MATHEMATICS

Paper 9758/01
Set I – Paper 1

Topic identification and short answers

Qn	Topic(s)	Part	Answers
1	Vectors (two dimensions); Graphs and transformations		Required sequence of transformation: 1st: Scaling by a scale factor of k^2 parallel to the $x-$axis 2nd: Translation of $k^2 - 1 - \dfrac{1}{k} + k$ units in the positive $y-$direction
2	Graphs (sketching using GC); Inequalities		• If $0 < a < 1$, then $-1 < x \leq -a$ or $a \leq x < 2$. • If $a = 1$, then $1 \leq x < 2$. • If $1 < a < 2$, then $-a \leq x < -1$ or $a \leq x < 2$. • If $a = 2$, then $-2 \leq x < -1$. • If $a > 2$, then $-a \leq x < -1$ or $2 < x \leq a$.
3	Sequences and series (arithmetic series)	**(i)**	$a = \dfrac{m^2 + (n-1)(m+n)}{mn}$ $d = -\dfrac{2(m+n)}{mn}$
		(ii)	[shown]
4	Differentiation (implicit functions, equations of tangents)	**(i)**	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x + 3y + 4}{2y - 3x}$
		(ii)	[shown]
		(iii)	$3x + 5y + 2 = 0$ $11x + 28y + 46 = 0$
5	Complex numbers (polar form, modulus and argument, Argand diagram)	**(i)**	$\dfrac{u-v}{u+v} = \mathrm{i}\tan\left(\dfrac{\alpha - \beta}{2}\right)$
		(ii)	[shown] Right angle at WOZ (or ZOW)
		(iii)	$\theta = \dfrac{\alpha - \beta}{2}$ $\lvert u - v\rvert = 2r\sin\theta$ $\lvert u + v\rvert = 2r\cos\theta$
6	Vectors (ratio theorem, angle between two vectors, scalar products)	**(i)**	$\lambda = \dfrac{\lvert\mathbf{b}\rvert}{\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert}$
		(ii)	[shown]

7	Functions (inverse, composite); System of linear equations	**(a) (i)**	$\mathrm{f}^{-1}(x)=\dfrac{x+1}{2x-1},\quad x\in\mathbb{R},\quad x\neq\dfrac{1}{2}$
		(a) (ii)	$\mathrm{g}(x)=\dfrac{4x-3}{x-1}$
		(b) (i)	$\mathrm{h}(2)=0.5$ $\mathrm{h}(0.5)=-1$ $\mathrm{h}(-1)=2.$
		(b) (ii)	$\mathrm{h}(x)=1-\dfrac{1}{x}$
8	Integration techniques (by substitution, by parts); Definite integrals (area of a region)	**(i)**	$I_0=\dfrac{\pi a^2}{8}$
		(ii)	[shown]
		(iii)	$\dfrac{5\pi}{16}$ units^2
9	Differentiation (parametric functions, stationary points, tangents); Graphs (parametric equation, transformations); Definite integrals (volume of revolution)	**(i)**	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{t^2-1}{2t}$
		(ii)	$t_1=\mathrm{e}^{-1}$ $t_2=\mathrm{e}$
		(iii)	Angle $=49.6°$
		(iv)	[shown] $k=\dfrac{a}{2}\left(\mathrm{e}+\mathrm{e}^{-1}\right)$
		(v)	Volume $=\dfrac{\pi a^3}{4}\left(8-e^2+5e^{-2}\right)$ units^3
10	Differential equations; Differentiation (local maxima and minima, connected rates of change)	**(i) (a)**	$\dfrac{\mathrm{d}R}{\mathrm{d}t}=0.6R-0.4RW$ $\dfrac{\mathrm{d}W}{\mathrm{d}t}=-0.8W+0.2RW$
		(i) (b)	[shown]
		(ii) (a)	Rabbit: smallest = 900, largest = 10,700. Wolf: smallest = 300, largest = 4,500.
		(ii) (b)	B_2
11	Sequences and series (geometric series)	**(i)**	End of July 2026
		(ii)	[shown] Amount = \$9,477.56
		(iii)	August 2029

Suggested solutions and post-mortem

Qn	Suggested Solutions	Post-mortem
1 **[4]**	Finding the cartesian equation of L_1, $\mathbf{r} = \begin{pmatrix} k \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ k \end{pmatrix}$ $\rightarrow x - k = \dfrac{y-1}{k}$ $\rightarrow y = kx - k^2 + 1$ Finding the cartesian equation of L_2, $\mathbf{r} = \begin{pmatrix} 1 \\ k \end{pmatrix} + \mu \begin{pmatrix} k \\ 1 \end{pmatrix}$ $\rightarrow \dfrac{x-1}{k} = y - k$ $\rightarrow y = \dfrac{1}{k}x - \dfrac{1}{k} + k$ Required sequence of transformation: 1st: Scaling by a scale factor of k^2 parallel to the $x-$axis 2nd: Translation of $k^2 - 1 - \dfrac{1}{k} + k$ units in the positive $y-$direction	This question mainly assesses on converting vector equations into cartesian equations. Admittedly, despite being within the expectations of the syllabus, two-dimensional vector equations rarely appear in A–Levels. The latter part concerning graph transformations should be relatively more routine. Note that when proposing translations involving unknown constants, **ensure that it is done (1) with a positive unit of translation, and (2) in the correct direction**. (In this case, it is graphically verifiable that $k^2 - 1 - \frac{1}{k} + k > 0$ for $k > 1$.)

2
[5]

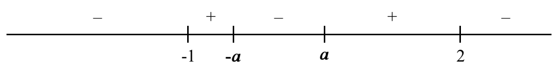
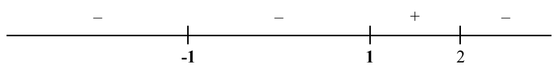
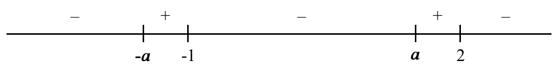
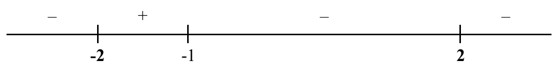
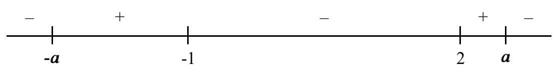
$$\frac{a^2 - x - 2}{x^2 - x - 2} \geq 1$$

$$\rightarrow \frac{a^2 - x - 2 - (x^2 - x - 2)}{x^2 - x - 2} \geq 0$$

$$\rightarrow \frac{a^2 - x^2}{x^2 - x - 2} \geq 0$$

$$\rightarrow \frac{(a + x)(a - x)}{(x - 2)(x + 1)} \geq 0$$

Principal values are $x = -1, 2$ and $\pm a$. Given $a > 0$, there are 5 possible solution intervals:

Case	Value of a	Number line (deduced from GC)	Solution interval
1	$0 < a < 1$		$-1 < x \leq -a$ or $a \leq x < 2$
2	$a = 1$		$1 \leq x < 2$
3	$1 < a < 2$		$-a \leq x < 1$ or $a \leq x < 2$
4	$a = 2$		$-2 \leq x < -1$
5	$a > 2$		$-a \leq x < -1$ or $2 < x \leq a$

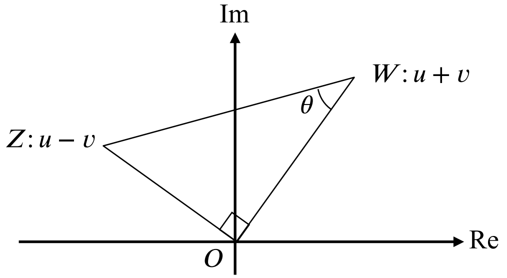
This question concerns graph sketching and solving inequalities. As this question does not explicitly prohibit the use of graphing calculators (GC), candidates may simply substitute different a values to observe the different cases of graphs and deduce the required solution intervals, all without sketching these graphs on paper as it is not required by the question.

