Jurong Pioneer Junior College
H2 Mathematics **JC2-2023**
Prelim Paper 1 Solution

1

(i)

$u_n = 2u_{n-1} - 15n^2 + 60n + A$

$u_2 = 2u_1 - 15(2)^2 + 60(2) + A$

$4 = 2(2) - 60 + 120 + A \Rightarrow A = -60$

$u_3 = 2u_2 - 15(3)^2 + 60(3) - 60 = 2(4) - 135 + 180 - 60 = -7$

(ii)

Given $u_1 = 2$, thus $2 = 4p - q + r$ ----Equation (1)

Given $u_2 = 4$, thus $4 = 8p - 4q + r$ ---Equation (2)

Given $u_3 = -7$, thus $-7 = 16p - 9q + r$ ---Equation (3)

Solving, $p = -\dfrac{43}{4},\ q = -15,\ r = 30$

2

(i)

$u_n = \ln\left(\dfrac{n}{n+1}\right) = \ln\left(1 - \dfrac{1}{n+1}\right)$

As $n \to \infty,\ \dfrac{1}{n+1} \to 0,\ u_n \to \ln 1 = 0$ (finite value)

Hence, sequence is convergent.

(ii)

Method 1

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{N} \ln n - \ln(n+1)$$

$$\begin{aligned}
&= \ln 1 - \cancel{\ln 2} \\
&\quad + \cancel{\ln 2} - \cancel{\ln 3} \\
&\quad + \cancel{\ln 3} - \cancel{\ln 4} \\
&\quad + \cancel{\ldots\ldots} \\
&\quad + \cancel{\ln(N-2)} - \cancel{\ln(N-1)} \\
&\quad + \cancel{\ln(N-1)} - \cancel{\ln N} \\
&\quad + \cancel{\ln N} - \ln(N+1) \\
&= \ln 1 - \ln(N+1) \\
&= -\ln(N+1)
\end{aligned}$$

Method 2

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} \ln\left(\frac{n}{n+1}\right)$$
$$= \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) + \ldots\ldots + \ln\left(\frac{N}{N-1}\right) + \ln\left(\frac{N-1}{N}\right) + \ln\left(\frac{N}{N+1}\right)$$
$$= \ln\left[\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\ldots\ldots\left(\frac{N}{N-1}\right)\left(\frac{N-1}{N}\right)\left(\frac{N}{N+1}\right)\right]$$
$$= \ln\left(\frac{1}{N+1}\right)$$
$$= -\ln(N+1)$$

3

$y = 1 - \dfrac{1}{x}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^{-2}$

At $x = a$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{a^2}, y = 1 - \dfrac{1}{a}$

Tangent at $x = a$

$y - 1 + \dfrac{1}{a} = \dfrac{1}{a^2}(x - a)$

$a^2 y - a^2 + a = x - a$

$a^2 y - x = a^2 - 2a$ (Shown)

If l is parallel to $9y - x = 1$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{a^2} = \dfrac{1}{9}$

$a = -3$ (reject as $a > 0$) or 3

When $a = 3$,

$3^2 y - x = 3^2 - 2(3)$

$9y - x = 3$

4

(i)

