Check your Understanding (Techniques of Differentiation)

For Questions1 to 6, differentiate the expressions with respect to x.

1 (a) $5x^3+6x^2-7$

(b) $3x^4-5x+\dfrac{1}{x^2}$

(c) $\sqrt{x}-\dfrac{3}{x}$

(d) $x^4-\dfrac{1}{2}x^2+\dfrac{5}{\sqrt{x}}$

1a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(5x^3+6x^2-7\right)$

$=5(3)x^2+6(2)x$

$=15x^2+12x$

1b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(3x^4-5x+x^{-2}\right)$

$=3(4)x^3-5+(-2)x^{-3}$

$=12x^3-5-\dfrac{2}{x^3}$

1c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{2}}-3x^{-1}\right)$

$=\dfrac{1}{2}x^{-\frac{1}{2}}+3x^{-2}$

$=\dfrac{1}{2\sqrt{x}}+\dfrac{3}{x^2}$

1d) Let $y=x^4-\dfrac{1}{2}x^2+\dfrac{5}{\sqrt{x}}$

$=x^4-\dfrac{1}{2}x^2+5x^{-\frac{1}{2}}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=4x^3-x-\dfrac{5}{2}x^{-\frac{3}{2}}$

$=4x^3-x-\dfrac{5}{2\sqrt{x^3}}$

2 (a) $(2x-1)^8$

(b) $\dfrac{2}{3}\left(x^4-3x^2+5\right)^3$

(c) $\sqrt{3x^2-1}$

(d) $\dfrac{1}{\sqrt{x-1}}$

(e) $\dfrac{1}{x^2+1}$

(f) $\sqrt[4]{2+3\sqrt{x}}$

2a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(2x-1)^8$

$=8(2x-1)^7(2)$

$=16(2x-1)^7$

2b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{2}{3}\left(x^4-3x^2+5\right)^3\right]$

$=\dfrac{2}{3}(3)\left(x^4-3x^2+5\right)^2(4x^3-6x)$

$=2\left(4x^3-6x\right)\left(x^4-3x^2+5\right)^2$

$=4x\left(2x^2-3\right)\left(x^4-3x^2+5\right)^2$

2c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(3x^2-1\right)^{\frac{1}{2}}$

$=\dfrac{1}{2}\left(3x^2-1\right)^{-\frac{1}{2}}(6x)$

$=3x\left(3x^2-1\right)^{-\frac{1}{2}}$

2d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[(x-1)^{-\frac{1}{2}}\right]$

$=-\dfrac{1}{2}(x-1)^{-\frac{3}{2}}$

$=-\dfrac{1}{2(x-1)^{\frac{3}{2}}}$

2e) Let $y=\dfrac{1}{x^2+1}=\left(x^2+1\right)^{-1}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=(-1)(x^2+1)^{-2}(2x)$

$=-\dfrac{2x}{(x^2+1)^2}$

2f) Let $y=\left(2+3\sqrt{x}\right)^{\frac{1}{4}}=\left(2+3x^{\frac{1}{2}}\right)^{\frac{1}{4}}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{4}\left(2+3x^{\frac{1}{2}}\right)^{-\frac{3}{4}}\left(\dfrac{3}{2}x^{-\frac{1}{2}}\right)$

$=\dfrac{3}{8}\left(2+3x^{\frac{1}{2}}\right)^{-\frac{3}{4}}\left(x^{-\frac{1}{2}}\right)$

3 (a) $2x(x-3)^2$

(b) $(2x-1)(5x^2+3)$

(c) $\dfrac{(3x-4)(6-x)}{x}$

(d) $\dfrac{4x^2+x-1}{\sqrt{x}}$

3a) Let $y=2x\left(x-3\right)^2=2x\left(x^2-6x+9\right)$

$y=2x^3-12x^2+18x$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=6x^2-24x+18$