Having said so, a manual attempt by hand is perfectly welcomed and encouraged. In fact, answers that rely on GC may also benefit from some manual prework which could reveal strategic choices of a values.

3 (i) [4]	$S_n = m \to \frac{n}{2}(2a+(n-1)d) = m \to 2a+(n-1)d = \frac{2m}{n}$ $S_m = n \to \frac{m}{2}(2a+(m-1)d) = n \to 2a+(m-1)d = \frac{2n}{m}$ Eliminating $2a$, $(n-1)d-(m-1)d = \frac{2m}{n}-\frac{2n}{m}$ $(n-m)d = 2\left(\frac{m^2-n^2}{mn}\right) = \frac{2(m+n)(m-n)}{mn}$ $\therefore d = -\frac{2(m+n)}{mn}$ Substituting d into S_n, $2a = \frac{2m}{n}-(n-1)\left(-\frac{2(m+n)}{mn}\right) = \frac{2m^2+2(n-1)(m+n)}{mn}$ $\therefore a = \frac{m^2+(n-1)(m+n)}{mn}$	This question mainly deals with arithmetic series. It should be relatively straightforward to find an expression for S_m and S_n. Afterwards, elimination and substitution can be done to yield a and d.
3 (ii) [2]	S_{m+n} $= \frac{m+n}{2}\left[2\left(\frac{m^2+(n-1)(m+n)}{mn}\right)+(m+n-1)\left(-\frac{2(m+n)}{mn}\right)\right]$ $= (m+n)\left[\frac{m^2}{mn}+\frac{(n-1)(m+n)}{mn}-\frac{(m+n-1)(m+n)}{mn}\right]$ $= (m+n)\left[\frac{m^2}{mn}+\frac{(m+n)(n-1-m-n+1)}{mn}\right]$ $= (m+n)\left[\frac{m^2}{mn}+\frac{(m+n)(-m)}{mn}\right]$ $= (m+n)\left[\frac{m^2-m^2-mn}{mn}\right]$ $= -(m+n)$	Like the first part, the second part entails making use of the formula for arithmetic series. As a side, a subsequential "show" question such as this one may sometimes prove to be a gainful hindsight, as it helps verify whether previously found results are correct (thereby securing marks in previous parts) so that no errors are carried forward to latter parts (thereby securing marks in latter parts).

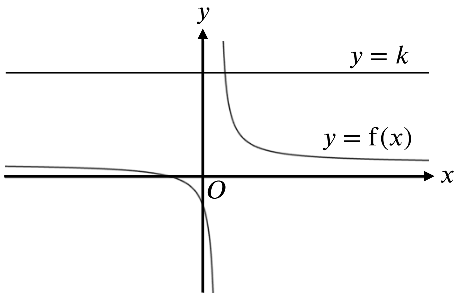
4 **(i)** **[1]**	By implicit differentiation, $2x+3y+3x\frac{\mathrm{d}y}{\mathrm{d}x}-2y\frac{\mathrm{d}y}{\mathrm{d}x}+4=0$ $2x+3y+4=\frac{\mathrm{d}y}{\mathrm{d}x}(2y-3x)$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x+3y+4}{2y-3x}$	Implicit differentiation should pose little challenge to careful candidates.
4 **(ii)** **[3]**	At the tangent point (x, y), the tangent equation is $y+4=m(x-6)$, where m is the gradient. Now since $\frac{\mathrm{d}y}{\mathrm{d}x}=m$, $\rightarrow \frac{2x+3y+4}{2y-3x}=\frac{y+4}{x-6}$ $\rightarrow (x-6)(2x+3y+4)=(2y-3x)(y+4)$ $\rightarrow 2x^2+3xy+4x-12x-18y-24=2y^2-3xy+8y-12x$ $\rightarrow 2\left(x^2+3xy-y^2\right)+4x-26y-24=0$ Since (x, y) lies on C, we use the equation of C to obtain $x^2+3xy-y^2=1-4x$. $\rightarrow 2(1-4x)+4x-26y-24=0$ $\rightarrow 2-4x-26y-24=0$ $\rightarrow 2x+3y=11$	This question assesses candidates on forming tangent line equations and finding relationships between their gradients and their intersection point. Successful candidates will recognise, after equating the gradient function with the gradient of a general line passing through $(6,-4)$, that **the equation of C itself helps eliminate implicit terms in the intermediate steps**. With some algebraic manipulation, the result follows.
4 **(iii)** **[3]**	$2x+3y=11 \rightarrow x=-\frac{13y+11}{2}$ Substituting x into the equation of C, $\left(-\frac{13y+11}{2}\right)^2+3\left(-\frac{13y+11}{2}\right)y-y^2=1-4\left(-\frac{13y+11}{2}\right)$ Using calculator, $y=-1$ or $y=-\frac{1}{3}$. Substituting these y values into $2x=3y=11$, we have the following: • When $y=-1$, $x=1$ and $m=-\frac{3}{5} \rightarrow$ equation of tangent: $3x+5y+2=0$ • When $y=-\frac{1}{3}$, $x=-\frac{10}{3}$ and $m=-\frac{11}{28} \rightarrow$ equation of tangent: $11x+28y+46=0$	After substituting into C accordingly, the resulting quadratic equation can be solved with ease using calculator, though proceeding manually is welcomed albeit not required. The remaining steps follow.

