Paper 2		Remarks
1	$\frac{dy}{dx} = 3ax^2 + 2bx$ At the turning points, $\frac{dy}{dx} = 0$. So, $3ax^2 + 2bx = 0$ $x(3ax + b) = 0$ $x = 0$ or $x = -\frac{2b}{3a}$. When $x = 0$, $y = c$. When $x = -\frac{2b}{3a}$, $y = a\left(-\frac{2b}{3a}\right)^3 + b\left(-\frac{2b}{3a}\right)^2 + c$ $= a\left(-\frac{8b^3}{27a^3}\right) + b\left(\frac{4b^2}{9a^2}\right) + c$ $= \left(-\frac{8}{27} + \frac{4}{9}\right)\frac{b^3}{a^2} + c$ $= \frac{4b^3}{27a^2} + c$ For the equation $ax^3 + bx^2 + c = 0$ to have 3 distinct real roots, the y-coordinates of the two turning points must have different signs. Thus $c\left(\frac{4b^3}{27a^2} + c\right) < 0$ $c\left(4b^3 + 27a^3c\right) < 0$ $27a^3c^2 + 4b^3c < 0$ (shown)	
2	**(a)** $u_{n+2} = \frac{1}{2}\left(u_{n+1} + u_n\right)$ $2u_{n+2} - u_{n+1} - u_n = 0$ The auxiliary equation of the RR is $2m^2 - m - 1 = 0$ $(2m+1)(m-1) = 0$ $m = -\frac{1}{2}$ or $m = 1$ $u_n = c\left(-\frac{1}{2}\right)^n + d(1)^n$ $u_n = c\left(-\frac{1}{2}\right)^n + d$ $u_1 = a$ $-\frac{1}{2}c + d = a \quad -(1)$ $u_2 = b$ $\frac{1}{4}c + d = b \quad -(2)$	

	(2) – (1): $c = \frac{4}{3}(b-a)$ (1) + 2(2): $d = \frac{1}{3}(a+2b)$ $\therefore u_n = \frac{4}{3}(b-a)\left(-\frac{1}{2}\right)^n + \frac{1}{3}(a+2b), \quad n \geq 1$ **(b)(i)** $u_n = \frac{4}{3}(b-a)\left(-\frac{1}{2}\right)^n + \frac{1}{3}(a+2b), \quad n \geq 1$ As $n \to \infty, u_n \to 3$ since $\left(-\frac{1}{2}\right)^n \to 0$ as $n \to \infty$, therefore $\frac{1}{3}(a+2b) = 3$ $a + 2b = 9$ Since a and b are distinct positive integers, hence possible pairs of values of a and b are: $a = 1$ and $b = 4$, $a = 5$ and $b = 2$, $a = 7$ and $b = 1$ **(b)(ii)** If $a = b$, then $u_n = \frac{4}{3}(0)\left(-\frac{1}{2}\right)^n + \frac{1}{3}(a+2a)$ $u_n = a, \quad n \geq 1$ Then the sequence is a constant sequence.	

