2018 H2 P2 Solutions
(Section A)

Q1
$$\frac{dy}{dx} = \left(\tfrac{1}{3}y-15\right)^{\frac{1}{3}}$$

(i)
$$\int \left(\tfrac{1}{3}y-15\right)^{-\frac{1}{3}} dy = \int dx$$

$$\frac{\left(\frac{1}{3}y-15\right)^{\frac{2}{3}}}{\left(\frac{1}{3}\right)\frac{2}{3}} = x + C$$

$$\left(\tfrac{1}{3}y-15\right)^{\frac{2}{3}} = \tfrac{2}{9}(x+C)$$

when $x=0$, $y=69$: $4 = \frac{2}{9}C$

$C = 18$

$$\left(\tfrac{1}{3}y-15\right)^{\frac{2}{3}} = \tfrac{2}{9}(x+18)$$
$$= \left(\tfrac{2}{9}x+4\right)$$

$$\tfrac{1}{3}y-15 = \left(\tfrac{2}{9}x+4\right)^{\frac{3}{2}}$$

$$\therefore f(x) = y = 3\left(\tfrac{2}{9}x+4\right)^{\frac{3}{2}} + 45.$$

(ii) $\frac{dy}{dx} = 4 \Rightarrow \left(\frac{1}{3}y-15\right)^{\frac{1}{3}} = 4$

$\Rightarrow y = 237.$

$$\frac{\left(\frac{1}{3}(237)-15\right)^{\frac{2}{3}}}{\frac{2}{9}} = x+18$$

$\therefore x = 54$

$\therefore$ coordinates $(54, 237)$.

Q2(a) Since coeff of polynomial are real, 2+3i is also a root.

$$4(2-3i)^4 - 20(2-3i)^3 + s(2-3i)^2 - 56(2-3i) + t = 0$$

$$332 + 828i + s(-5-12i) + t = 0.$$

Comparing Re & Im parts,

$$\begin{cases} -5s + t + 332 = 0 \\ 828 - 12s = 0 \end{cases}$$

$\therefore$ $s = 69$; $t = 13$.

$$4x^4 - 20x^3 + 69x^2 - 56x + 13 = (x-(2-3i))(x-(2+3i))(ax^2+bx+c)$$

$$= ((x-2)^2 - (3i)^2)(ax^2+bx+c)$$

$$= (x^2-4x+13)(ax^2+bx+c)$$

$a = 4$

$c = 1$

$b - 4a = -20 \Rightarrow b = -4$

$4x^2 - 4x + 1 = 0$

$x = \frac{1}{2}$.

$\therefore$ other roots are $\frac{1}{2}$ and $2+3i$.

An easier way...

$4x^4 - 20x^3 + 69x^2 - 56x + 13 = 0$

By GC,

$x = \frac{1}{2}, \ 2 \pm 3i$

(b)(i) $w^3 = 27$.

* non-calculator method!!

$w^3 - 27 = 0$

$(w-3)(w^2+3w+9) = 0$

$$w = \frac{-3 \pm \sqrt{3^2 - 4(1)(9)}}{2}$$

$$= \frac{-3 \pm \sqrt{-27}}{2}$$

$$= \frac{-3 \pm 3\sqrt{3}i}{2}$$

$$= -\frac{3}{2} \pm \frac{3}{2}\sqrt{3}i$$

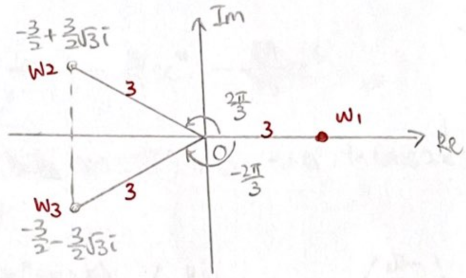
(ii) $|w| = 3$.

$\arg\left(-\frac{3}{2}+\frac{3}{2}\sqrt{3}i\right) = \frac{2\pi}{3}$.

$\arg\left(-\frac{3}{2}-\frac{3}{2}\sqrt{3}i\right) = -\frac{2\pi}{3}$.

$\therefore -\frac{3}{2}+\frac{3}{2}\sqrt{3}i = 3\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

$-\frac{3}{2}-\frac{3}{2}\sqrt{3}i = 3\left(\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}\right)$.



