Suggested Solution to 2015 SH2 H2 Mathematics Preliminary Examination Paper 1

1	Let P_N be the statement $\sum_{n=0}^{N}\frac{3n+2}{(n+1)!3^{n+1}}=1-\frac{1}{(N+1)!3^{N+1}}$ for all $N\geq 0$. When $N=0$, $\text{LHS}=\sum_{n=0}^{0}\frac{3n+2}{(n+1)!3^{n+1}}=\frac{3(0)+2}{(0+1)!3^{0+1}}=\frac{2}{3}$ $\text{RHS}=1-\frac{1}{(0+1)!3^{0+1}}=\frac{2}{3}=\text{LHS}$ Thus, P_0 is true. Assume P_k is true for some $k\geq 0$, i.e. $\sum_{n=0}^{k}\frac{3n+2}{(n+1)!3^{n+1}}=1-\frac{1}{(k+1)!3^{k+1}}$. Consider P_{k+1}: To show $\sum_{n=0}^{k+1}\frac{3n+2}{(n+1)!3^{n+1}}=1-\frac{1}{(k+2)!3^{k+2}}$ $\text{LHS of }\mathrm{P}_{k+1}=\sum_{n=0}^{k+1}\frac{3n+2}{(n+1)!3^{n+1}}$ $=\sum_{n=0}^{k}\frac{3n+2}{(n+1)!3^{n+1}}+\frac{3(k+1)+2}{(k+2)!3^{k+2}}$ $=1-\frac{1}{(k+1)!3^{k+1}}+\frac{3k+5}{(k+2)!3^{k+2}}$ $=1-\frac{(k+2)(3)}{(k+1)!(k+2)3^{k+1}\cdot 3}+\frac{3k+5}{(k+2)!3^{k+2}}$ $=1-\frac{3k+6}{(k+2)!3^{k+2}}+\frac{3k+5}{(k+2)!3^{k+2}}$ $=1+\frac{3k+5-3k-6}{(k+2)!3^{k+2}}$ $=1-\frac{1}{(k+2)!3^{k+2}}=\text{RHS of }\mathrm{P}_{k+1}$ P_k is true $\Rightarrow \mathrm{P}_{k+1}$ is true. Since P_0 is true, and P_k is true $\Rightarrow \mathrm{P}_{k+1}$ is true, by mathematical induction, P_N is true for all $N\geq 0$.

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<td>2 (i)</td>
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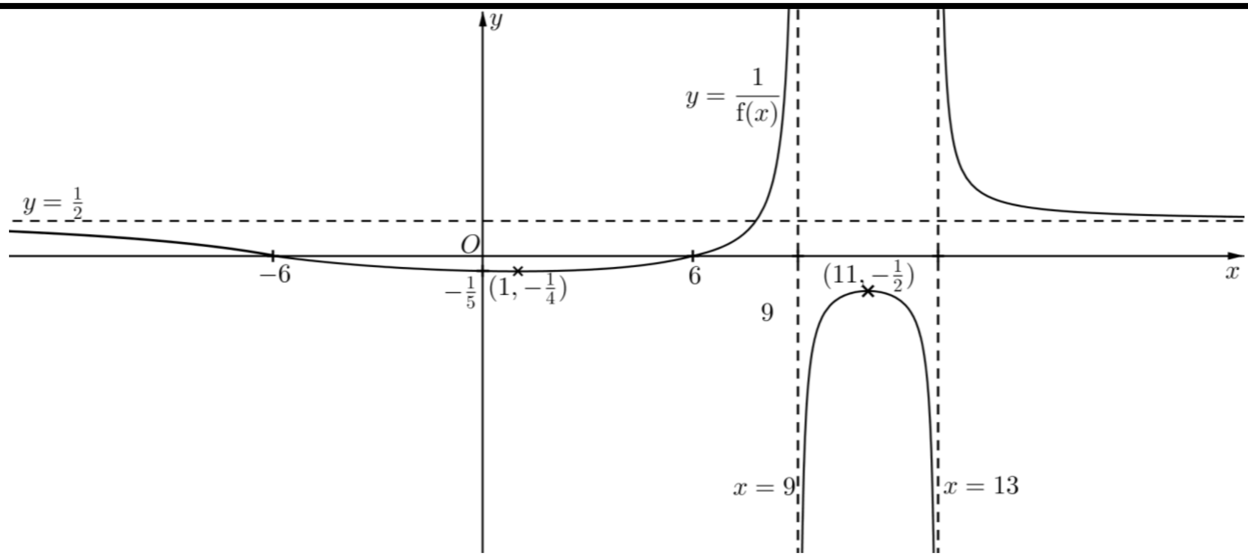