3	**(a)** Let $\mathrm{f}(x,y)=\dfrac{y(x+\ln y)}{x}$, $x_0=1, y_0=1$. $u_1=y_0+0.5\mathrm{f}(x_0,y_0)=1+0.5\left(\dfrac{1+\ln 1}{1}\right)=1.5$ $y_1=y_0+\dfrac{0.5}{2}\left[\mathrm{f}(x_0,y_0)+\mathrm{f}(x_1,u_1)\right]=1+0.25\left(1+\dfrac{1.5(1.5+\ln 1.5)}{1.5}\right)$ $=1.72637...$ $u_2=y_1+0.5\mathrm{f}(x_1,y_1)=2.90376...$ $y_2=y_1+\dfrac{0.5}{2}\left[\mathrm{f}(x_1,y_1)+\mathrm{f}(x_2,u_2)\right]=3.42793...=3.43$ (3sf) **(b)(i)** $y=\dfrac{x^x}{\sin x}, 0<x<\pi$ Taking logarithms, we have $\ln y=x\ln x-\ln\sin x$. $\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x}=x\left(\dfrac{1}{x}\right)+\ln x-\dfrac{\cos x}{\sin x}$ At turning point, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$. $\cot x=1+\ln x$ $\tan x=\dfrac{1}{1+\ln x}$ $x=\tan^{-1}\left(\dfrac{1}{1+\ln x}\right)$ (shown) **(b)(ii)** $x_{n+1}=\mathrm{g}(x_n)$, with $x_0=0.9$ From GC, $x_1=0.84095, x_2=0.87994,\cdots,x_n$ converges to 0.864 (3sf). $\therefore$ x-coordinate of the turning point is 0.864, y-coordinate is 1.16 (3sf)	
4	**(a)(i)** $y=A\sin\sqrt{x}+B\cos\sqrt{x}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\left(A\cos\sqrt{x}\right)\left(\dfrac{1}{2}x^{-\frac{1}{2}}\right)-\left(B\sin\sqrt{x}\right)\left(\dfrac{1}{2}x^{-\frac{1}{2}}\right)$ $2\sqrt{x}\dfrac{\mathrm{d}y}{\mathrm{d}x}=A\cos\sqrt{x}-B\sin\sqrt{x}$ $4x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\left(A\sin\sqrt{x}+B\cos\sqrt{x}\right)$ $4x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\dfrac{\mathrm{d}y}{\mathrm{d}x}+y=0$ (shown) $2\sqrt{x}\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(x^{-\frac{1}{2}}\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}=\left(-A\sin\sqrt{x}\right)\left(\dfrac{1}{2}x^{-\frac{1}{2}}\right)-\left(B\cos\sqrt{x}\right)\left(\dfrac{1}{2}x^{-\frac{1}{2}}\right)$ **(a)(ii)** Let P_n be the statement $4x\dfrac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}+(4n+2)\dfrac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}+\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}=0$ for all integers $n\geq 0$. LHS of $P_0=4x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+[4(0)+2]\dfrac{\mathrm{d}y}{\mathrm{d}x}+y=4x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\dfrac{\mathrm{d}y}{\mathrm{d}x}+y$	

	RHS of $P_0 = 0$ LHS = RHS (by **(i)**) $\therefore P_0$ is true. Assume that P_k is true for some integers $k \geq 0$, i.e., $4x\frac{d^{k+2}y}{dx^{k+2}} + (4k+2)\frac{d^{k+1}y}{dx^{k+1}} + \frac{d^k y}{dx^k} = 0$ We want to prove that P_{k+1} is true, i.e., $4x\frac{d^{k+3}y}{dx^{k+3}} + (4k+6)\frac{d^{k+2}y}{dx^{k+2}} + \frac{d^{k+1}y}{dx^{k+1}} = 0$ Consider P_k, $4x\frac{d^{k+2}y}{dx^{k+2}} + (4k+2)\frac{d^{k+1}y}{dx^{k+1}} + \frac{d^k y}{dx^k} = 0$ $4x\frac{d^{k+3}y}{dx^{k+3}} + 4\frac{d^{k+2}y}{dx^{k+2}} + (4k+2)\frac{d^{k+2}y}{dx^{k+2}} + \frac{d^{k+1}y}{dx^{k+1}} = 0$ $4x\frac{d^{k+3}y}{dx^{k+3}} + (4k+6)\frac{d^{k+2}y}{dx^{k+2}} + \frac{d^{k+1}y}{dx^{k+1}} = 0$ (shown) $\therefore$ P_k is true $\Rightarrow P_{k+1}$ is true Since (1) P_0 is true, and (2) P_k is true $\Rightarrow P_{k+1}$ is true, by mathematical induction, P_n is true for all integers $n \geq 0$. **(b)** $4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 - 25$ Let $Q(x) = Dx^2 + Ex + F$, then $Q'(x) = 2Dx + E$ $Q''(x) = D$ Hence, $4xD + 2(2Dx + E) + (Dx^2 + Ex + F) = x^2 - 25$ $Dx^2 + (12D + E)x + (2E + F) = x^2 - 25$ By comparing coefficients, $D = 1, \quad 12D + E = 0, \quad 2E + F = -25$ $D = 1, E = -12$ and $F = -1$ $\therefore Q(x) = x^2 - 12x - 1$ Let $F(x) = A\sin\sqrt{x} + B\cos\sqrt{x}$ and $y = Q(x) + F(x)$, then $\text{LHS} = 4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y$ $= 4x[Q''(x) + F''(x)] + 2[Q'(x) + F'(x)] + [Q(x) + F(x)]$ $= [4xQ''(x) + 2Q'(x) + Q(x)] + [4xF''(x) + 2F'(x) + F(x)]$ $= (x^2 - 25) + 0$ $= x^2 - 25 = \text{RHS}$	