(iii) Let $w_1 = 3$, $w_2 = -\frac{3}{2}+\frac{3}{2}\sqrt{3}i$, $w_3 = -\frac{3}{2}-\frac{3}{2}\sqrt{3}i$

$w_1 + w_2 + w_3 = 0$.

$w_1 w_2 w_3 = 3\left(3e^{i\frac{2\pi}{3}}\right)\left(3e^{i\left(-\frac{2\pi}{3}\right)}\right)$

$= 27$.

Q3 (i) $\vec{AD} = \vec{BC}$

$\vec{OD} - \begin{pmatrix}5\\-4\\1\end{pmatrix} = \begin{pmatrix}-5\\4\\2\end{pmatrix} - \begin{pmatrix}5\\4\\0\end{pmatrix}$

$\vec{OD} = \begin{pmatrix}-10\\0\\2\end{pmatrix} + \begin{pmatrix}5\\-4\\1\end{pmatrix} = \begin{pmatrix}-5\\-4\\3\end{pmatrix}$

$\therefore D(-5, -4, 3)$.

(ii) $\vec{BC} = \begin{pmatrix} -10 \\ 0 \\ 2 \end{pmatrix}$

$\vec{BE} = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 10 \end{pmatrix}$.

$\underline{n} = \begin{pmatrix} -10 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} -5 \\ -4 \\ 10 \end{pmatrix} = \begin{pmatrix} 8 \\ 90 \\ 40 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 45 \\ 20 \end{pmatrix}$

plane BCE : $\underline{r} \cdot \begin{pmatrix} 4 \\ 45 \\ 20 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 45 \\ 20 \end{pmatrix} = 200.$

$\therefore\ 4x + 45y + 20z = 200.$

(iii) $\vec{BA} = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -8 \\ 1 \end{pmatrix}$

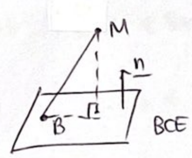
$\underline{n}$ of base $= \begin{pmatrix} 0 \\ -8 \\ 1 \end{pmatrix} \times \begin{pmatrix} -10 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -16 \\ -10 \\ -80 \end{pmatrix} = -2\begin{pmatrix} 8 \\ 5 \\ 40 \end{pmatrix}.$

$$\cos\theta = \frac{\begin{pmatrix} 4 \\ 45 \\ 20 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 5 \\ 40 \end{pmatrix}}{\sqrt{2441}\ \sqrt{1689}} = \frac{1057}{\sqrt{2441}\ \sqrt{1689}}$$

$\theta = 58.6^\circ$ (to 3sf.)

(iv) let midpoint of AD be M.

$\vec{OM} = \frac{1}{2}(\vec{OA} + \vec{OD}) = \frac{1}{2}\left(\begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix}.$



$\vec{BM} = \begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -8 \\ 2 \end{pmatrix}.$

$\therefore$ shortest dist $= |\vec{BM} \cdot \hat{n}| = \left| \begin{pmatrix} -5 \\ -8 \\ 2 \end{pmatrix} \cdot \frac{\begin{pmatrix} 4 \\ 45 \\ 20 \end{pmatrix}}{\sqrt{2441}} \right| = \frac{340}{\sqrt{2441}} = 6.88$ units. (to 3sf).

(i) $\ln(\cos 2x)$

$$= \ln\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots\right)$$

$$= \ln\left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \dots\right) \approx \ln\left(1 + \left(-2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right)\right)$$

$$= \left(-2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right) - \frac{\left(-2x^2 + \frac{2}{3}x^4\right)^2}{2} + \frac{(-2x^2)^3}{3} + \dots$$

$$= -2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 - \left(2x^4 - \frac{4}{3}x^6\right) + \left(-\frac{8}{3}\right)x^6 + \dots$$

$$\approx -2x^2 - \frac{4}{3}x^4 - \frac{64}{45}x^6 \,//$$

$x \neq \frac{\pi}{4}$ since $\cos 2(\frac{\pi}{4}) = 0$ thus $\ln(\cos 2x)$ will be undefined.