5 **(i)** **[4]**	Write $u = r(\cos\alpha + \mathrm{i}\sin\alpha)$ and $v = r(\cos\beta + \mathrm{i}\sin\beta)$. $\dfrac{u-v}{u+v}$ $= \dfrac{r(\cos\alpha + \mathrm{i}\sin\alpha) - r(\cos\beta + \mathrm{i}\sin\beta)}{r(\cos\alpha + \mathrm{i}\sin\alpha) + r(\cos\beta + \mathrm{i}\sin\beta)}$ $= \dfrac{r[\cos\alpha - \cos\beta + \mathrm{i}(\sin\alpha - \mathrm{i}\sin\beta)]}{r[\cos\alpha + \cos\beta + \mathrm{i}(\sin\alpha + \mathrm{i}\sin\beta)]}$ $= \dfrac{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) + \mathrm{i}\left[2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right]}{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + \mathrm{i}\left[2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]}$ $= \dfrac{2\sin\left(\frac{\alpha-\beta}{2}\right)\left[-\sin\left(\frac{\alpha+\beta}{2}\right) + \mathrm{i}\cos\left(\frac{\alpha+\beta}{2}\right)\right]}{2\cos\left(\frac{\alpha-\beta}{2}\right)\left[\cos\left(\frac{\alpha+\beta}{2}\right) + \mathrm{i}\sin\left(\frac{\alpha+\beta}{2}\right)\right]}$ $= \dfrac{\mathrm{i}\tan\left(\frac{\alpha-\beta}{2}\right)\left[\cos\left(\frac{\alpha+\beta}{2}\right) + \mathrm{i}\sin\left(\frac{\alpha+\beta}{2}\right)\right]}{\left[\cos\left(\frac{\alpha+\beta}{2}\right) + \mathrm{i}\sin\left(\frac{\alpha+\beta}{2}\right)\right]}$ $= \mathrm{i}\tan\left(\dfrac{\alpha-\beta}{2}\right)$	This question assesses candidates’ capacity in manipulating complex number expressions using the sum-to-product trigonometric identities in the List of Formulae (MF26). (Incidentally, note that all the four identities are eventually used in the intermediate steps.) A small manipulation is required in the last bit, in which an imaginary number is factored out as a common term in the nominator to allow for cancelation.

5 (ii) [1]	From **(i)**, it can be deduced that: $\rightarrow \arg\left(\frac{u-v}{u+v}\right) = \arg\left(\mathrm{i}\tan\left(\frac{\alpha-\beta}{2}\right)\right) = \arg(\mathrm{i}) = \frac{\pi}{2}$ $\rightarrow \arg(u-v) - \arg(u+v) = \frac{\pi}{2}$ $\therefore$ Triangle OWZ is a right triangle with right angle at WOZ (or ZOW).	This part mainly assesses on complex number operations involving arguments. It should be apparent that triangle OWZ is a right triangle once candidates eventually obtain the value $\frac{\pi}{2}$. Candidates then only need to deduce where the angle is.
5 (ii) [3]	Using information in **(i)** to roughly sketch triangle OWZ on an Argand diagram:  Using the sketch and the result in **(i)**, we have: $\rightarrow \tan(\theta) = \frac{OZ}{OW} = \frac{\lvert u-v\rvert}{\lvert u+v\rvert} = \left\lvert\frac{u-v}{u+v}\right\rvert = \left\lvert\mathrm{i}\tan\left(\frac{\alpha-\beta}{2}\right)\right\rvert = \tan\left(\frac{\alpha-\beta}{2}\right)$ $\therefore \theta = \frac{\alpha-\beta}{2}$ Using trigonometry to find $\lvert u-v\rvert$ and $\lvert u+v\rvert$, $\rightarrow \lvert u-v\rvert = \lvert ZW\rvert \sin\theta$ and $\lvert u+v\rvert = \lvert ZW\rvert\cos\theta$ Notice that $\lvert ZW\rvert = \lvert(u+v)-(u-v)\rvert = \lvert 2v\rvert = 2r$ $\therefore \lvert u-v\rvert = 2r\sin\theta$ and $\lvert u+v\rvert = 2r\cos\theta$	This part mainly assesses candidates' ability to ascribe features to shapes on an Argand diagram, but also deals with operations on complex modulus. After drawing the Argand diagram and deducing the length ratio with suitable operations, the relationship between angle θ and the arguments α and β eventually becomes apparent. Once the angle θ is identified, further results can be obtained using standard trigonometric properties on right triangles.

6 (i) [4]	Using the formula in List of Formulae (MF26), $\overrightarrow{OC} = \dfrac{\lambda\mathbf{a} + (1-\lambda)\mathbf{b}}{\lambda + 1 - \lambda} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$ Since OP bisects angle AOB, $\cos\angle AOC = \cos\angle BOC$ $\rightarrow \dfrac{\overrightarrow{OC}\cdot\mathbf{a}}{\lvert\overrightarrow{OC}\rvert\lvert\mathbf{a}\rvert} = \dfrac{\overrightarrow{OC}\cdot\mathbf{b}}{\lvert\overrightarrow{OC}\rvert\lvert\mathbf{b}\rvert}$ $\rightarrow \dfrac{\lambda\mathbf{a}\cdot\mathbf{a} + (1-\lambda)\mathbf{b}\cdot\mathbf{a}}{\lvert\mathbf{a}\rvert} = \dfrac{\lambda\mathbf{a}\cdot\mathbf{b} + (1-\lambda)\mathbf{b}\cdot\mathbf{b}}{\lvert\mathbf{b}\rvert}$ $\rightarrow \dfrac{\lambda\lvert\mathbf{a}\rvert^2 + (1-\lambda)\lvert\mathbf{b}\rvert\lvert\mathbf{a}\rvert\cos\angle AOB}{\lvert\mathbf{a}\rvert} = \dfrac{\lambda\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\angle AOB + (1-\lambda)\lvert\mathbf{b}\rvert^2}{\lvert\mathbf{b}\rvert}$ $\rightarrow \lambda\lvert\mathbf{a}\rvert + (1-\lambda)\lvert\mathbf{b}\rvert\cos\angle AOB = \lambda\lvert\mathbf{a}\rvert\cos\angle AOB + (1-\lambda)\lvert\mathbf{b}\rvert$ $\rightarrow \lambda\lvert\mathbf{a}\rvert(1-\cos\angle AOB) + \left((1-\lambda)\lvert\mathbf{b}\rvert\right)(\cos\angle AOB - 1) = 0$ $\rightarrow (\lambda\lvert\mathbf{a}\rvert - (1-\lambda)\lvert\mathbf{b}\rvert)(1-\cos\angle AOB) = 0$ Since O, A and B are not collinear, $\cos\angle AOB \neq 1$ $\rightarrow \lambda\lvert\mathbf{a}\rvert - (1-\lambda)\lvert\mathbf{b}\rvert = 0$ $\rightarrow \lambda\lvert\mathbf{a}\rvert - \lvert\mathbf{b}\rvert + \lambda\lvert\mathbf{b}\rvert = 0$ $\rightarrow \lambda(\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert) = \lvert\mathbf{b}\rvert$ $\rightarrow \lambda = \dfrac{\lvert\mathbf{b}\rvert}{\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert}$	This part requires candidates' aptitude in vector product operations. To write an equation regarding the bisection, an expression for $\overrightarrow{OC}$ must be obtained first, which can be done using the ratio theorem given in the List of Formulae (MF26). Subsequently, candidates can use the fact that a bisected angle yields two angles with equal cosine values. The resulting equation eventually simplifies to a product of two separate factors, one of which expressed in terms of the constant λ to be solved. While the other factor $(1-\cos\angle AOB)$ is irrelevant towards finding λ, **candidates must still justify why the root of this factor is rejected**. Details provided in the question that initially seem irrelevant but later turn significant must always be referred to in the answer, lest they become a reason for penalty under a strict marking scheme.