$y = \dfrac{x^2 + x + 3}{x - 2}$

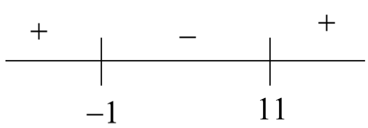
$yx - 2y = x^2 + x + 3$

$x^2 + (1 - y)x + 3 + 2y = 0$

Consider Discriminant ≥ 0

$(1 - y)^2 - 4(1)(3 + 2y) \geq 0$

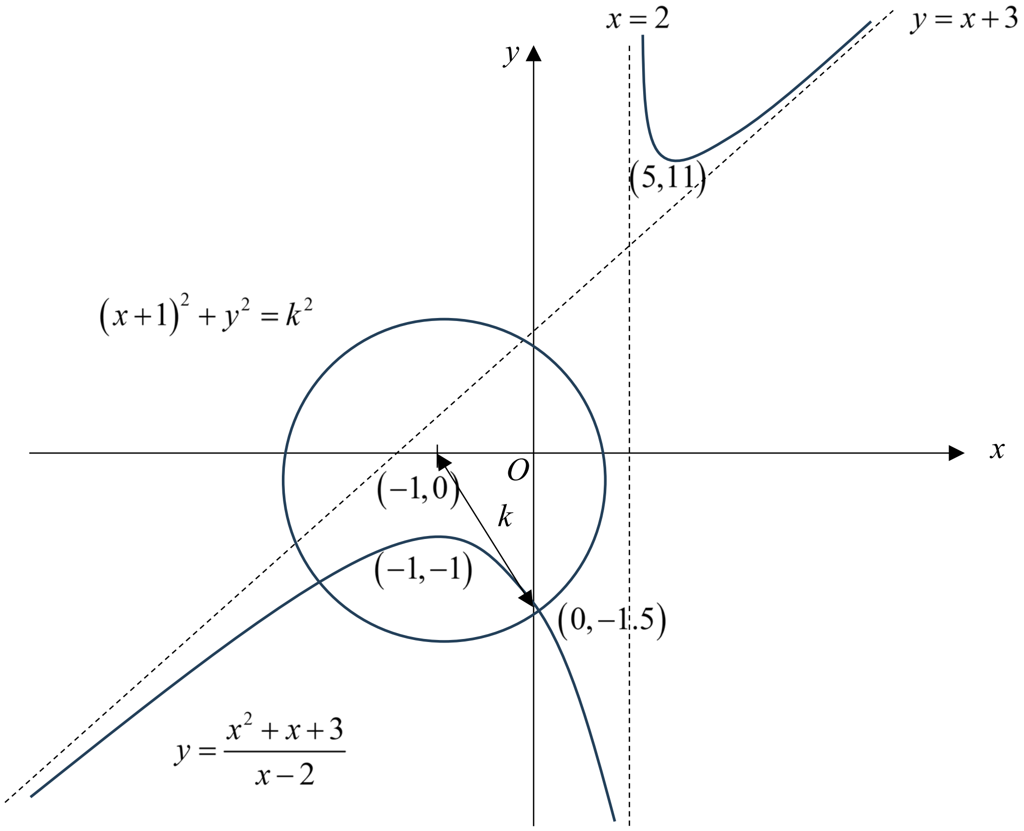
$y^2 - 10y - 11 \geq 0$



$(y+1)(y-11) \geq 0$

$y \leq -1$ or $y \geq 11$

(ii)



(iii)

$(x+1)^2(x-2)^2+\left(x^2+x+3\right)^2=k^2\left(x-2\right)^2$

$(x+1)^2+\left(\dfrac{x^2+x+3}{x-2}\right)^2=k^2$

$(x+1)^2+y^2=k^2$

For $(x+1)^2(x-2)^2+\left(x^2+x+3\right)^2=k^2\left(x-2\right)^2$ to have at least 1 positive real root,

Since $k>0$, $k>\sqrt{\left(-1-0\right)^2+\left(0-(-1.5)\right)^2}$

$k>\sqrt{3.25}$

$k>1.803$ (3 d.p)

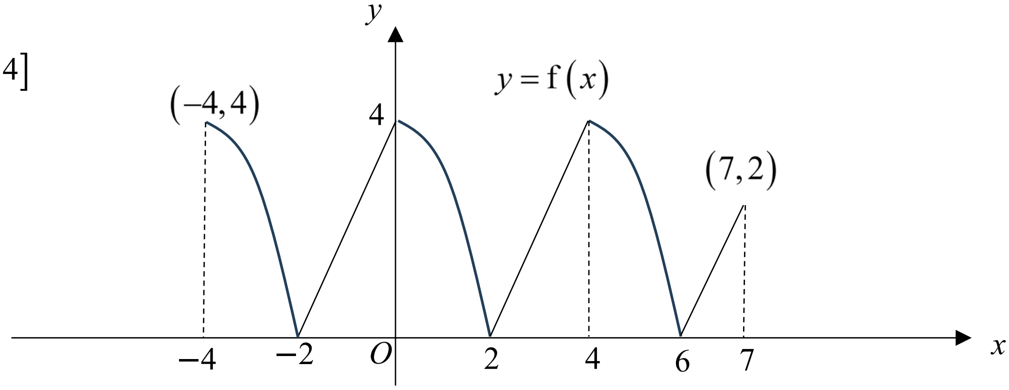
5

(i)

$\mathrm{f}(45)=\mathrm{f}(41)=\mathrm{f}(37)=......=\mathrm{f}(1)=4-(1)^2=3$

(ii)

$R_f=[0,4]$



(iii)

$R_f=[0,4]\not\subset D_g=(0,1)$
Hence, gf does not exist.