(ii) $$\int \frac{\ln(\cos 2x)}{x^2}\,dx$$

$$\approx \int -2 - \frac{4}{3}x^2 - \frac{64}{45}x^4 \, dx$$

$$= -2x - \frac{4}{3}\frac{x^3}{3} - \frac{64}{45}\frac{x^5}{5} + C$$

$$= -2x - \frac{4}{9}x^3 - \frac{64}{225}x^5 + C.$$

$$\int_0^{0.5} \frac{\ln(\cos 2x)}{x^2}\,dx = \left[-2x - \frac{4}{9}x^3 - \frac{64}{225}x^5\right]_0^{0.5}$$

$$= -1.0644 \text{ (to 4d.p)}$$

(iii) $$\int_0^{0.5} \frac{\ln(\cos 2x)}{x^2}\,dx = -1.0670 \text{ (to 4 dp) by GC.}$$

(section B)

Q5 (i) As the TTF is not normally distributed and population variance is unknown, a sample of at least 30 fans needs to be taken in order for CLT to be applied ie. to approximate the MTTF to a normal distribution. These fans should be randomly chosen.

(ii) Let μ be the MTTF of a fan, in hours.

$H_0 : \mu = 65\,000$

$H_1 : \mu < 65000$

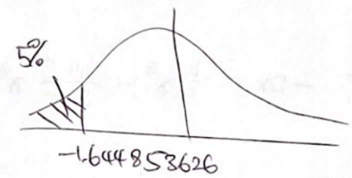
given $n = 43$, $\bar{x} = 64230$.

(iii) Under H_0, since $n = 43$ (large), by CLT, $\bar{X} \sim N\left(65000, \frac{\sigma^2}{43}\right)$ approx.

$$Z = \frac{\bar{X} - 65000}{\frac{\sigma}{\sqrt{43}}} \sim N(0,1) \text{ approx.}$$

do not rej H_0 at 5% sig level :

$Z_{calc} \geq -1.64485.$



$$\frac{64230 - 65000}{\frac{\sigma}{\sqrt{43}}} \geq -1.644853626$$

$$\sigma \geq 3069.712456$$

$$\sigma^2 \geq 9423134.561$$

2 possible answers accepted : $\sigma^2 \geq 9430000$ (logically) OR $\sigma^2 \geq 9420000$ (rounding)

Q6(i)

i) Let X be the number of left forks out of 8 junctions.

$P(X=5) = \binom{8}{5} p^5 q^3$ (Since the bug makes 5 left turns & 3 right turns to arrive at D)

$= 56p^5q^3$.

need to explain!

(ii) Given $x=5$ is the mode

$P(X=4) = 70p^4q^4$

$P(X=6) = 28p^6q^2$

$P(X=4) < P(X=5) > P(X=6)$

$70p^4q^4 < 56p^5q^3 > 28p^6q^2$

$\cancel{70p^4q^4} < 56\cancel{p^5q^3}$ and $56\cancel{p^5q^3} > 28\cancel{p^6q^2}$

$70(1-p) < 56p$ $\quad$ $56(1-p) > 28p$

$70 < 126p$ $\quad$ $56 > 84p$.

$\frac{5}{9} < p$ and $\frac{2}{3} > p$.

$\therefore \frac{5}{9} < p < \frac{2}{3}$.

(iii) P(not being swallowed by black hole) $= 1-0.1 = 0.9$.
(at each step)

Since there are 8 steps in total,

P(bug reaches A–I without being swallowed) $= (0.9)^8 = 0.430$ (to 3sf.).

Q7(i) $P(A' \cap B') = 1 - P(A \cup B)$

$= 1 - (P(A) + P(B) - P(A \cap B))$

$= 1 - (a + b - ab)$ since $P(A \cap B) = P(A)P(B) = ab$. (indep)

$= 1 - a - b + ab$.

Since $P(A') \times P(B') = (1-a)(1-b) = 1 - b - a + ab = P(A' \cap B')$,

$\therefore$ A' and B' are independent events.

(ii) $P(A' \cap C') = 1 - P(A \cup C)$

$= 1 - (P(A) + P(C))$ since $P(A \cap C) = 0$.

$= 1 - (a + c)$

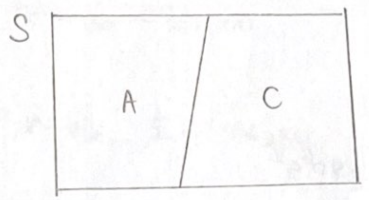
$= 1 - a - c$.