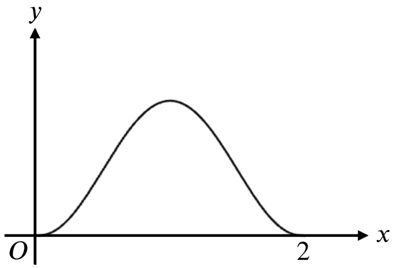
6 (ii) [4]	Given that $AC = BD$, $\overrightarrow{OD} = \lambda\mathbf{b} + (1-\lambda)\mathbf{a}$ Using dot product to find OC^2 and OD^2, $OC^2 = \overrightarrow{OC}\cdot\overrightarrow{OC} = \lambda^2\mathbf{a}\cdot\mathbf{a} + 2\lambda(1-\lambda)\mathbf{a}\cdot\mathbf{b} + (1-\lambda)^2\mathbf{b}\cdot\mathbf{b}$ $OD^2 = \overrightarrow{OD}\cdot\overrightarrow{OD} = \lambda^2\mathbf{b}\cdot\mathbf{b} + 2\lambda(1-\lambda)\mathbf{b}\cdot\mathbf{a} + (1-\lambda)^2\mathbf{a}\cdot\mathbf{a}$ $\therefore OD^2 - OC^2$ $= \left(\lambda^2 - (1-\lambda)^2\right)\mathbf{b}\cdot\mathbf{b} - \left(\lambda^2 - (1-\lambda)^2\right)\mathbf{a}\cdot\mathbf{a}$ $= \left(\lambda^2 - (1-\lambda)^2\right)(\mathbf{b}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{a})$ $= (\lambda + 1 - \lambda)(\lambda - 1 + \lambda)\left(\lvert\mathbf{b}\rvert^2 - \lvert\mathbf{a}\rvert^2\right)$ $= (1)(2\lambda - 1)\left(\lvert\mathbf{b}\rvert^2 - \lvert\mathbf{a}\rvert^2\right)$ $= \left(\dfrac{2\lvert\mathbf{b}\rvert}{\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert} - \dfrac{\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert}{\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert}\right)\left(\lvert\mathbf{b}\rvert^2 - \lvert\mathbf{a}\rvert^2\right)$ $= \left(\dfrac{\lvert\mathbf{b}\rvert - \lvert\mathbf{a}\rvert}{\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert}\right)(\lvert\mathbf{b}\rvert + \lvert\mathbf{a}\rvert)(\lvert\mathbf{b}\rvert - \lvert\mathbf{a}\rvert)$ $= (\lvert\mathbf{b}\rvert - \lvert\mathbf{a}\rvert)^2$	Finding an expression for $\overrightarrow{OD}$ might prove to be a hurdle. A sketch may help illustrate the similarity between $\overrightarrow{OC}$ and $\overrightarrow{OD}$. As hinted by the question, the values of OC^2 and OD^2 can be deduced using dot product. The remaining parts of the proof follow with some algebraic manipulation.

7 (a) (i) [3]	Consider the sketch of the graph of $y = \mathrm{f}(x)$:  In the sketch, the line $y = k$, where $k \in \mathbb{R}$, intersects the graph $y = \mathrm{f}(x)$ at most once. $\therefore$ f is a one-one function, and hence f has an inverse. Let $y = \mathrm{f}(x)$. $2y = 2\left(\frac{x+1}{2x-1}\right) = \frac{2x+2}{2x-1} = \frac{2x-1+3}{2x-1} = 1 + \frac{3}{2x-1}$ $2y - 1 = \frac{3}{2x-1}$ $2x - 1 = \frac{3}{2y-1}$ $x = \frac{1}{2}\left(1 + \frac{3}{2y-1}\right) = \frac{1}{2}\left(\frac{2y+2}{2y-1}\right) = \frac{y+1}{2y-1}$ $\therefore \mathrm{f}^{-1}(x) = \frac{x+1}{2x-1}, \quad x \in \mathbb{R}, \quad x \neq \frac{1}{2}.$	Questions on functions and inverses are relatively standard in A–Levels. Once a sketch of $y = \mathrm{f}(x)$ is produced, the explanation for why the inverse of f exists is straightforward, as is finding an expression for this inverse. As a side, it might be interesting to note that the given function $\mathrm{f}(x)$ is coincidentally a *self-inverse* function, i.e., it satisfies $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. Self-inverse functions, or otherwise called *involutions* or *involutory functions*, often appear in A–Levels and school papers, and among the questions that are asked involving them is, most popularly, telescoping (e.g., evaluating $\mathrm{f}^{2024}(x)$).
7 (a) (ii) [2]	$\mathrm{gf}(x) = \frac{2x-7}{x-2} \rightarrow \mathrm{gff}^{-1}(x) = \frac{2\mathrm{f}^{-1}(x)-7}{\mathrm{f}^{-1}(x)-2}$ Using the identity $\mathrm{ff}^{-1}(x) = x$, we have $\mathrm{g}(x) = \frac{2\left(\frac{x+1}{2x-1}\right)-7}{\frac{x+1}{2x-1}-2} = \frac{2(x+1)-7(2x-1)}{x+1-2(2x-1)} = \frac{2x+2-14x+7}{x+1-4x+2} = \frac{9-12x}{3-3x} = \frac{4x-3}{x-1}$	This part may prove challenging. An intuitive approach to this part requires recognising the identity $\mathrm{ff}^{-1}(x) = x$, which can be used to reduce $\mathrm{gf}(x)$ to $\mathrm{g}(x)$. Alternatively, since it is previously discovered that f is self-inverse, it is possible to reduce $\mathrm{gf}(x)$ by writing $\mathrm{gff}(x)$ to obtain $\mathrm{g}(x)$.

7 (b) (i) [3]	$x = 2 \rightarrow \mathrm{h}(2) - 2\mathrm{h}(0.5) + \mathrm{h}(-1) = \frac{9}{2}$ $x = 0.5 \rightarrow \mathrm{h}(0.5) - 2\mathrm{h}(-1) + \mathrm{h}(2) = -\frac{9}{2}$ $x = -1 \rightarrow \mathrm{h}(-1) - 2\mathrm{h}(2) + \mathrm{h}(0.5) = 0$ Using GC, $\mathrm{h}(2) = 0.5$, $\mathrm{h}(0.5) = -1$ and $\mathrm{h}(-1) = 2$.	The values of $\mathrm{h}(2)$, $\mathrm{h}(0.5)$ and $\mathrm{h}(-1)$ can be found most directly using a system of linear equations. The relevant equations should become apparent after substituting the x values of interest into the given functional equation.
7 (b) (ii) [2]	$\mathrm{h}(2) = 0.5$ $\mathrm{hh}(2) = \mathrm{h}(0.5) = -1$ $\mathrm{hhh}(2) = \mathrm{h}(-1) = 2$ $\therefore \mathrm{hhh}(x) = x$ $\mathrm{hhh}(x) = \mathrm{h}\left(\frac{1}{1-x}\right) = x$ Let $u = \frac{1}{1-x}$ $\therefore 1 - x = \frac{1}{u} \rightarrow x = 1 - \frac{1}{u}$ $\therefore \mathrm{h}(u) = 1 - \frac{1}{u}$ Rewriting variable $u \rightarrow x$, we have $\mathrm{h}(x) = 1 - \frac{1}{x}$	It should be rather evident from **(b)(i)** that self-composing h enough times will yield the input of the function. Afterwards, candidates may encounter a function h with its input being a complicated expression in x, which can be simplified using a substitution – essentially similar to the standard method of finding inverses. As a side, it might be interesting to note that, like the function f in **(a)**, h is also "self-inverse" – or rather more accurately an *iterated function* with period 3 (and involutions like f are said to be iterated functions with period 2).