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<td>2 (ii)</td>
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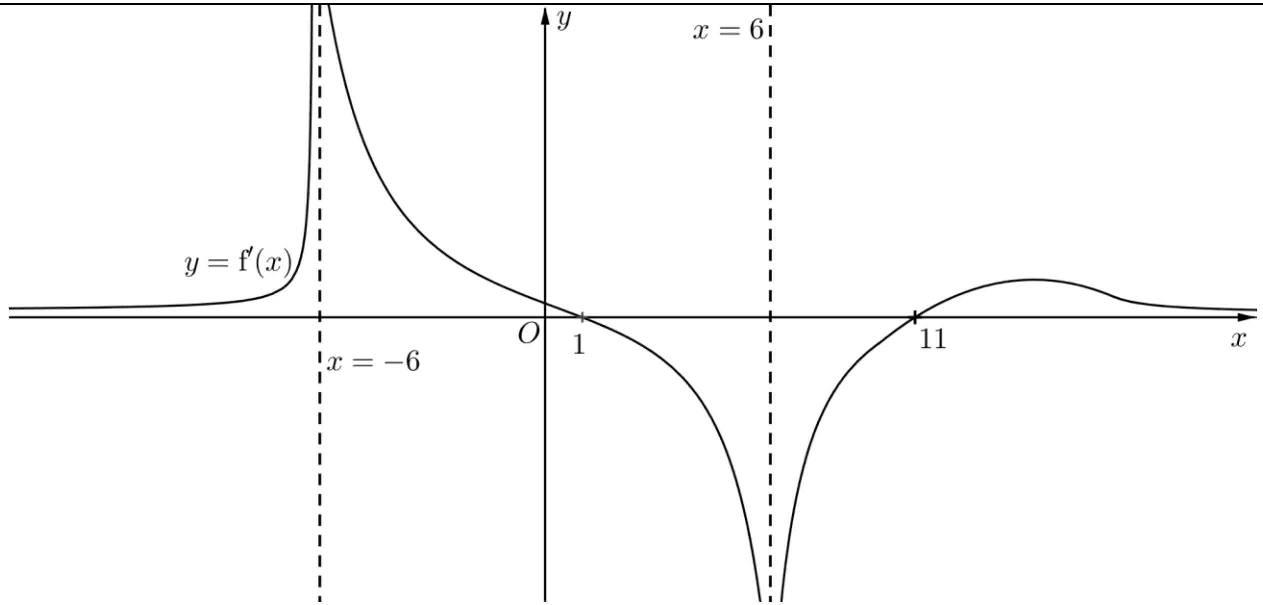

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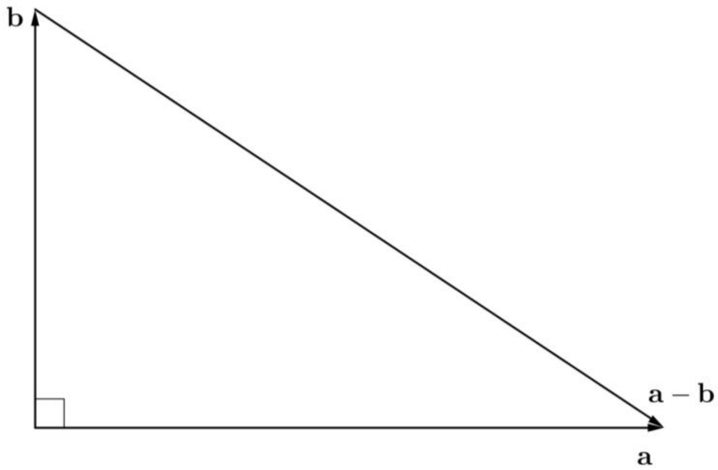
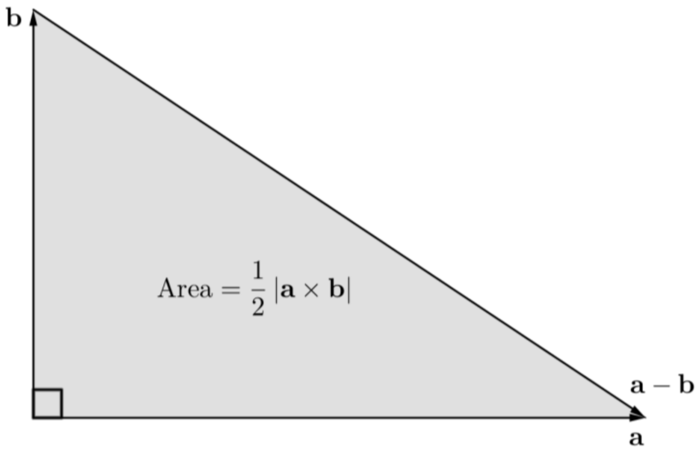
3 (i)	$$\frac{\cos 2x}{1-x^2} = (\cos 2x)\left(1-x^2\right)^{-1}$$ $$= \left(1-\frac{(2x)^2}{2}+\frac{(2x)^4}{4!}+\ldots\right)\left(1+x^2+x^4+\ldots\right)$$ <u>Method 1:</u> $$= \left(1-2x^2+\frac{2x^4}{3}+\ldots\right)\left(1+x^2+x^4+\ldots\right)$$ $$= \left(1+x^2+x^4\right)+\left(-2x^2-2x^4\right)+\frac{2x^4}{3}+\ldots$$ $$\text{or } \left(1-2x^2+\frac{2x^4}{3}\right)+\left(x^2-2x^4\right)+x^4+\ldots$$ $$\approx 1-x^2-\frac{x^4}{3}$$
3 (ii)	$$\frac{\cos\left[2\left(\frac{1}{3}\right)\right]}{1-\left(\frac{1}{3}\right)^2} \approx 1-\left(\frac{1}{3}\right)^2-\frac{1}{3}\left(\frac{1}{3}\right)^4 = 1-\frac{1}{9}-\frac{1}{243} = \frac{215}{243}$$ $$\cos\left(\frac{2}{3}\right) \approx \frac{215}{243}\times\frac{8}{9}$$ $$= \frac{1720}{2187}$$