(iv)

$g: x \mapsto (x-1)^2+2 \quad \text{for } x\in\mathbb{R},\ 0<x<1.$
$R_g=(2,3)$
$\mathrm{fg}(x)=2\left[(x-1)^2+2\right]-4=2(x-1)^2$

Let $(\mathrm{fg})^{-1}(\frac{1}{2})=\alpha$

$\mathrm{fg}(\alpha)=\frac{1}{2}$

$2(\alpha-1)^2=\frac{1}{2}$

$(\alpha-1)^2=\frac{1}{4}$

$(\alpha-1)=\frac{1}{2} \quad \text{or} \quad -\frac{1}{2}$

$\alpha=\frac{3}{2} \quad \text{or} \quad \frac{1}{2}$

Since $D_{fg}=D_g=(0,1)$, $\alpha=\frac{1}{2}$

Alternative Method

$y = 2(x-1)^2,\ 0 < x < 1$

$(x-1)^2 = \dfrac{y}{2}$

$(x-1) = \pm\sqrt{\dfrac{y}{2}}$

$x = 1 \pm \sqrt{\dfrac{y}{2}}$

Since $0 < x < 1,\ x = 1 - \sqrt{\dfrac{y}{2}}$

$(\mathrm{fg})^{-1}(x) = 1 - \sqrt{\dfrac{x}{2}}$

$(\mathrm{fg})^{-1}(\dfrac{1}{2}) = 1 - \sqrt{\dfrac{\left(\dfrac{1}{2}\right)}{2}} = \dfrac{1}{2}$

6
(i)

Given p: $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 12$.

Equation of the line AN: $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ where $\lambda \in \mathbb{R}$.

Then $\overrightarrow{ON} = \begin{pmatrix} 2+\lambda \\ 1-2\lambda \\ -3+2\lambda \end{pmatrix}$ for a specific value of λ.

Since the point N lies on the plane p, $\begin{pmatrix} 2+\lambda \\ 1-2\lambda \\ -3+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 12$

$2 + \lambda - 2(1-2\lambda) + 2(-3+2\lambda) = 12$

Solving, $\lambda = 2$

Thus $\overrightarrow{ON} = \begin{pmatrix} 2+2 \\ 1-4 \\ -3+4 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$. The coordinates of the point N is $(4, -3,\ 1)$

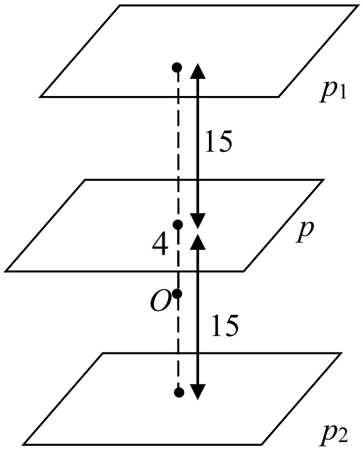
Let A' be the point of reflection of A.

By Ratio Theorem, $\overrightarrow{ON} = \dfrac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$

$$\overrightarrow{OA'} = 2\overrightarrow{ON} - \overrightarrow{OA} = 2\begin{pmatrix}4\\-3\\1\end{pmatrix} - \begin{pmatrix}2\\1\\-3\end{pmatrix} = \begin{pmatrix}6\\-7\\5\end{pmatrix}$$

The coordinates of the reflection of A in p is $(6, -7,\ 5)$.

(ii)



For plane p: $\dfrac{\mathbf{r}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix}}{3} = \dfrac{12}{3}$

Distance from O to the plane $p = \dfrac{12}{3} = 4$ $\quad$ (since $\left\|\begin{pmatrix}1\\-2\\2\end{pmatrix}\right\| = 3$)

$\therefore$ The two planes with perpendicular distance 15 to p are

$\dfrac{\mathbf{r}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix}}{3} = 4 - 15 = -11$ and $\dfrac{\mathbf{r}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix}}{3} = 4 + 15 = 19$