8 **(i)** **[4]**	$x = a\sin^2\theta$ $\rightarrow \mathrm{d}x = 2a\sin\theta\cos\theta\,\mathrm{d}\theta = a\sin 2\theta$ $\rightarrow$ When $x = 0$, $\theta = 0$ $\rightarrow$ When $x = a$, $\theta = \frac{1}{2}\pi$ $\therefore I_0$ $= \int_0^a \sqrt{ax - x^2}\,\mathrm{d}x$ $= \int_0^{\frac{1}{2}\pi} \sqrt{a^2\sin^2\theta - a^2\sin^4\theta}\,(a\sin 2\theta)\,\mathrm{d}\theta$ $= \int_0^{\frac{1}{2}\pi} \sqrt{a^2\sin^2\theta\,(1 - \sin^2\theta)}(a\sin 2\theta)\,\mathrm{d}\theta$ $= \int_0^{\frac{1}{2}\pi} (a\sin\theta\cos\theta)(a\sin 2\theta)\,\mathrm{d}\theta$ $= \frac{a^2}{2}\int_0^{\frac{1}{2}\pi} \sin^2 2\theta\,\mathrm{d}\theta$ $= \frac{a^2}{2}\int_0^{\frac{1}{2}\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right)\mathrm{d}\theta$ $= \frac{a^2}{2}\left[\frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\right]_0^{\frac{1}{2}\pi}$ $= \frac{\pi a^2}{8}$	Integration by trigonometric substitution is fairly routine in A–Levels. Like most integration questions involving trigonometric functions, this one involves the use of several identities to arrive at the exact answer.

8 (ii) [4]	Using integration by parts, I_{n+1} $= \int_0^a x^{n+1}\sqrt{ax-x^2}\,\mathrm{d}x$ $= \int_0^a x^{n+\frac{3}{2}}(a-x)^{\frac{1}{2}}\,\mathrm{d}x$ $= \left[\left(x^{n+\frac{3}{2}}\right)\left(-\frac{2}{3}(a-x)^{\frac{3}{2}}\right)\right]_0^a - \int_0^a \left(\frac{2n+3}{2}x^{n+\frac{1}{2}}\right)\left(-\frac{2}{3}(a-x)^{\frac{3}{2}}\right)\mathrm{d}x$ $= \left[\left(a^{\frac{2n+3}{2}}\right)(0) - (0)(-\frac{2}{3}a^{\frac{3}{2}})\right] + \frac{2n+3}{3}\int_0^a \left[\left(x^{n+\frac{1}{2}}\right)(a-x)^{\frac{1}{2}}\right](a-x)\,\mathrm{d}x$ $= \frac{2n+3}{3}\int_0^a a\left[\left(x^{n+\frac{1}{2}}\right)(a-x)^{\frac{1}{2}}\right] - x\left[\left(x^{n+\frac{1}{2}}\right)(a-x)^{\frac{1}{2}}\right]\mathrm{d}x$ $= \frac{2n+3}{3}\left[a\int_0^a \left[\left(x^{n+\frac{1}{2}}\right)(a-x)^{\frac{1}{2}}\right]\mathrm{d}x - \int_0^a \left[\left(x^{n+\frac{3}{2}}\right)(a-x)^{\frac{1}{2}}\right]\mathrm{d}x\right]$ $= \frac{2n+3}{3}\left[a\int_0^a x^n\sqrt{ax-x^2}\,\mathrm{d}x - \int_0^a x^{n+1}\sqrt{ax-x^2}\,\mathrm{d}x\right]$ $= \frac{2n+3}{3}\left[aI_n - I_{n+1}\right]$ $\to 3I_{n+1} = (2n+3)aI_n - (2n+3)I_{n+1}$ $\to (2n+6)I_{n+1} = (2n+3)aI_n$ $\therefore I_{n+1} = \left(\frac{2n+3}{2n+6}\right)aI_n$	Integral equations of this form are known as *reduction formula* of an integral. While reduction formula is not formally tested in A–Levels (and in fact barred in future syllabus), it has often appeared in preliminary exams as an application of integration by parts. Admittedly, this part of the question may prove to be challenging. To proceed, there are key observations to be recognised: • Both I_{n+1} and I_n are in the equation → the term x^n will undergo integration or differentiation, signalling the need for **integration by parts**. • The constant a is in the equation → ***a* must somehow be extracted out** from the integrand, either by algebraic manipulation, calculus, or any other way. This suggested answer begins with I_{n+1} and rewriting it to obtain the term $x^{n+\frac{3}{2}}$ to be differentiated, and the term $(a-x)^{\frac{1}{2}}$ to be integrated. It is also possible to begin with I_n and integrating x^n to obtain I_{n+1} – both approaches will nonetheless require some degree of manipulation.

8 (iii) [4]	Consider the sketch of $y=\sqrt{(2x-x^2)^5}$ below:  $\therefore$ Required area $=\int_0^2 \sqrt{(2x-x^2)^5}\,\mathrm{d}x$ $=\int_0^2 \left(2x-x^2\right)^2\sqrt{2x-x^2}\,\mathrm{d}x$ $=\int_0^2 \left(4x^2-4x^3+x^4\right)\sqrt{2x-x^2}\,\mathrm{d}x$ $=4\int_0^2 x^2\sqrt{2x-x^2}\,\mathrm{d}x-4\int_0^2 x^3\sqrt{2x-x^2}\,\mathrm{d}x+\int_0^2 x^4\sqrt{2x-x^2}\,\mathrm{d}x$ $=4I_2-4I_3+I_4$ $=4I_2-4\times\left(\frac{2(2)+3}{2(2)+6}\right)(2)\times I_2+\left(\frac{2(3)+3}{2(3)+6}\right)(2)\times\left(\frac{2(2)+3}{2(2)+6}\right)(2)\times I_2$ $=\left(4-\frac{28}{5}+\frac{63}{30}\right)\times I_2$ $=\left(4-\frac{28}{5}+\frac{63}{30}\right)\times\left(\frac{2(1)+3}{2(1)+6}\right)(2)\times\left(\frac{2(0)+3}{2(0)+6}\right)(2)\times\frac{\pi(2)^2}{8}$ $=\frac{1}{2}\times\frac{5}{4}\times 1\times\frac{\pi}{2}$ $=\frac{5\pi}{16}$ units^2	The most appropriate way to start finding the area is to first sketch the region of interest. This will first and foremost reveal the boundary values of the integrals. When integrating, there may be an initial temptation to factor out some power of x from $\sqrt{(2x-x^2)^5}$ to obtain $x^n\sqrt{(2x-x^2)}$, which is not clearly feasible and may potentially lead to a dead-end. A more feasible approach is to interpret the expression as being some power away from the root function $\sqrt{2x-x^2}$, yielding the form $\left(2x-x^2\right)^n\sqrt{2x-x^2}$ which is far more straightforward to execute. It is eventually apparent after several manipulation that the area can be expressed as a combination of I_2, I_3 and I_4. The remaining calculations follow, with further simplification done by collecting I_2 first before collapsing further to I_0.