Hence, $Q(x)+A\sin\sqrt{x}+B\cos\sqrt{x}$ for any arbitrary constants A and B, is a solution of this differential equation.

5

(a)

$$2x^2+2xy+y^2=1$$

$$2r^2\cos^2\theta+2\left(r\cos\theta\right)\left(r\sin\theta\right)+r^2\sin^2\theta=1$$

$$r^2\left(2\cos^2\theta+2\cos\theta\sin\theta+r^2\sin^2\theta\right)=1$$

$$\frac{r^2\left(2\cos^2\theta+2\cos\theta\sin\theta+\sin^2\theta\right)}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

$$r^2\left(2+\frac{2\sin\theta}{\cos\theta}+\frac{\sin^2\theta}{\cos^2\theta}\right)=\sec^2\theta$$

$$r^2\left(2+2\tan\theta+\tan^2\theta\right)=\sec^2\theta$$

$$r^2\left(1+1+2\tan\theta+\tan^2\theta\right)=\sec^2\theta$$

$$r^2\left[1+\left(1+\tan\theta\right)^2\right]=\sec^2\theta$$

$$r^2=\frac{\sec^2\theta}{1+\left(1+\tan\theta\right)^2}\ \text{(shown)}$$

(b)

$$\text{Area}=\frac{1}{2}\int_0^{\frac{1}{4}\pi}\frac{\sec^2\theta}{1+\left(1+\tan\theta\right)^2}\,\mathrm{d}\theta$$

$$=\frac{1}{2}\left[\tan^{-1}\left(1+\tan\theta\right)\right]_0^{\frac{1}{4}\pi}$$

$$=\frac{1}{2}\tan^{-1}\left(1+\tan\frac{\pi}{4}\right)-\frac{1}{2}\tan^{-1}\left(1+0\right)$$

$$=\frac{1}{2}\tan^{-1}2-\frac{1}{2}\times\frac{\pi}{4}$$

$$=\frac{1}{2}\tan^{-1}2-\frac{\pi}{8}$$

(c)

$$r^2=\frac{\sec^2\theta}{1+(1+\tan\theta)^2}=\ldots=\frac{1}{1+2\sin\theta\cos\theta+\cos^2\theta}$$

$$2r\frac{\mathrm{d}r}{\mathrm{d}\theta}=\frac{2\sin\theta\cos\theta+2\sin^2\theta-2\cos^2\theta}{(1+2\sin\theta\cos\theta+\cos^2\theta)^2}$$