Cartesian equations are $x - 2y + 2z = -33$ and $x - 2y + 2z = 57$

7

(a)

$$\int \frac{2x^3}{1+x^4}\,\mathrm{d}x$$
$$= \frac{1}{2}\int \frac{4x^3}{1+x^4}\,\mathrm{d}x$$
$$= \frac{1}{2}\ln\left|1+x^4\right| + c$$
$$= \frac{1}{2}\ln\left(1+x^4\right) + c \qquad \text{since } 1+x^4 > 0$$

(b)

$$\int \cos 4x \sin 10x \text{ d}x$$
$$= \int \sin 10x \cos 4x \text{ d}x$$
$$= \frac{1}{2}\int \sin 14x + \sin 6x \text{ d}x$$
$$= -\frac{1}{2}\left(\frac{\cos 14x}{14} + \frac{\cos 6x}{6}\right) + c$$

(c)

$$x = \tan\theta$$
$$\frac{\text{d}x}{\text{d}\theta} = \sec^2\theta$$
$$x = 1, \theta = \frac{\pi}{4}$$
$$x = 0, \theta = 0$$

$$\int_0^1 \frac{x^3}{\left(x^2+1\right)^3} \text{ d}x$$
$$= \int_0^{\frac{\pi}{4}} \frac{\tan^3\theta}{\left(\tan^2\theta+1\right)^3} \sec^2\theta \text{ d}\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{\tan^3\theta}{\left(\sec^2\theta\right)^3} \sec^2\theta \text{ d}\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{\tan^3\theta}{\sec^4\theta} \text{ d}\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{\sin^3\theta\cos^4\theta}{\cos^3\theta} \text{ d}\theta$$
$$= \int_0^{\frac{\pi}{4}} \cos\theta\sin^3\theta \text{ d}\theta \text{ (Shown)}$$

$$\int_0^{\frac{\pi}{4}} \cos\theta\sin^3\theta \text{ d}\theta$$
$$= \left[\frac{\sin^4\theta}{4}\right]_0^{\frac{\pi}{4}} = \frac{1}{16}$$

8

(i)

For the sprinter: 56 + 62 + 68 + … + $[56+(n-1)(6)]$
AP: First term = 56, common difference = 6

The time the sprinter takes to complete n laps $= \frac{n}{2}\left[2(56)+(n-1)(6)\right]$
$= n\left(56+3n-3\right) = 3n^2+53n$ (Shown)

(ii)

For the marathon runner: 60 + $60(1.04)+60(1.04)^2+60(1.04)^3+...+60(1.04)^{n-1}$

GP: First term = 60, common ratio = 1.04

The time the marathon runner takes to complete n laps = $\frac{60\left(1-1.04^n\right)}{1-1.04}$ or $\frac{60\left(1.04^n-1\right)}{1.04-1}$

$$=1500\left(1.04^n-1\right)$$

(iii)

Consider $1500\left(1.04^n-1\right)>2400$

Method 1

$$1.04^n-1>1.6$$
$$1.04^n>2.6$$

Taking lg on both sides: $n>\dfrac{\lg 2.6}{\lg 1.04}$

$\therefore\ n>24.362$ (correct to 5 s.f.)

The marathon runner needs to complete 25 laps.

Method 2

n	$1500\left(1.04^n-1\right)$	
24	2345	<2400
25	2498.8	>2400
26	2658.7	>2400

The marathon runner needs to complete 25 laps.

(iv)

$$n=\frac{12000}{400}=30$$

Time taken by the sprinter = $3(30)^2+53(30)=4290$ seconds

Time taken by the marathon runner = $1500\left(1.04^{30}-1\right)=3365.1$ seconds (correct to 5 s.f.)

The marathon runner will be the first to complete the race.

(v)

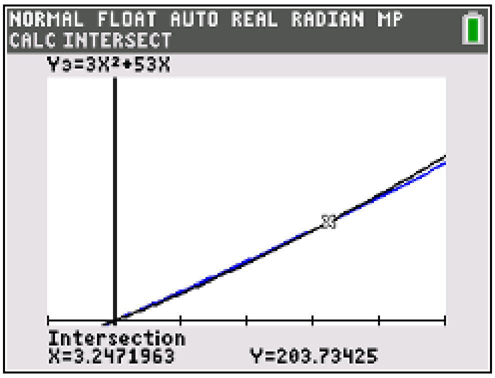
Consider $1500\left(1.04^n-1\right)<3n^2+53n$

Method 1

n	$1500\left(1.04^n-1\right)$	$3n^2+53n$
3	187.3	186
4	254.79	260
5	324.98	340

Method 2

(Sketch $Y_1=1500\left(1.04^X-1\right)$, $Y_2=3X^2+53X$)



Since $n \geq 3.25$ (3 s.f.), $\therefore\ n = 4$
Hence, the marathon runner first overtakes the sprinter on his 4th lap.