3b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[(2x-1)(5x^2+3)\right]$

$=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(10x^3-5x^2+6x-3\right)$

$=30x^2-10x+6$

3c) Let

$$y=\frac{(3x-4)(6-x)}{x}=\frac{22x-3x^2-24}{x}$$

$$=22-3x-24x^{-1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-3-24(-1)x^{-2}$$

$$=\frac{24}{x^2}-3$$

3d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{4x^2+x-1}{\sqrt{x}}\right)$

$$=\frac{\mathrm{d}}{\mathrm{d}x}\left(4x^{\frac{3}{2}}+x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)$$

$$=6x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{-\frac{3}{2}}$$

4 (a) $7\mathrm{e}^{\frac{1}{2}x}-2\mathrm{e}^{-\frac{1}{2}}$

(b) e^{x^2+3x-2}

(c) $\left(\mathrm{e}^{x}-\mathrm{e}^{-x}\right)^{-2}$

(d) $\mathrm{e}^{x}\left(e^{x}+e^{2}\right)$

(e) $\dfrac{3\mathrm{e}^{4x}+5\mathrm{e}^{2}}{2\mathrm{e}^{x-1}}$

(f) $\dfrac{\mathrm{e}^{x}-1}{\mathrm{e}^{3x+1}}$

4a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(7\mathrm{e}^{\frac{1}{2}x}-2\mathrm{e}^{-\frac{1}{2}}\right)$

$$=7\left(\frac{1}{2}\right)\mathrm{e}^{\frac{1}{2}x}-0$$

$$=\frac{7}{2}\mathrm{e}^{\frac{1}{2}x}$$

4b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2+3x-2}\right)$

$$=\left(2x+3\right)\mathrm{e}^{x^2+3x-2}$$

4c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\left(\mathrm{e}^{x}-\mathrm{e}^{-x}\right)^{-2}\right]$

$$=-2\left(\mathrm{e}^{x}-\mathrm{e}^{-x}\right)^{-3}\left(\mathrm{e}^{x}+\mathrm{e}^{-x}\right)$$

4d) Let $y=\mathrm{e}^{x}\left(e^{x}+e^{2}\right)=\mathrm{e}^{2x}+\mathrm{e}^{x+2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2\mathrm{e}^{2x}+\mathrm{e}^{x+2}$$

4e) Let $y = \dfrac{3e^{4x} + 5e^{2}}{2e^{x-1}}$

$$= \frac{3e^{4x}}{2e^{x-1}} + \frac{5e^{2}}{2e^{x-1}}$$

$$= \frac{3}{2}e^{3x+1} + \frac{5}{2}e^{3-x}$$

$$\frac{dy}{dx} = \frac{9}{2}e^{3x+1} - \frac{5}{2}e^{3-x}$$

$$= \frac{1}{2}\left(9e^{3x+1} - 5e^{3-x}\right)$$

4f) $\dfrac{d}{dx}\left(\dfrac{e^{x}-1}{e^{3x+1}}\right)$

$$= \frac{d}{dx}\left(e^{-2x-1} - e^{-3x-1}\right)$$

$$= -2e^{-2x-1} - (-3)e^{-3x-1}$$

$$= -2e^{-2x-1} + 3e^{-3x-1}$$

5 (a) $6\ln(3x-5)$

(b) $\ln\left(2x^{2}+3\right)$

(c) $\ln\dfrac{2}{x+1}$

(d) $3\ln\sqrt{x^{2}+2}$

5a) $\dfrac{d}{dx}\left[6\ln(3x-5)\right]$

$$= \frac{6(3)}{3x-5}$$

$$= \frac{18}{3x-5}$$

5b) $\dfrac{d}{dx}\left[\ln(2x^{2}+3)\right]$

$$= \frac{1}{2x^{2}+3}(4x)$$

$$= \frac{4x}{2x^{2}+3}$$

5c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\ln\dfrac{2}{x+1}\right]$

$=\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\ln 2-\ln(x+1)\right]$

$=0-\dfrac{1}{x+1}$

$=-\dfrac{1}{x+1}$

5d) Let $y=3\ln\sqrt{(x^2+2)}=3\ln\left(x^2+2\right)^{\frac{1}{2}}$

$=\dfrac{3}{2}\ln\left(x^2+2\right)$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3}{2}\left[\dfrac{2x}{(x^2+2)}\right]$

$=\dfrac{3x}{x^2+2}$

6 (a) $\ln\left(3x^2-5x+2\right)^3$

(b) $\ln\left[\left(x^2+8\right)\sqrt{3x^3-7}\right]$

(c) $\ln\left(\dfrac{5}{\sqrt[3]{x^2+4}}\right)$

(d) $\ln\sqrt{\dfrac{1-3x}{2x+1}}$

6a) Let $y=\ln\left(3x^2-5x+2\right)^3$

$=3\ln\left(3x^2-5x+2\right)$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3(6x-5)}{3x^2-5x+2}$