4 (i)	Either **a** = **0** OR **b** = **0** OR **a** and **b** are perpendicular to each other.
4 (ii)	Projection vector of **a** – **b** onto **a** $=\left((\mathbf{a}-\mathbf{b})\cdot\hat{\mathbf{a}}\right)\hat{\mathbf{a}}$ $=\left(\mathbf{a}\cdot\hat{\mathbf{a}}-\mathbf{b}\cdot\hat{\mathbf{a}}\right)\hat{\mathbf{a}}$ $=\left(\mathbf{a}\cdot\frac{\mathbf{a}}{\lvert\mathbf{a}\rvert}-0\right)\frac{\mathbf{a}}{\lvert\mathbf{a}\rvert}$ $(\mathbf{b}\cdot\hat{\mathbf{a}}=0$ since $\mathbf{b}\cdot\mathbf{a}=0)$ $=\left(\frac{\mathbf{a}\cdot\mathbf{a}}{\lvert\mathbf{a}\rvert^2}\right)\mathbf{a}$ $=\mathbf{a}$ (since $\mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2$) **Alternatively**,  From the diagram, since $\mathbf{a}\perp\mathbf{b}$, the projection vector of vector of **a** – **b** onto **a** is **a** itself.
4 (iii)	<u>**Method 1: Geometrical Definition of Cross Product**</u> Since $\mathbf{a}\perp\mathbf{b}$, the angle between **a** and **b** is 90°. Hence, $\lvert\mathbf{a}\times\mathbf{b}\rvert$ $=\left\lvert\left(\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin 90^\circ\right)\hat{\mathbf{n}}\right\rvert$ $=\left\lvert\left(\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin 90^\circ\right)\right\rvert$ $(\because \lvert\hat{\mathbf{n}}\rvert=1)$ $=\left\lvert\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert(1)\right\rvert$ $=\left\lvert\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\right\rvert$ $=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert$ (shown) <u>**Method 2: Area of Triangle or Rectangle**</u>

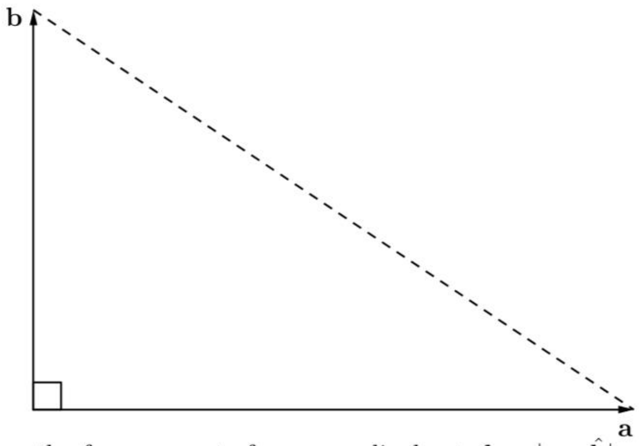


Area of triangle with sides **a** and **b** $= \frac{1}{2}|\mathbf{a}||\mathbf{b}|$

$$\frac{1}{2}|\mathbf{a}\times\mathbf{b}| = \frac{1}{2}|\mathbf{a}||\mathbf{b}|$$

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}| \text{ (shown)}$$

<u>Method 3: Length of Perpendicular Component</u>



Length of component of **a** perpendicular to **b** $= |\mathbf{a}\times\hat{\mathbf{b}}|$

Since $\mathbf{a}\perp\mathbf{b}$, the component of **a** perpendicular to **b** is **a** itself. Hence,

$$|\mathbf{a}\times\hat{\mathbf{b}}| = |\mathbf{a}|$$

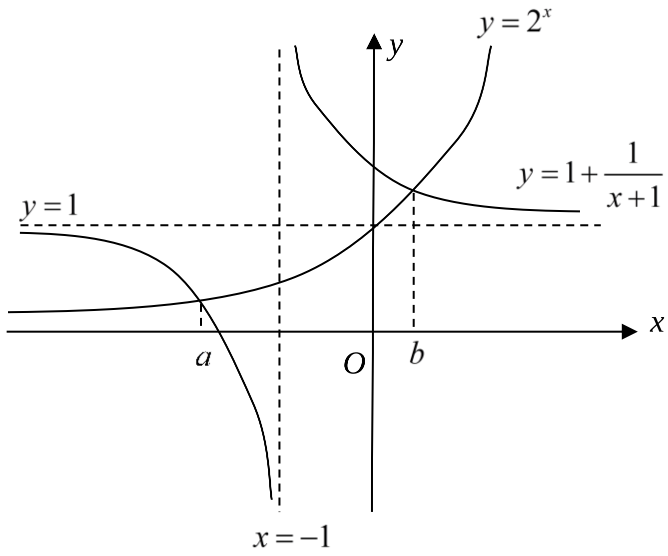
$$\left|\mathbf{a}\times\frac{\mathbf{b}}{|\mathbf{b}|}\right| = |\mathbf{a}|$$

