathbf{r}=\begin{pmatrix}-1\\3\\0\end{pmatrix}+\alpha\begin{pmatrix}1\\-1\\2\end{pmatrix}$ (iii) $m=-\dfrac{7}{4}$
7	**2009**	**CJC**	(ii) 75.7° (iii) $\sqrt{22}$ (iv) b = 6 or 10
8	**2009**	**MJC**	(i) $\alpha=-1,\beta=3$ (ii) $l_1:\mathbf{r}=\begin{pmatrix}-1\\3\\0\end{pmatrix}+\lambda\begin{pmatrix}3b-a\\3a-b\\-8\end{pmatrix},\lambda\in\mathbb{R}$ (iii) $a=2,\quad b=-2$
9	**2014**	**SRJC**	(i) 4 (ii) 3 (iii) $\dfrac{9}{16}$ (iv) $\dfrac{1}{25}\begin{pmatrix}32\\25+16\sqrt{5}\\11\end{pmatrix}$