6b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left((x^2+8)\sqrt{(3x^3-7)}\right)\right]$

$=\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x^2+8)+\ln\left(3x^3-7\right)^{\frac{1}{2}}\right]$

$=\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x^2+8)+\dfrac{1}{2}\ln\left(3x^3-7\right)\right]$

$=\dfrac{2x}{x^2+8}+\dfrac{9x^2}{2(3x^3-7)}$

6c) Let

$y=\ln\left(\dfrac{5}{\sqrt[3]{x^2+4}}\right)=\ln 5-\ln\sqrt[3]{x^2+4}$

$=\ln 5-\dfrac{1}{3}\ln\left(x^2+4\right)$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=0-\dfrac{1}{3}\left(\dfrac{2x}{x^2+4}\right)=-\dfrac{2x}{3\left(x^2+4\right)}$

6d) Let $y=\ln\sqrt{\dfrac{1-3x}{2x+1}}$

$=\dfrac{1}{2}\left[\ln(1-3x)-\ln(2x+1)\right]$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{2}\left[\dfrac{-3}{1-3x}-\dfrac{2}{2x+1}\right]$

$=-\dfrac{1}{2}\left[\dfrac{3}{1-3x}+\dfrac{2}{2x+1}\right]$

7 Differentiate with respect to x

(i) $\ln(e^{\sqrt{x}} - x)$ [4]

(ii) $\dfrac{1}{(2x^2+1)^2}$ [2]

(i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln\left(e^{\sqrt{x}} - x\right) = \dfrac{1}{e^{\sqrt{x}} - x}\left[\dfrac{1}{2\sqrt{x}}e^{\sqrt{x}} - 1\right]$

$= \dfrac{e^{\sqrt{x}} - 2\sqrt{x}}{2\sqrt{x}e^{\sqrt{x}} - 2x^{\frac{3}{2}}}$

(ii) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{1}{(2x^2+1)^2}\right] = \dfrac{-8x}{(2x^2+1)^2}$

8 Differentiate each of the following expressions with respect to x.

(i) $(3x-2)^4$, [3]

(ii) $\dfrac{(x-1)^2}{x}$, [3]

(iii) $3\ln(2-5x)$. [3]

(i) Let $y = (3x-2)^4$

$\dfrac{dy}{dx} = (4)(3x-2)^3(3)$

$= 12(3x-2)^3$

(ii) $\text{Let } y = \dfrac{(x-1)^2}{x}$

$$y = \frac{x^2 - 2x + 1}{x}$$

$$y = x - 2 + x^{-1}$$

$$\frac{dy}{dx} = 1 + (-1)x^{-2}$$

$$= 1 - \frac{1}{x^2}$$

(iii) $\text{Let } y = 3\ln(2-5x)$

$$\frac{dy}{dx} = 3\frac{-5}{(2-5x)}$$

$$= \frac{-15}{(2-5x)}$$

9 Differentiate $(\ln x)^{2006}$ with respect to x. [2]

$$\frac{d}{dx}(\ln x)^{2006} = \frac{2006}{x}(\ln x)^{2005}$$

10 Differentiate the following with respect to x:

(i) $\dfrac{1}{2x-1}$, [2]

(ii) $\dfrac{1}{2}e^{-x}(e^{-3x} - e^{3x})$, [3]

(iii) $\dfrac{(1+x)^2}{1-x}$. [3]

(i) $\dfrac{d}{dx}\left(\dfrac{1}{2x-1}\right) = -\dfrac{2}{(2x-1)^2}$

(ii) $$\frac{d}{dx}\left[\frac{1}{2}e^{-x}\left(e^{-3x}-e^{3x}\right)\right]=\frac{1}{2}\frac{d}{dx}\left(e^{-4x}-e^{2x}\right)$$

$$=\frac{1}{2}\left(-4e^{-4x}-2e^{2x}\right)$$

$$=-2e^{-4x}-e^{2x}$$

(iii) $$\frac{d}{dx}\left[\frac{(1+x)^2}{1-x}\right]=\frac{d}{dx}\left(\frac{1+2x+x^2}{1-x}\right)$$

$$=\frac{d}{dx}\left(-x-3+\frac{4}{1-x}\right)$$

$$=\frac{4}{(1-x)^2}-1$$