$$\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|} = |\mathbf{a}|$$

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}| \text{ (shown)}$$

5 (i)	$\begin{aligned} \sum_{r=1}^{n}\frac{1}{(2r-1)(2r+3)} &= \sum_{r=1}^{n}\left[\frac{1}{4(2r-1)}-\frac{1}{4(2r+3)}\right] \\ &= \frac{1}{4}\left[\frac{1}{1}\cancel{-\frac{1}{5}}\right. \\ &\quad +\frac{1}{3}\cancel{-\frac{1}{7}} \\ &\quad \cancel{+\frac{1}{5}}\cancel{-\frac{1}{9}} \\ &\quad +\ldots \\ &\quad \cancel{+\frac{1}{2(n-2)-1}}\cancel{-\frac{1}{2(n-2)+3}} \\ &\quad \cancel{+\frac{1}{2(n-1)-1}}-\frac{1}{2(n-1)+3} \\ &\quad \left.\cancel{+\frac{1}{2n-1}}-\frac{1}{2n+3}\right] \\ &= \frac{1}{4}\left(1+\frac{1}{3}-\frac{1}{2n+1}-\frac{1}{2n+3}\right) \\ &= \frac{1}{4}\left[\frac{4}{3}-\frac{2n+3+2n+1}{(2n+1)(2n+3)}\right] \\ &= \frac{1}{3}-\frac{n+1}{(2n+1)(2n+3)} \end{aligned}$
5 (ii) (a)	$\begin{aligned} & \sum_{r=5}^{\infty}\frac{1}{(2r-1)(2r+3)} \\ = \quad & \sum_{r=1}^{\infty}\frac{1}{(2r-1)(2r+3)}-\sum_{r=1}^{4}\frac{1}{(2r-1)(2r+3)} \\ = \quad & \lim_{n\to\infty}\left[\frac{1}{3}-\frac{n+1}{(2n+1)(2n+3)}\right]-\left[\frac{1}{3}-\frac{5}{(9)(11)}\right] \\ = \quad & \frac{1}{3}-0-\frac{1}{3}+\frac{5}{99} \\ = \quad & \frac{5}{99} \end{aligned}$

5 (ii) (b)	Let $k = r + 1$. Then $\displaystyle\sum_{r=0}^{n}\frac{1}{(2r+1)(2r+5)} = \sum_{r=0}^{r=n}\frac{1}{(2r+1)(2r+5)}$ $\displaystyle = \sum_{r+1=1}^{r+1=n+1}\frac{1}{\left[2(r+1)-1\right]\left[2(r+1)+3\right]}$ $\displaystyle = \sum_{k=1}^{k=n+1}\frac{1}{(2k-1)(2k+3)} = \sum_{k=1}^{n+1}\frac{1}{(2k-1)(2k+3)}$ $\displaystyle = \frac{1}{3} - \frac{n+2}{(2n+3)(2n+5)}$
	Alternatively, $\displaystyle\sum_{r=0}^{n}\frac{1}{(2r+1)(2r+5)}$ $\displaystyle = \frac{1}{1\cdot 5} + \frac{1}{3\cdot 7} + \frac{1}{5\cdot 9} + \cdots + \frac{1}{(2n+1)(2n+5)}$ $\displaystyle = \frac{1}{(2\cdot 1-1)(2\cdot 1+3)} + \frac{1}{(2\cdot 2-1)(2\cdot 2+3)} + \cdots$ $\displaystyle + \frac{1}{(2(n+1)-1)(2(n+1)+3)}$ $\displaystyle = \sum_{r=1}^{n+1}\frac{1}{(2r-1)(2r+3)}$ $\displaystyle = \frac{1}{3} - \frac{n+2}{(2n+3)(2n+5)}$
	For students who ignored the "Hence" condition $\displaystyle\sum_{r=0}^{n}\frac{1}{(2r+1)(2r+5)} = \sum_{r=0}^{n}\left[\frac{1}{4(2r+1)} - \frac{1}{4(2r+5)}\right]$ $\displaystyle = \frac{1}{4}\left[\frac{1}{1} - \cancel{\frac{1}{5}}\right.$ $\displaystyle + \frac{1}{3} - \cancel{\frac{1}{7}}$ $\displaystyle + \cancel{\frac{1}{5}} - \cancel{\frac{1}{9}}$ $+ \ldots$ $\displaystyle + \cancel{\frac{1}{2(n-1)-1}} - \cancel{\frac{1}{2(n-1)+3}}$ $\displaystyle + \cancel{\frac{1}{2n-1}} - \cancel{\frac{1}{2n+3}}$ $\displaystyle \left. + \cancel{\frac{1}{2(n+1)-1}} - \frac{1}{2n+3}\right]$ $\displaystyle = \frac{1}{4}\left(1 + \frac{1}{3} - \frac{1}{2n+3} - \frac{1}{2n+5}\right) = \frac{1}{4}\left(\frac{4}{3} - \frac{1}{2n+3} - \frac{1}{2n+5}\right)$