When $\frac{\mathrm{d}r}{\mathrm{d}\theta}=0$, $\sin\theta\cos\theta=\cos^2\theta-\sin^2\theta$

$$\Rightarrow\quad \frac{1}{2}\sin 2\theta=\cos 2\theta$$

$$\Rightarrow\quad \tan 2\theta=2$$

Let $m=\tan\theta$

$$\frac{2m}{1-m^2}=2\quad\Rightarrow\quad m^2+m-1=0$$

$$\therefore\quad m=\frac{-1\pm\sqrt{5}}{2}$$

5 **(d)** Exact area enclosed by E

$$= 4 \times \frac{1}{2} \int_{\tan^{-1}\left(\frac{-1-\sqrt{5}}{2}\right)}^{\tan^{-1}\left(\frac{-1+\sqrt{5}}{2}\right)} \frac{\sec^2\theta}{1+(1+\tan\theta)^2}\, \mathrm{d}\theta$$

$$= 2\left[\tan^{-1}(1+\tan\theta)\right]_{\tan^{-1}\left(\frac{-1-\sqrt{5}}{2}\right)}^{\tan^{-1}\left(\frac{-1+\sqrt{5}}{2}\right)}$$

$$= 2\tan^{-1}\left(\frac{1+\sqrt{5}}{2}\right) - 2\tan^{-1}\left(\frac{1-\sqrt{5}}{2}\right)$$

$$= 2\left[\tan^{-1}\left(\frac{1+\sqrt{5}}{2}\right) - \tan^{-1}\left(\frac{1-\sqrt{5}}{2}\right)\right]$$

$$= 2 \times \frac{\pi}{2}$$

$$= \pi$$

6 **(a)** Let T denote the time spent by Chan waiting at the bus stop in minutes. Thus, $T \sim \text{Exp}(0.2)$.

Note

$$\mathrm{E}(T) = \frac{1}{\lambda} = 5 \Rightarrow \lambda = \frac{1}{5}$$

$$\mathrm{f}(t) = \frac{1}{5}\mathrm{e}^{-\frac{1}{5}t}, \ t > 0$$

Not allowed to quote lack of memory property. Question is basically asking you to show.

$$\mathrm{P}(T \ge t+6 \mid T \ge t) = \frac{\mathrm{P}(T \ge t+6 \text{ and } T \ge t)}{\mathrm{P}(T \ge t)}$$

$$= \frac{\mathrm{P}(T \ge t+6)}{\mathrm{P}(T \ge t)}$$

Need to integrate. Cannot just quote cdf and use cdf to show!

$$= \frac{\int_{t+6}^{\infty} 0.2\mathrm{e}^{-0.2x}\,\mathrm{d}x}{\int_{t}^{\infty} 0.2\mathrm{e}^{-0.2x}\,\mathrm{d}x}$$

$$= \frac{\left[-\mathrm{e}^{-0.2x}\right]_{t+6}^{\infty}}{\left[-\mathrm{e}^{-0.2x}\right]_{t}^{\infty}}$$

$$= \frac{0-\left(-\mathrm{e}^{-0.2(t+6)}\right)}{0-\left(-\mathrm{e}^{-0.2t}\right)}$$

$$= \mathrm{e}^{-1.2}$$

which is independent of t.

(b)

NO need to integrate!

$$Y = 0.8T$$

$$\mathrm{Var}(Y) = \mathrm{Var}(0.8T)$$

$$= 0.8^2\,\mathrm{Var}(T)$$

$$= 0.64\left(5^2\right)$$

$$= 16 \text{ litres}^2.$$

7 **(a)** The Wilcoxon test is appropriate in this case as it takes into account both the **magnitude and sign of the difference** in the amount each customer spends on Friday and on Saturday, which are available from the data. Moreover, the **distribution of the difference** in the amount spent by each customer on Friday and on Saturday **is not known and is unlikely to follow a normal distribution**.

	(b) Let D = amount spent by customer on Friday − amount spent on Saturday. Let m_D be the population median of D. (NOT differences of median) $H_0 : m_D = 0$ $H_1 : m_D > 0$	

Customer	A	B	C	D	E	F	G	H	I	J
Amount spent on Friday	190	152	139	100	90	88	60	55	42	20
Amount spent on Saturday	150	120	132	103	99	83	66	40	32	24
D	40	32	7	–3	–9	5	–6	15	10	–4
Rank	10	9	5	–1	–6	3	–4	8	7	–2

$P =$ sum of positive ranks = 42

$Q =$ sum of negative ranks = 13

$T = \min(P, Q) = 13$

From MF26, for n = 10, 1-tailed test at 5% significance level, critical region is $T \le 10$.