If A' & C' are mutually exclusive, $P(A' \cap C') = 0$

$1 - a - c = 0$

$a + c = 1$.

$P(A) + P(C) = 1$



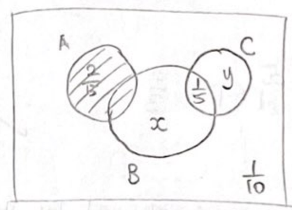
Given,

$P(A) = \frac{2}{5}$, $P(B \cap C) = \frac{1}{5}$, $P(A' \cap B' \cap C') = \frac{1}{10}$, $P(A' \cap C') \neq 0$. $\Rightarrow$ it all dep on the locat of C.

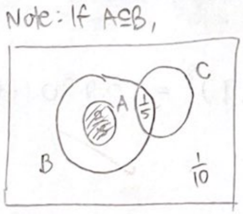
★ (iii) $P(A \cup B \cup C) = 1 - \frac{1}{10} = \frac{9}{10}$.

use venn diagrams!

*min case $P(A \cap C) = 0$



Note: If $A \subseteq B$,



$P(A \cap B) = P(A) = \frac{2}{5} = \frac{2}{5} P(B)$

$\Rightarrow P(B) = 1 \; (\rightarrow\leftarrow)$

$P(A \cup B \cup C) = 1 - \frac{1}{10} = \frac{9}{10}$

$\frac{2}{5} + x + \frac{1}{5} + y = \frac{9}{10}$

$x + y = \frac{3}{10}$

$x = \frac{3}{10} - y$ (largest when $y = 0$, smallest when $y = \frac{3}{10}$)

when $y = \frac{3}{10}$, $x = 0$:

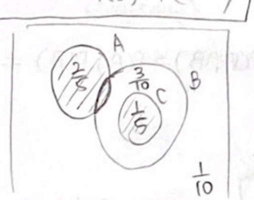
since $P(A \cap B) = \frac{2}{5} P(B)$,

$\frac{2}{5} P(B) + \frac{1}{5} = P(B) \Rightarrow P(B) = \frac{1}{3}$

$\Rightarrow P(A \cap B) = \frac{2}{5}\left(\frac{1}{3}\right) = \frac{2}{15}$

if $y = 0$,

* max case



$\therefore \frac{2}{15} \leq P(A \cap B) \leq \frac{1}{3}$.

$P(A \cap B) = \frac{2}{5} P(B)$

$P(B) = \frac{2}{5} P(B) + \frac{3}{10} + \frac{1}{5} \Rightarrow P(B) = \frac{5}{6} \Rightarrow P(A \cap B) = \frac{1}{3}$.

Q8(i)

$S =$ sum of numbers on 2 balls

S	6	7	8	9	10
$P(S=s)$	$\frac{2}{(n+5)(n+4)}$	$\frac{12}{(n+5)(n+4)}$	$\frac{6+4n}{(n+5)(n+4)}$	$\frac{6n}{(n+5)(n+4)}$	$\frac{n(n-1)}{(n+5)(n+4)}$

$3+3 \quad \frac{2}{n+5}\,\frac{1}{n+4}$

$3+4 \quad \frac{2}{n+5}\,\frac{3}{n+4} \times 2!$

$4+4 \quad \frac{3}{n+5}\,\frac{2}{n+4}$ $\qquad 3+5 \quad \frac{2}{n+5}\,\frac{n}{n+4} \times 2!$

$4+5 \quad \frac{3}{n+5}\,\frac{n}{n+4} \times 2!$

$5+5 \quad \frac{n}{n+5}\,\frac{n-1}{n+4}$

* without replacement!

(ii) When $n=1$, $P(S=10)=0$. The total of 2 balls can only be 10 if there were 2 balls numbered "5", and that contradicts $n=1$.