6 (a)	 Using GC, $a \approx -2.2631$ $b \approx 0.67529$. For $1 + \frac{1}{x+1} < 2^x$, $-2.26 < x < -1$ or $x > 0.676$ OR $-2.26 < x < -1$ or $x > 0.675$
6 (b)	Let the distances for the swimming, cycling and running stage be *s* km, *c* km and *r* km respectively. Kandy: $\frac{s}{3} + \frac{c}{26} + \frac{r}{12} = 11.7$ Landy: $\frac{s}{3} + \frac{c}{28} + \frac{r}{9} = 12.4$ Mandy: $\frac{s}{2} + \frac{c}{30} + \frac{r}{7} = 13.9$ By GC, $s = 3.37$, $c = 181$, $r = 43.1$ (3s.f). Therefore, the distances for the swimming, cycling and running stage are 3.37 km, 181 km and 43.1 km respectively. *s* = 29133/8635 *c* = 313404/1727 *r* = 372582/8635

7 (a)

$$y=\frac{(x-1)(x-2)}{x-3},\ x\in\mathbf{R},\ x\neq 3$$

$$(x-3)y=(x-1)(x-2)$$

$$xy-3y=x^2-3x+2$$

$$x^2-(y+3)x+(3y+2)=0$$

For the values of *y* for which the graph exists, the equation $x^2-(y+3)x+(3y+2)=0$ has solutions for *x*. Hence, discriminant (with respect to *x*) ≥ 0

$$(y+3)^2-4(3y+2)\geq 0$$

$$y^2+6y+9-12y-8\geq 0$$

$$y^2-6y+1\geq 0$$

Solving $y^2-6y+1=0$,

$$y=\frac{6\pm\sqrt{6^2-4(1)(1)}}{2}$$

$$=\frac{6\pm\sqrt{32}}{2}$$

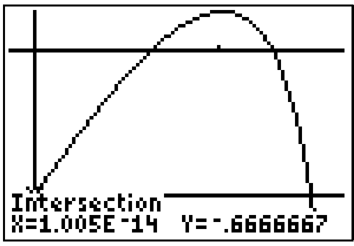
$$=\frac{6\pm 4\sqrt{2}}{2}$$

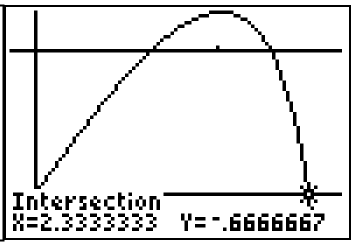
$$=3\pm 2\sqrt{2}$$

Hence, $y^2-6y+1\geq 0 \Rightarrow y\leq 3-2\sqrt{2}$ or $y\geq 3+2\sqrt{2}$

7 (b)

Equation of translated curve is $y=\frac{(x-1)(x-2)}{x-3}-\left(-\frac{2}{3}\right)$





At the intersection between *C* and the line $y=-\frac{2}{3}$,

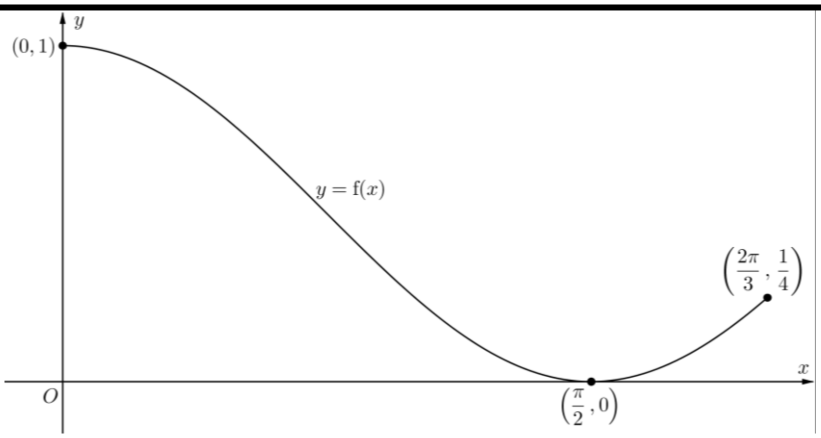
x = 0 or *x* = 2.33333 $\left(\text{or } \frac{7}{3}\right)$.

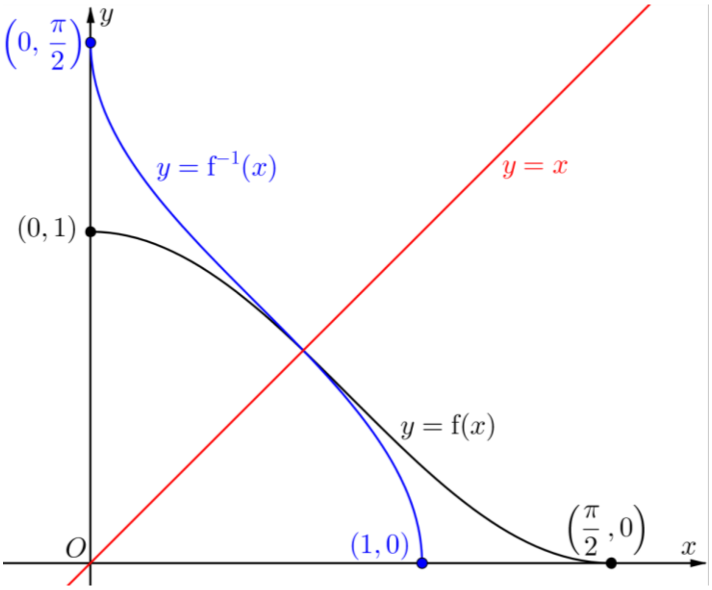
Hence volume of revolution

$$=\pi\int_0^{2.33333}\left[\frac{(x-1)(x-2)}{x-3}-\left(-\frac{2}{3}\right)\right]^2\ \mathrm{d}x$$