Since T = 13 lies outside the critical region, we do not reject H_0 and conclude that there is insufficient evidence at 5% significance level that on average, customers who shop regularly on both Fridays and Saturdays spend more on Fridays than on Saturdays.

8

(a) Unbiased estimate of the population mean = $\bar{t} = \frac{\sum t}{10} = \frac{513.00}{10} = 51.300$

Unbiased estimate of population variance, s^2

$$= \frac{1}{9}\left(26524.26 - \frac{(513.00)^2}{10}\right) = 23.04$$

A 95% confidence interval for μ is

$$\left(\bar{t} - t_{0.025}\frac{s}{\sqrt{10}},\ \bar{t} + t_{0.025}\frac{s}{\sqrt{10}}\right) \quad \text{where } \mathrm{P}(T \ge t_{0.025}) = 0.025,\ T \sim t_9 .$$

i.e. $\left(51.300 - 2.262157\frac{\sqrt{23.04}}{\sqrt{10}},\ 51.300 + 2.262157\frac{\sqrt{23.04}}{\sqrt{10}}\right)$

i.e. (47.87, 54.73) (to 4 s.f.)

By symmetry, the sample mean for the second sample of 10 is $\frac{44.88 + 52.32}{2} = \frac{97.2}{2} = 48.6$

So for the second sample of 10, $\sum t_2 = 48.6 \times 10 = 486$

For this second sample of 10,

$2(2.262157)\frac{s_2}{\sqrt{10}} = 52.32 - 44.88$ where s_2 is an unbiased estimate of the population variance for the second sample.

$$s_2^2 = 27.04209873 = \frac{1}{9}\left(\sum t_2^2 - \frac{(486)^2}{10}\right)$$

	$\sum t_2^2 = 23862.97889$ Combining the two samples of 10 $\sum t_c = \sum t + \sum t_2 = 513.00 + 486 = 999$ $\sum t_c^2 = \sum t^2 + \sum t_2^2 = 26524.26 + 23862.97889 = 50387.23889$ $n = 10 + 10 = 20$ New sample mean $= \bar{t_c} = \frac{999}{20} = 49.95$ New unbiased estimate of population variance $= s_c^2 = \frac{1}{19}\left(50387.23889 - \frac{(999)^2}{20}\right) \approx 25.64152$ $\left(\bar{t_c} - t_{0.025}\frac{s_c}{\sqrt{20}},\ \bar{t_c} + t_{0.025}\frac{s_c}{\sqrt{20}}\right)$ where $\mathrm{P}(T \geq t_{0.025}) = 0.025,\ T \sim t_{19}$. i.e. $\left(49.95 - 2.09302\frac{\sqrt{25.64152}}{\sqrt{20}},\ 49.95 + 2.09302\frac{\sqrt{25.64152}}{\sqrt{20}}\right)$ i.e. (47.6, 52.3) (to 3 s.f.)	
9	**(a)** Foxes are found independently in the city and at a constant average rate. The following statements are incorrect: • The number of foxes in one square kilometre is independent of the number of foxes in another square kilometre. • the probability of a given number in any square kilometre is constant • the number of foxes of any given square kilometre had to be constant" **(b)** $F \sim \mathrm{Po}(\lambda)$ Required probability = $\mathrm{P}(F=1) + \mathrm{P}(F=3) + \mathrm{P}(F=5)$ $= \mathrm{e}^{-\lambda}\lambda + \frac{\mathrm{e}^{-\lambda}\lambda^3}{3!} + \frac{\mathrm{e}^{-\lambda}\lambda^5}{5!}$ $= \mathrm{e}^{-\lambda}\left(\lambda + \frac{\lambda^3}{6} + \frac{\lambda^5}{120}\right)$	Do not leave as three separate answers. You need to add them up