9 (i) [1]	$\frac{dx}{dt} = \frac{a}{t}$ $\frac{dy}{dt} = \frac{a}{2}\left(1 - \frac{1}{t^2}\right)$ $\therefore \frac{dy}{dx} = \frac{\frac{a}{2}\left(1-\frac{1}{t^2}\right)}{\frac{a}{t}} = \frac{\frac{a}{2t^2}(t^2-1)}{\frac{a}{2t^2}(2t)} = \frac{t^2-1}{2t}$	Finding the gradient function of a curve defined parametrically is standard in A–Levels. Candidates should be able to arrive at the result with little to no issue.
9 (ii) [3]	For the lowest point on the rope, $\frac{dy}{dx} = \frac{t^2-1}{2t} = 0 \rightarrow t = \pm 1$ Finding the horizontal position x of the lowest point with $t = 1$ (since $\ln t$ is only defined for $t > 1$), $x = a(\ln 1) = 0$ Since the lowest point is horizontally in the middle of both persons standing $2a$ units apart, Person 1 is at $x = -a \rightarrow t_1 = e^{-1}$ Person 2 is at $x = a \rightarrow t_2 = e$	Finding the lowest point on the rope is a classic minima problem. It follows that $x = 0$ for the lowest point. The subsequent part on finding t_1 and t_2 requires proper interpretation of the question. It should be relatively easy to visualise, without a sketch, that the two persons can only stand on $x = -a$ and $x = a$ for them to stand $2a$ units apart with the low point in the middle at $x = 0$.
9 (iii) [2]	Angle at $t_1 = e$ $= \tan^{-1}\left(\frac{e^2-1}{2e}\right) \approx 49.6°$ Angle at $t_2 = e^{-1}$ $= \tan^{-1}\left(\frac{e^{-2}-1}{2e^{-1}}\right) \approx -49.6°$ Both ends of the rope make an angle of 49.6° with the horizontal.	It is becoming a trend in A–Levels to inquire about angles given the gradient of a line. Although not squarely within the syllabus, the relationship between the **gradient $\boldsymbol{m}$** of a line and the principal **angle $\boldsymbol{\alpha}$** it makes with the horizontal satisfies $\boldsymbol{\alpha = \tan^{-1} m}$**.** In the context of this question, the angle subtended by both ends of the rope with the horizontal is equal due to the symmetry of the curve.

9 (iv) [2]	$x = a\ln t \to t = \mathrm{e}^{\frac{x}{a}}$ $\therefore y = \frac{a}{2}\left(\mathrm{e}^{\frac{x}{a}} + \frac{1}{\mathrm{e}^{\frac{x}{a}}}\right) = \frac{a}{2}\left(\mathrm{e}^{\frac{x}{a}} + \mathrm{e}^{-\frac{x}{a}}\right)$ Using t_1 (or t_2) to find the height at which the rope is being held, $k = \frac{a}{2}\left(\mathrm{e} + \mathrm{e}^{-1}\right)$	Finding the cartesian equation of this curve should be straightforward, as there is a simple relationship between x and t. Finding k in terms of a afterwards is routine
9 (v) [5]	Considering transformation, volume required $= \pi \int_{-a}^{a} (y-k)^2 \,\mathrm{d}x$ $= \pi \int_{-a}^{a} \left[\frac{a}{2}\left(\mathrm{e}^{\frac{x}{a}} + \mathrm{e}^{-\frac{x}{a}}\right) - \frac{a}{2}\left(\mathrm{e} + \mathrm{e}^{-1}\right)\right]^2 \mathrm{d}x$ $= \frac{\pi a^2}{4}\int_{-a}^{a}\left[\left(\mathrm{e}^{\frac{x}{a}} + \mathrm{e}^{-\frac{x}{a}}\right)^2 - 2\left(\mathrm{e}+\mathrm{e}^{-1}\right)\left(\mathrm{e}^{\frac{x}{a}} + \mathrm{e}^{-\frac{x}{a}}\right) + \left(\mathrm{e}+\mathrm{e}^{-1}\right)^2\right]\mathrm{d}x$ $= \frac{\pi a^2}{4}\int_{-a}^{a}\left[\mathrm{e}^{\frac{2x}{a}} + 2 + \mathrm{e}^{-\frac{2x}{a}} - 2\left(\mathrm{e}+\mathrm{e}^{-1}\right)\left(\mathrm{e}^{\frac{x}{a}} + \mathrm{e}^{-\frac{x}{a}}\right) + \left(\mathrm{e}+\mathrm{e}^{-1}\right)^2\right]\mathrm{d}x$ $= \frac{\pi a^2}{4}\left[\frac{a}{2}\mathrm{e}^{\frac{2x}{a}} + 2x - \frac{a}{2}\mathrm{e}^{-\frac{2x}{a}} - 2\left(\mathrm{e}+\mathrm{e}^{-1}\right)\left(a\mathrm{e}^{\frac{x}{a}} - a\mathrm{e}^{-\frac{x}{a}}\right) + \left(\mathrm{e}+\mathrm{e}^{-1}\right)^2 x\right]_{-a}^{a}$ $= \frac{\pi a^2}{4}\left[a\mathrm{e}^2 + 4a - a\mathrm{e}^{-2} - 4a\left(\mathrm{e}+\mathrm{e}^{-1}\right)\left(\mathrm{e}-\mathrm{e}^{-1}\right) + 2a\left(\mathrm{e}+\mathrm{e}^{-1}\right)^2\right]$ $= \frac{\pi a^3}{4}\left[\mathrm{e}^2 + 4 - \mathrm{e}^{-2} - 4e^2 + 4e^{-2} + 2e^2 + 4 + 2e^{-2}\right]$ $= \frac{\pi a^3}{4}\left(8 - e^2 + 5e^{-2}\right)$ $\therefore A = \frac{\pi}{4},\ \ B = 8,\ \ C = -1\ \text{ and }\ D = 5$	This part may prove challenging. There are some key observations to be made. Most importantly, it must be noted that **the axis of rotation for the volume of revolution is not the x–axis as per usual**, but instead the line $y = k$. Most of the heavy lifting in this question comes from integrating $(y-k)^2$, which unfortunately simplifies to a myriad of terms to be integrated – though it *is* straightforward to integrate these terms individually. When evaluating the integral, candidates may simplify their working should they recognise that the integral expression returns the same value for both the upper bound and the lower bound, and thus the integral value is simply double of the integral expression. This is indeed possible as the integrand is in the form of an *even function*, which satisfies $\mathrm{f}(x) = \mathrm{f}(-x)$ for all x (and this can easily be proven for this curve). The context of this question also lends itself to the evenness of this integrand. With careful manipulation, the remaining steps follow.