$$=2.6434$$

$$=2.643 \text{ units}^3 \text{ (to 3 decimal places)}$$

<table>
<tr><td>8 (i)</td><td>

From the graph, the range of f is [0, 1]</td></tr>
<tr><td>8 (ii)</td><td>Largest possible value of k is $\dfrac{\pi}{2}$.</td></tr>
<tr><td>8 (iii)</td><td>
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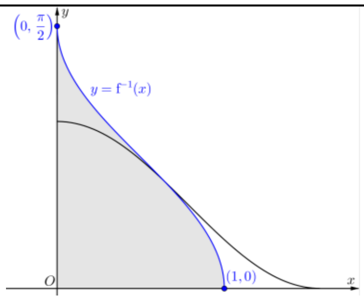
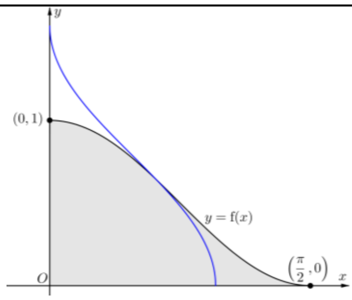
<tr><td>8 (iv)</td><td>



From the symmetry of the sketch, required area
$=\int_0^1 \mathrm{f}^{-1}(x)\,\mathrm{d}x=\int_0^{\frac{\pi}{2}} \mathrm{f}(x)\,\mathrm{d}x$
$=\int_0^{\frac{\pi}{2}} \cos^2 x\,\mathrm{d}x$
$=\int_0^{\frac{\pi}{2}} \dfrac{1+\cos 2x}{2}\,\mathrm{d}x$
$=\left[\dfrac{x}{2}+\dfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}=\dfrac{\pi}{4}$ units^2</td></tr>
<tr><td>8 (iv)</td><td>Alteratively, let $y=\mathrm{f}(x)$. Then</td></tr>
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	$y=\cos^2 x$ $\Rightarrow x=\cos^{-1}\sqrt{y}$ $\Rightarrow f^{-1}(x)=\cos^{-1}\sqrt{x}$ Hence required area $=\int_0^1 f^{-1}(x)\ dx$ $=\int_0^1 \cos^{-1}\sqrt{x}\ dx$ $=\int_0^1 1\cdot\cos^{-1}\sqrt{x}\ dx$ $=\left[x\cos^{-1}\sqrt{x}\right]_0^1-\int_0^1 x\cdot\frac{-1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\cdot\frac{1}{2\sqrt{x}}\ dx$ $=\frac{1}{2}\int_0^1\sqrt{\frac{x}{1-x}}\ dx$ $=\frac{1}{2}\int_0^1\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\left(2\sin\theta\cos\theta\right)\ d\theta$ (using the substitution $x=\sin^2\theta$) $=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sin\theta}{\cos\theta}\left(2\sin\theta\cos\theta\right)\ d\theta$ $=\int_0^{\frac{\pi}{2}}\sin^2\theta\ d\theta$ $=\int_0^{\frac{\pi}{2}}\frac{1-\cos 2\theta}{2}\ d\theta$ $=\left[\frac{\theta}{2}-\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}=\frac{\pi}{4}$ units^2

9 (i)	A = area of base + area of walls $=\frac{\theta r^2}{2}+\frac{r}{6}(r+r+r\theta)=\frac{r^2}{6}(2+4\theta)=\frac{r^2}{3}(1+2\theta)$
	<u>Method 1 : Expressing *V* in terms of *r*</u> $\theta=\frac{3A}{2r^2}-\frac{1}{2}$ $V=\left(\frac{\theta r^2}{2}\right)\left(\frac{r}{6}\right)$ $=\frac{r^3}{12}\left(\frac{3A}{2r^2}-\frac{1}{2}\right)$ $=\frac{Ar}{8}-\frac{r^3}{24}$ $\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{A}{8}-\frac{r^2}{8}$. For stationary point(s), $\frac{\mathrm{d}V}{\mathrm{d}r}=0$, so we have $r=\sqrt{A}$ (reject $r=-\sqrt{A}$ as $r>0$). $\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=-\frac{r}{4}$ When $r=\sqrt{A}$, $\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=-\frac{\sqrt{A}}{4}<0.$ i.e. V ix maximum when $r=\sqrt{A}\Rightarrow\theta=\frac{3A}{2\left(\sqrt{A}\right)^2}-\frac{1}{2}=1.$
	<u>Method 2 : Expressing *V* in terms of *θ*</u> $r=\sqrt{\frac{3A}{1+2\theta}}$ (reject $r=-\sqrt{\frac{3A}{1+2\theta}}$ as $r>0$) $V=\left(\frac{\theta r^2}{2}\right)\left(\frac{r}{6}\right)=\frac{\theta r^3}{12}$ $=\frac{\theta}{12}\left(\frac{3A}{1+2\theta}\right)^{\frac{3}{2}}$ $=\frac{(3A)^{\frac{3}{2}}}{12}\left[\theta(1+2\theta)^{-\frac{3}{2}}\right]$