(iii)
$$E(S) = \frac{12+84+48+32n+54n+10n(n-1)}{(n+5)(n+4)}$$

$$= \frac{144+76n+10n^2}{(n+5)(n+4)}$$

$$= \frac{(10n+36)(n+4)}{(n+5)(n+4)}$$

$$= \frac{10n+36}{n+5}.$$

$$E(S^2) = \frac{72+588+384+256n+486n+100(n^2-n)}{(n+5)(n+4)}$$

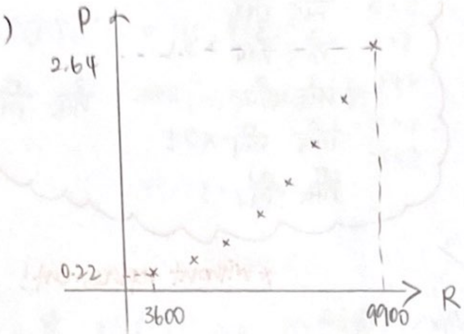
$$= \frac{100n^2+642n+1044}{(n+5)(n+4)}$$

$$Var(S) = \frac{100n^2+642n+1044}{(n+5)(n+4)} - \frac{(10n+36)^2}{(n+5)^2}$$

$$= \frac{(100n^2+642n+1044)(n+5)-(100n^2+720n+1296)(n+4)}{(n+5)^2(n+4)}$$

$$= \frac{22n^2+78n+36}{(n+5)^2(n+4)}.$$

Q9 (i)



The scatter diagram shows a curvilinear relationship between P & R.
As R increases, P increases at an increasing rate.
∴ $P = aR + b$ is not suitable.

(ii) $P = aR + b \rightarrow r = 0.969$

$P = aR^2 + b \rightarrow r = 0.993$.

Since $|r|$ closer to 1 for model of $P = aR^2 + b$, ∴ $P = aR^2 + b$ is better.

$P = -0.28260761 + (2.84577445 \times 10^{-8}) R^2$.

$= -0.283 + (2.85 \times 10^{-8}) R^2$.

(iii) when $P = 0.9$

$R = 6446.476177$

≈ 6450 revolutions per min.

estimate is reliable as P is within data range $0.22 \leq P \leq 2.64$ and $|r|$ close to 1. Also since R is an indep var and P dep on R, thus the line of P on R^2 is suitably used

(iv) when $R = 3300$,

$P = 0.0273$ W.

estimate is not reliable as $R = 3300$ is not within data range $3600 \leq R \leq 9900$.

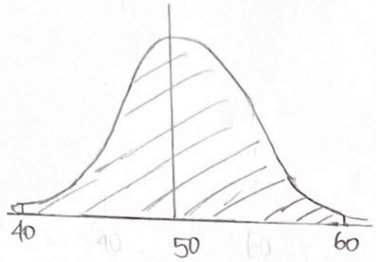
$\boxed{\text{(v)}}$ Let R' be the no. of revolutions per second

$\frac{R}{60} = R'$

$P = -0.28260761 + 2.8457445 \times 10^{-8} (60R')^2$

$= -0.283 + 1.02 \times 10^{-4} (R')^2$.

Q10 (i)



must shade!!
area shaded should be almost the whole area.
must label the mean & the 2 boundaries.

(ii) Let X be mass in grams of a light bulb

$X \sim N(50, 1.5^2)$

$P(X < 50.4) = 0.605$

(iii) Let Y be mass in grams of an empty box.

$Y \sim N(75, 2^2)$ ← (qn did not say mass is in grams!)

$Y_1 + Y_2 + Y_3 + Y_4 \sim N(300, 16)$

$P(Y_1 + Y_2 + Y_3 + Y_4 > 297) = 0.773$.

(iv) $X + Y \sim N(125, 6.25)$

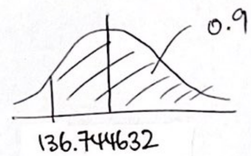
$P(124.9 < X + Y < 125.7) = 0.126$.

(v) Let $W = \underbrace{X_1}_{\text{bulb}} + \underbrace{0.3 X_2}_{\text{padding}} + \underbrace{Y}_{\text{box}} \sim N(140, 6.4525)$

$P(W > k) = 0.9$.

$k = 136.744632$

≈ 137



(vi) $W_1 + W_2 + W_3 + W_4 \sim N(140(4), 6.4525(4))$

$\Rightarrow W_1 + W_2 + W_3 + W_4 \sim N(560, 25.81)$

$P(W_1 + W_2 + W_3 + W_4 > 565) = 0.163$