9
(a)(i)

<u>Method 1</u>
$(k\mathrm{i})^4 - 4(k\mathrm{i})^3 + a(k\mathrm{i})^2 - 16(k\mathrm{i}) + b = 0$
$k^4 + 4k^3\mathrm{i} - ak^2 - 16k\mathrm{i} + b = 0$
$\left(k^4 - ak^2 + b\right) + \left(4k^3 - 16k\right)\mathrm{i} = 0$
Comparing real and imaginary parts,
$4k^3 - 16k = 0$
$k^2 = 4 \quad (k \neq 0)$
$k = \pm 2$
The two roots are 2i and −2i.

$k^4 - ak^2 + b = 0$
$(4)^2 - a(4) + b = 0$
$16 - 4a + b = 0$ (shown)
<u>Method 2</u>
Since the coefficients of the polynomial are all real, as $z = k\mathrm{i}$ is a root, its conjugate $z = -k\mathrm{i}$ is also a root.
$x^4 - 4x^3 + ax^2 - 16x + b = (x - k\mathrm{i})(x + k\mathrm{i})(x^2 + cx + d)$
$= (x^2 + k^2)(x^2 + cx + d)$
Comparing constant, $b = dk^2 \Rightarrow d = \dfrac{b}{k^2}$ …(1)
Comparing coefficient of x, $-16 = ck^2$ …(2)
Comparing coefficient of x^2, $a = d + k^2$ … (3)
Comparing coefficient of x^3, $-4 = c$ … (4)

Substitute (4) into (2),
$-16 = -4k^2$

$k^2 = 4$

$k = \pm 2$

The two roots are 2i and −2i.

Substitute (1) and $k^2 = 4$ into (3),

$a = \frac{b}{4} + 4$

$4a = b + 16$

$16 - 4a + b = 0$ (shown)

(ii)

Sub (0, 20) into $\mathrm{f}(x) = x^4 - 4x^3 + ax^2 - 16x + b$, $b = 20$

$16 - 4a + 20 = 0$

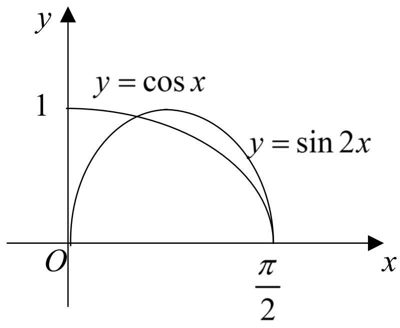
$a = 9$

$$\begin{aligned} \therefore \mathrm{f}(x) &= x^4 - 4x^3 + 9x^2 - 16x + 20 \\ &= (x - 2\mathrm{i})(x + 2\mathrm{i})(x^2 + cx + d) \\ &= (x^2 + 4)(x^2 - 4x + 5) \end{aligned}$$

(b)

$$\begin{aligned} \left| \frac{\mathrm{z}\left(\mathrm{i} - z^*\right)}{3\mathrm{i}\left(z^2 + \mathrm{i}z\right)} \right| &= \frac{|\mathrm{z}|\left|\mathrm{i} - z^*\right|}{|3\mathrm{i}|\left|(z)(z + \mathrm{i})\right|} \\ &= \frac{|\mathrm{z}|\left|\mathrm{i} - z^*\right|}{3|z||z + \mathrm{i}|} \\ &= \frac{\left|\mathrm{i} - (a - b\mathrm{i})\right|}{3\left|a + b\mathrm{i} + \mathrm{i}\right|} \qquad \text{let } z = a + b\mathrm{i} \Rightarrow z^* = a - b\mathrm{i} \\ &= \frac{\left|-a + (b + 1)\mathrm{i}\right|}{3\left|a + (b + 1)\mathrm{i}\right|} \\ &= \frac{1}{3} \end{aligned}$$