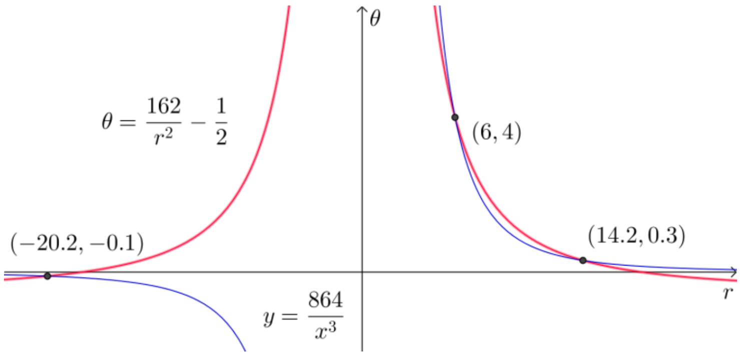
	$\dfrac{\mathrm{d}V}{\mathrm{d}\theta}=\dfrac{(3A)^{\frac{3}{2}}}{12}\left[(1+2\theta)^{-\frac{3}{2}}-1.5(2)\theta(1+2\theta)^{-\frac{5}{2}}\right]$ $=\dfrac{(3A)^{\frac{3}{2}}}{12}(1+2\theta-3\theta)(1+2\theta)^{-\frac{5}{2}}$ $=\dfrac{(3A)^{\frac{3}{2}}(1-\theta)(1+2\theta)^{-\frac{5}{2}}}{12}$ $=\dfrac{(1-\theta)}{12}\sqrt{\dfrac{(3A)^3}{(1+2\theta)^5}}$ For stationary point(s), $\dfrac{\mathrm{d}V}{\mathrm{d}\theta}=0$, so we have $\theta=1$. Since $\left.\dfrac{\mathrm{d}^2V}{\mathrm{d}\theta^2}\right\rvert_{\theta=1}$ $=\left.\dfrac{(3A)^{\frac{3}{2}}\left[(1-\theta)2(-2.5)(1+2\theta)^{-\frac{7}{2}}-(1+2\theta)^{-\frac{5}{2}}\right]}{12}\right\rvert_{\theta=1}=-\dfrac{(3A)^{\frac{3}{2}}\left(3^{-\frac{5}{2}}\right)}{12}<0$, $\theta=1$ gives a maximum V.
9 (ii)	$\dfrac{r^2}{3}(1+2\theta)=108\ \ldots(1)$ and $\left(\dfrac{\theta r^2}{2}\right)\left(\dfrac{r}{6}\right)=72\ \ldots(2)$ From (2), $\theta=\dfrac{864}{r^3}$. Substituting it into (1), we have $\dfrac{r^2}{3}\left(1+\dfrac{1728}{r^3}\right)=108$ $r^2+\dfrac{1728}{r}=324$ $r^3-324r+1728=0$ By GC, $r=-20.234$ (reject as $r>0$) or $r=14.234$ or $r=6$. When $r=6$, $\theta=\dfrac{864}{6^3}=4$ (rejected since $4>\pi$) When $r=14.234$, $\theta\approx\dfrac{864}{14.234^3}\approx0.300<\pi$ Therefore r = 14.2 is the only solution.
	Alternatively,

$\frac{r^2}{3}(1+2\theta)=108$ and

$\left(\frac{\theta r^2}{2}\right)\left(\frac{r}{6}\right)=72$

$\theta=\frac{162}{r^2}-\frac{1}{2}$ and $\theta=\frac{864}{r^3}$

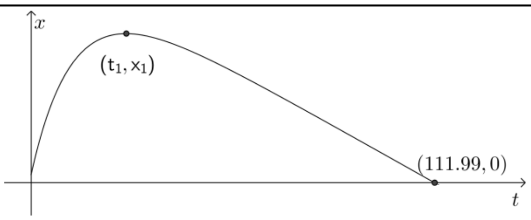
By sketching both graphs in the same diagram,

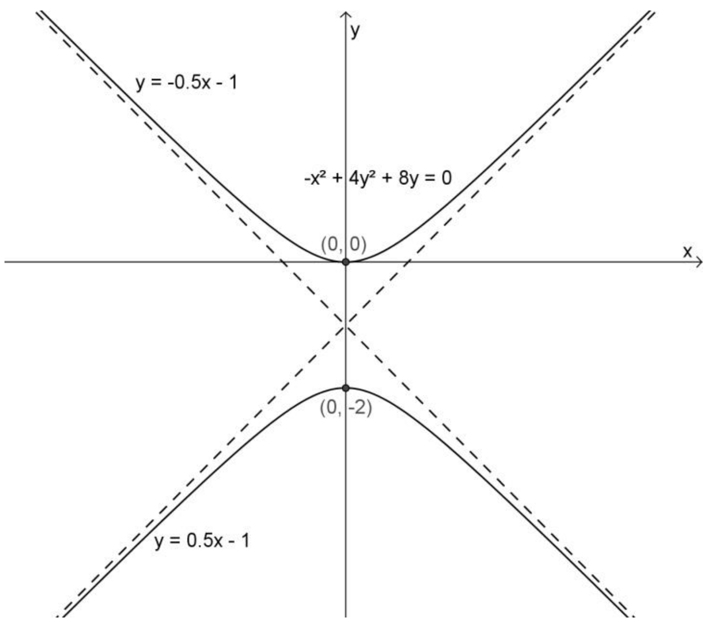


The points of intersections are
$(-20.2,-0.1)$, $(6,4)$ and $(-14.2,0.3)$.