10 (i) (a) [2]	$\dfrac{\mathrm{d}R}{\mathrm{d}t} = 0.6R - 0.4RW$ $\dfrac{\mathrm{d}W}{\mathrm{d}t} = -0.8W + 0.2RW$	Writing down differential equations in context is classic in A–Levels. Such contextual questions usually allocate a few marks in the initial parts to ensure proper interpretation before any mathematical work is done further in subsequent parts.
10 (i) (b) [5]	$\dfrac{\mathrm{d}W}{\mathrm{d}R} = \dfrac{\frac{\mathrm{d}W}{\mathrm{d}t}}{\frac{\mathrm{d}R}{\mathrm{d}t}} = \dfrac{-0.8W + 0.2RW}{0.6R - 0.4RW} = \dfrac{W(-0.8 + 0.2R)}{R(0.6 - 0.4W)}$ $\int \left(\dfrac{0.6 - 0.4W}{W}\right) \mathrm{d}W = \int \left(\dfrac{-0.8 + 0.2R}{R}\right) \mathrm{d}R$ $\int \left(\dfrac{0.6}{W} - 0.4\right) \mathrm{d}W = \int \left(\dfrac{-0.8}{R} + 0.2\right) \mathrm{d}R$ $0.6 \ln W - 0.4W = -0.8 \ln R + 0.2R + C, \quad C \in \mathbb{R}$ Substituting initial population of $W = 1$ and $R = 1$, $0.6 \ln(1) - 0.4(1) = -0.8 \ln(1) + 0.2(1) + C$ $C = -0.4 - 0.2 = -0.6$ $0.6 \ln W - 0.4W = -0.8 \ln R + 0.2R - 0.6$ $-3 \ln W + 2W = 4 \ln R - R + 3$ $2W + R - 3 \ln W - 4 \ln R = 3$	Lotka–Volterra equations make use of a system of non-homogeneous differential equations – such differential equations are a form that hardly yields a 'nice' expression in terms of for each population (and it would also require mathematical tools outside of the syllabus). It is instead much more manageable to find an implicit relationship governing R and W. The absence of t in the equation should sufficiently hint at **connected rates of change**. After obtaining an expression for $\frac{\mathrm{d}W}{\mathrm{d}R}$ (or $\frac{\mathrm{d}R}{\mathrm{d}W}$) in terms of R and W, subsequent steps involving separable differential equations follow as per usual. As the time variable t is no longer relevant to the equation, the initial populations $R = 1$ and $W = 1$ can be directly substituted to solve for the constant of integration C. This part also serves as a hindsight checkpoint. If the differential equations in **(i)** are indeed correct, they should yield an implicit equation that is consistent with what needs to be shown.

10 (ii) (a) [4]	Consider the equations in **(i)(a)** to find the extreme values of R and W. For smallest and largest rabbit population size, $\frac{dR}{dt} = 0 \to R(0.6 - 0.4W) = 0$ Since $R \neq 0$ as seen from the graph, $0.6 - 0.4W = 0 \to W = 1.5$ Substituting $W = 1.5$ into the equation in **(i)(b)**, $3 + R - 3\ln 1.5 - 4\ln R = 3$ $R - 4\ln R = 3\ln 1.5$. Using GC, $R = 0.915 \approx 0.9$ or $R = 10.696 \approx 10.7$ $\therefore$ The smallest rabbit population size is 900, and the largest rabbit population size is 10,700. For smallest and largest wolf population size, $\frac{dW}{dt} = 0 \to W(-0.8 + 0.2R) = 0$ Since $W \neq 0$ as seen from the graph, $-0.8 + 0.2R = 0 \to R = 4$ Substituting $R = 4$ into the equation in **(i)(b)**, $2W + 4 - 3\ln W - 4\ln 4 = 3$. $2W - 3\ln W = 4\ln 4 - 1$ Using GC, $W = 0.262 \approx 0.3$ or $W = 4.543 \approx 4.5$ $\therefore$ The smallest wolf population size is 300, and the largest wolf population size is 4,500.	The given sketch of $2W + R - 3\ln W - 4\ln R = 3$ shows that the curve contains turning points which correspond to the minimum and maximum values of R and W respectively. Once recognised, this question is reduced to a standard question on maxima-minima. Again, care must be taken to **justify rejecting roots** from certain factors, as such display of awareness are likely to be considered for a few marks. Upon substituting relevant values into the equation in **(i)(b)**, a calculator can be used subsequently to obtain two values corresponding to the maxima and minima. **Care must be taken to convert these values into the level of precision that is demanded by the question.**
10 (ii) (b) [2]	Considering the equations in **(i)(a)**, and substituting the initial population $R = 1$ and $W = 1$, we have $\frac{dR}{dt} = 0.6 - 0.4 = 0.2 > 0$ $\frac{dW}{dt} = -0.8 + 0.2 = -0.6 < 0$ $\therefore$ After the initial year, R increases while W decreases. $\therefore$ Point B_2 corresponds to the population in the year after the initial.	This question is pertaining rates of change, particularly the ability to apply rate equations to predict, in context, the behaviour of a relevant variable. Substituting the initial population into the rate of change of R and W respectively in **(i)(a)** will yield sufficient information to help determine the behaviour of each population size in the upcoming year (unit time t).

11 (i) [4]

Tabulate amounts as follows:

Month	Amount at the start of this month	Amount at the end of this month
1	\$100,000	\$100,000(1.02)
2	\$100,000(1.02) + \$1,000	$\$100{,}000(1.02)^2 + \$1{,}000(1.02)$
3	$\$100{,}000(1.02)^2 + \$1{,}000(1.02+1)$	$\$100{,}000(1.02)^3 + \$1{,}000\left(1.02^2 + 1.02\right)$
$\vdots$	$\vdots$	$\vdots$
n	S_n	

The amount at the start of the nth month after January 2023 in Mr Wong's account, S_n

$= \$100{,}000(1.02)^{n-1} + \$1{,}000\left(1.02^{n-2} + 1.02^{n-3} + \cdots + 1\right)$

$= \$100{,}000(1.02)^{n-1} + \$1{,}000\left(\dfrac{1.02^{n-1}-1}{1.02-1}\right)$

$= \$100{,}000(1.02)^{n-1} + \$50{,}000\left(1.02^{n-1}-1\right)$

$= \$150{,}000(1.02)^{n-1} - \$50{,}000$

Using GC to find the least n such that $S_n > \$300{,}000$,

n	S_n
42	\$287,830.07 (< \$300,000)
43	\$294,586.67 (< \$300,000)
44	\$301,478.40 (> \$300,000)

At the start of the 44th month after January 2023, the total is \$301,478.40, which exceeds \$300,000. At the end of the 43rd month after January 2023, the total is \$300,478.40, since Mr Wong is yet to put \$1,000 more into the account. This also exceeds \$300,000.