10(i)



(ii)

$\sin 2x > \cos x$

Consider $\sin 2x = \cos x$

$\sin 2x - \cos x = 0$

$2\sin x \cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ or $2\sin x - 1 = 0$

$x = \dfrac{\pi}{2}$ $\quad$ $\sin x = \dfrac{1}{2}$

$x = \dfrac{\pi}{6}$

From sketch, $\dfrac{\pi}{6} < x < \dfrac{\pi}{2}$

(iii)

$$\int_0^{\frac{\pi}{2}} \left|\sin 2x - \cos x\right| \, dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin 2x - \cos x \, dx - \int_0^{\frac{\pi}{6}} \sin 2x - \cos x \, dx$$

$$= \left[-\frac{1}{2}\cos 2x - \sin x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \left[-\frac{1}{2}\cos 2x - \sin x \right]_0^{\frac{\pi}{6}}$$

$$= \left(\frac{1}{2} - 1 + \frac{1}{4} + \frac{1}{2} \right) - \left(-\frac{1}{4} - \frac{1}{2} + \frac{1}{2} - 0 \right)$$

$$= \frac{1}{2}$$

(iv)

$$V=\pi\int_0^{\frac{\pi}{6}}\cos^2 x-\sin^2 2x\ \mathrm{d}x$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{6}}\cos 2x+1-\left(1-\cos 4x\right)\ \mathrm{d}x$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{6}}\cos 2x+\cos 4x\ \mathrm{d}x$$

$$=\frac{\pi}{2}\left[\frac{\sin 2x}{2}+\frac{\sin 4x}{4}\right]_0^{\frac{\pi}{6}}$$

$$=\frac{\pi}{2}\left[\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{8}\right]$$

$$=\frac{3\sqrt{3}}{16}\pi$$

11

(i)

$h^2+r^2=8^2$

$r^2=64-h^2$

$V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(64-h^2\right)h=\frac{1}{3}\pi\left(64h-h^3\right)$

$\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{1}{3}\pi\left(64-3h^2\right)$

At Stat point, $\frac{\mathrm{d}V}{\mathrm{d}h}=0$

$64-3h^2=0$

$h^2=\frac{64}{3}$

$h=\frac{8\sqrt{3}}{3}$ (since $h>0$)

$\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=-2\pi h$

When $h=\frac{8\sqrt{3}}{3},\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=-2\pi\left(\frac{8\sqrt{3}}{3}\right)=-\frac{16}{3}\sqrt{3}\pi<0$

$\therefore h=\frac{8\sqrt{3}}{3}$ will give a maximum V

$\therefore V=\frac{1}{3}\pi\left[64\left(\frac{8\sqrt{3}}{3}\right)-\left(\frac{8\sqrt{3}}{3}\right)^3\right]=\pi\left(\frac{8\sqrt{3}}{9}\right)\left(64-\left(\frac{8\sqrt{3}}{3}\right)^2\right)$

$=\frac{1024\sqrt{3}}{27}\pi$

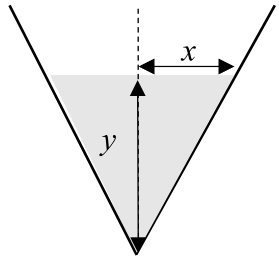
(ii)

$$r^2 = 64 - h^2 = 64 - \frac{64}{3} = \frac{128}{3}$$

$$\frac{h}{r} = \sqrt{\frac{64}{3}} \div \sqrt{\frac{128}{3}} = \frac{1}{\sqrt{2}}$$

$r = \sqrt{2}h$ (shown)

(iii)



Let the height and volume of the water be y cm and V' cm^3 respectively.
By similar triangle,

$$\frac{y}{x} = \frac{h}{r}$$

$$x = \sqrt{2}y$$

$$V' = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi\left(\sqrt{2}y\right)^2 y = \frac{2}{3}\pi y^3$$

$$\frac{dV'}{dy} = 2\pi y^2$$

When $t = 6$, $\quad V' = 18\pi$

$$18\pi = \frac{2}{3}\pi y^3$$

$$y^3 = 27$$

$$y = 3$$

$$\frac{dV'}{dt} = \frac{dV'}{dy} \times \frac{dy}{dt}$$

$$3\pi = 2\pi(3)^2 \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{6} \text{cm/s}$$