	(c) $P(X \text{ is odd}) = e^{-\lambda}\left(\lambda + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots\right)$ Consider e^{λ} and $e^{-\lambda}$: $e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \frac{\lambda^5}{5!} + \cdots$ $e^{-\lambda} = 1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} - \frac{\lambda^5}{5!} + \cdots$ $\Rightarrow e^{\lambda} - e^{-\lambda} = 2\left(\lambda + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots\right)$ $\therefore \left(\lambda + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots\right) = \frac{1}{2}\left(e^{\lambda} - e^{-\lambda}\right)$ $\therefore P(X \text{ is odd}) = e^{-\lambda} \cdot \frac{1}{2}\left(e^{\lambda} - e^{-\lambda}\right) = \frac{1}{2}\left(1 - e^{-2\lambda}\right)$ (shown) **(d)** $P(F = 0) = 0.1108 \Rightarrow e^{-\lambda} = 0.1108$ $P(F \text{ is odd}) = \frac{1}{2}\left[1 - (0.1108)^2\right] \approx 0.49386 = 0.494$ (3sf)	
10	Let k denote the number in the cell (Grade D, Men). Then, the numbers in the remaining missing cells are given as follows.	

	Total	Grade A	Grade B	Grade C	Grade D	Grade E
Total	500	153	96	121	70	60
Men	300	81	54	77	k	$88-k$
Women	200	72	42	44	$70-k$	$k-28$

H_0 : There is no association between gender and grade.

H_1 : There is association between gender and grade.

Under H_0, the expected frequencies are

	Total	Grade A	Grade B	Grade C	Grade D	Grade E
Total	500	153	96	121	70	60
Men	300	$\frac{153\times300}{500}$ = 91.8	$\frac{96\times300}{500}$ = 57.6	$\frac{121\times300}{500}$ = 72.6	$\frac{70\times300}{500}$ = 42	$\frac{60\times300}{500}$ = 36
Women	200	$\frac{153\times200}{500}$ = 61.2	$\frac{96\times200}{500}$ = 38.4	$\frac{121\times200}{500}$ = 48.4	$\frac{70\times200}{500}$ = 28	$\frac{60\times200}{500}$ = 24

Since no cell has a value of less than 5, $\text{d.f.} = (5-1)(2-1) = 4$.

Test statistic: $\chi^2 = \sum_{i=1}^{10} \frac{(O_i - E_i)^2}{E_i} \sim \chi^2(4)$.

Observed test statistic,

$$\chi^2 = \sum_{i=1}^{10} \frac{(o_i - e_i)^2}{e_i}$$

$$= \frac{(81-91.8)^2}{91.8} + \frac{(72-61.2)^2}{61.2} + \frac{(54-57.6)^2}{57.6} + \frac{(42-38.4)^2}{38.4}$$

$$+ \frac{(77-72.6)^2}{72.6} + \frac{(44-48.4)^2}{48.4} + \frac{(k-42)^2}{42} + \frac{(70-k-28)^2}{28}$$

$$+ \frac{(88-k-36)^2}{36} + \frac{(k-28-24)^2}{24}$$

$$= 4.4057 + \frac{(k-42)^2}{42} + \frac{(k-42)^2}{28} + \frac{(k-52)^2}{36} + \frac{(k-52)^2}{24}$$

$$= 4.4057 + \frac{5}{84}(k-42)^2 + \frac{5}{72}(k-52)^2$$

Level of significance: 5%.

Critical region: $\chi^2 \geq 9.488$.

Since the null hypothesis is not rejected at the 5% significance level,

$$4.4057 + \frac{5}{84}(k-42)^2 + \frac{5}{72}(k-52)^2 < 9.488.$$

By GC, $43.6 < k < 51.2$.

Therefore, the possible numbers in the cell (Grade *D*, Men) are 44, 45, 46, 47, 48, 49, 50 or 51.