$\therefore$ The total in the account will first exceed the price of the car at the end of July 2026.

Financial mathematics involving compound interest is a classic application of geometric progression in A–Levels. When approaching questions involving interest and monthly investment, it is best to tabulate the monthly amounts for clarity. This will greatly help in finding the expression for the account total in terms of the months passed n.

Just as important, **great care must be taken when writing the ratio and avoid mistaking the percentage as the ratio itself** (i.e., "$r = 0.02$" is incorrect).

The context behind financial mathematics question such as this one may vary slightly across questions, most notably the day of the month in which the monthly investment and the interest rate occurs. While it may seem no different to sum the amounts on either day, **it is recommended to find the sum on days the monthly investment occurs**. Here are some reasons:

- The newly added investment adds one (+1) to the geometric portion of the sum – that portion becomes $\left(r^{n-1} + r^{n-2} + \cdots + r + 1\right)$, which has a more direct sum expression as opposed to the incomplete $\left(r^{n-1} + r^{n-2} + \cdots + r\right)$.
- It is easier to use this sum to justify answers involving the "first day vs. last day" of the month. The question is simply reduced to "does the amount exceed with or without this '+1?'"

The final part on counting months can be done by expressing n as $12a + b$, where $a, b \in \mathbb{Z}^+$ and $b < 12$, and simply add a years and b months to the initial time.

11 (ii) [5]

After deposit, the remaining amount to be paid is \$300,000 – \$75,000 = \$225,000

Tabulate amounts as follows:

Month	Outstanding at the start of this month	Outstanding at the end of this month
1	\$225,000	\$225,000(1.005)
2	\$225,000(1.005) – \$15,000	$\$225{,}000(1.005)^2 - \$15{,}000(1.005)$
3	$\$225{,}000(1.005)^2 - \$15{,}000(1.005 + 1)$	$\$225{,}000(1.005)^3 - \$15{,}000(1.005^2 + 1.005)$
⋮	⋮	⋮
n	R_n	

The outstanding amount to be paid at the start of the nth payment month, R_n

$$= \$225{,}000(1.005)^{n-1} - \$15{,}000\left(1.005^{n-2} + 1.005^{n-3} + \cdots + 1\right)$$
$$= \$225{,}000(1.005)^{n-1} - \$15{,}000\left(\frac{1.005^{n-1} - 1}{1.005 - 1}\right)$$
$$= \$225{,}000(1.005)^{n-1} - \$3{,}000{,}000\left(1.005^{n-1} - 1\right)$$
$$= \$3{,}000{,}000 - \$2{,}775{,}000(1.005)^{n-1}$$

Using GC to find the least n such that $R_n < \$0$,

n	R_n
15	\$24,308.86 (> \$0)
16	\$9,430.40 (> \$0)
17	−\$5,522.44 (< \$0)

At the beginning of the seventeenth payment month, R_n starts to become negative. This means that the monthly instalments of \$15,000 first exceeds the outstanding amount to be paid at this month. Hence, he would have completely paid for the car at the beginning of the seventeenth payment month.

∴ The amount to be paid at this month = \$15,000 – \$5,522.44 = \$9,477.56.

This part onwards regarding the hire purchase scheme requires proper question interpretation, though the approach is identical to the previous part.

Great care must be taken to address several contextual information:

- **Initial payment**: After the deposit, the amount affected by the interest is \$225,000 (not \$300,000).
- **Interest**: The 0.5% interest converts to a ratio of 1.005 (not 0.5 or 1.05).
- **Instalments**: Monthly instalments are paid “on subsequent months”, so apart from the deposit, there is no further deduction of \$15,000 for the first payment month.
- **Final amount**: A good initial assumption is that the final amount due is most likely less than \$15,000.

11 (iii) [5]

Let the amount at this first payment month be S_k.

At the first payment month, Mr Wong pays a deposit of \$75,000 and no longer puts additional \$1,000.

$\therefore S_k = \$150{,}000(1.02)^{k-1} - \$50{,}000 - \$75{,}000 - \$1{,}000 = \$150{,}000(1.02)^{k-1} - \$126{,}000$

Using the number of month and the amount paid found in **(iii)**,

$\therefore S_{k+17} = \$527{,}328.50 + \$9{,}477.56 = \$536{,}806.06$

Tabulate amounts as follows:

Month	Amount at the start of this month	Amount at the end of this month
k	$\$150{,}000(1.02)^{k-1}$ $-\$126{,}000$	$\$150{,}000(1.02)^{k}$ $-\$126{,}000(1.02)$
$k+1$	$\$150{,}000(1.02)^{k}$ $-\$126{,}000(1.02)$ $-\$15{,}000$	$\$150{,}000(1.02)^{k+1}$ $-\$126{,}000(1.02)^2$ $-\$15{,}000(1.02)$
$k+2$	$\$150{,}000(1.02)^{k+1}$ $-\$126{,}000(1.02)^2$ $-\$15{,}000(1.02+1)$	$\$150{,}000(1.02)^{k+2}$ $-\$126{,}000(1.02)^3$ $-\$15{,}000\left(1.02^2+1.02\right)$
$\vdots$	$\vdots$	$\vdots$
$k+17$	S_{k+17}	

S_{k+17}

$= \$150{,}000(1.02)^{k+16} - \$126{,}000(1.02)^{17} - \$15{,}000\left(1.02^{16} + 1.02^{15} + \cdots + 1\right)$

$= \$150{,}000(1.02)^{k+16} - \$126{,}000(1.02)^{17} - \$15{,}000\left(\dfrac{1.02^{17}-1}{1.02-1}\right)$

$= \$150{,}000(1.02)^{k+16} - \$126{,}000(1.02)^{17} - \$750{,}000\left(1.02^{17}-1\right)$

$= \$150{,}000(1.02)^{k+16} - \$876{,}000(1.02)^{17} + \$750{,}000 = \$536{,}806.06$

Using GC, $k = 80$

$\therefore$ Mr Wong made the first payment in August 2029.

This part requires the expression for the amount in the bank account pre-purchase, which will be used accordingly as a starting point. Once the expression for the starting capital and the fixed amount to be deducted monthly are found, the approach to this question becomes identical to the previous part.

Great care must be taken to address several contextual information:

- **No more monthly investment**: Avoid using a new expression that assumes Mr Wong is still investing \$1,000 monthly during the hire purchase. Also, **make sure to deduct \$1,000 accordingly from S_n**, should it include the last monthly investment.
- **Final payment**: This must be added back to the given balance to obtain the correct total after interest.
- **Interest**: Note that bank interest still applies throughout the deduction, and as such **S_n is not as simple to find as adding back the total amount paid to the given balance, as this would ignore interest being applied while deduction is still occurring**. Also, care must be taken to not confuse the interest value from the bank (2%) with that from the hire purchase (0.5%).

This part also requires counting months, which can be done by expressing n in the form $12a + b$, as recommended in **(ii)**.