Since $0<\theta<\pi$,we reject $(r,\theta)=(-20.2,-0.1)$ and $(r,\theta)=(6,4)$.

Therefore $r=14.234$ is the only solution.

10 (1st Part)	$\frac{dx}{dt}=k\left(200-2t-x\right)$ When $t=0$, $x=8$ and $\frac{dx}{dt}=16$. $16=k\left(200-8\right)$ $k=\frac{16}{192}=\frac{1}{12}\Rightarrow\frac{dx}{dt}=\frac{200-2t-x}{12}$.
10 (2nd Part)	$u=2t+x\Rightarrow\frac{du}{dt}=2+\frac{dx}{dt}$ $\frac{du}{dt}-2=\frac{200-u}{12}$ $\frac{du}{dt}=\frac{224-u}{12}$ $\frac{1}{224-u}\frac{du}{dt}=\frac{1}{12}$ Integrating both sides with respect to t, $\int\frac{1}{224-u}\,du=\int\frac{1}{12}\,dt$ $-\ln\lvert 224-u\rvert=\frac{t}{12}+C$ $\ln\lvert 224-u\rvert=-\frac{t}{12}-C$ $\lvert 224-u\rvert=e^{-\frac{t}{12}-C}$ $224-u=\pm e^{-\frac{t}{12}-C}=Ae^{-\frac{t}{12}}$ where $A=\pm e^{-C}$ $u=224-Ae^{-\frac{t}{12}}$ $x=224-2t-Ae^{-\frac{t}{12}}$ When $t=0$, $x=8$ $8=224-A\Rightarrow A=216$ Thus, $x=224-2t-216e^{-\frac{t}{12}}$ By GC, when $x=0$, $t\approx 112$ (years)
10 (3rd Part)	When $\frac{dx}{dt}=0$, $200-2t-x=0$ $x=200-2t$  So, x_1 is the maximum population size of the fish. t_1 is the number of years for the population to reach its maximum.

11 (i)	$4y^2+8y-x^2=0$ By applying implicit differentiation, $8y\dfrac{dy}{dx}+8\dfrac{dy}{dx}-2x=0$ $8\dfrac{dy}{dx}(y+1)=2x \Rightarrow \dfrac{dy}{dx}=\dfrac{x}{4(y+1)}$
11 (ii)	At $\left(-\dfrac{3}{2},\dfrac{1}{4}\right)$: $\dfrac{dy}{dx}=\dfrac{-3/2}{4(1/4+1)}=-\dfrac{3}{10}$ Gradient of normal = $-\dfrac{1}{-3/10}=\dfrac{10}{3}$ Equation of normal at $\left(-\dfrac{3}{2},\dfrac{1}{4}\right)$ is $\left(y-\dfrac{1}{4}\right)=\dfrac{10}{3}\left(x+\dfrac{3}{2}\right)$ $y=\dfrac{10}{3}x+\dfrac{21}{4}$
11 (iii)	By completing the square, $4y^2+8y-x^2=0$ $4(y^2+2y)-x^2=0$ $4\left[(y+1)^2-1\right]-x^2=0$ $(y+1)^2-1-\dfrac{x^2}{4}=0$ $\dfrac{(y-(-1))^2}{1^2}-\dfrac{x^2}{2^2}=1$ 

12 (a)(i)

<u>Method 1: Using Exponential Forms</u>

$z_1 = \sqrt{2} - \sqrt{2}\mathrm{i} = 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}$; $z_2 = 1 + \sqrt{3}\mathrm{i} = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$

$$
\begin{aligned}
z_3 &= -\frac{{z_2}^2}{{z_1}^*} \\
&= \mathrm{e}^{\mathrm{i}\pi}\,\frac{\left(2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right)^2}{\left(2\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\right)^*} \\
&= \mathrm{e}^{\mathrm{i}\pi}\left(\frac{4\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}}{2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}\right) \\
&= \frac{2\mathrm{e}^{\mathrm{i}\frac{5\pi}{3}}}{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}} \\
&= 2\mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{3}-\frac{\pi}{4}\right)} \\
&= 2\mathrm{e}^{\mathrm{i}\frac{17\pi}{12}} = 2\mathrm{e}^{\mathrm{i}\left(-\frac{7\pi}{12}\right)}
\end{aligned}
$$

<u>Method 2: Applying Laws of Modulus & Argument</u>

$$
\begin{aligned}
|z_1| &= \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2} & \qquad |z_2| &= \sqrt{(1)^2 + \left(\sqrt{3}\right)^2} \\
&= 2, & &= 2, \\
\arg(z_1) &= -\tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) & \qquad \arg(z_2) &= \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) \\
&= -\frac{\pi}{4} & &= \frac{\pi}{3}
\end{aligned}
$$

Hence

$$\left|-\frac{{z_2}^2}{{z_1}^*}\right| = \frac{|z_2|^2}{|z_1|} = \frac{2^2}{2} = 2$$

$$
\begin{aligned}
\arg\left(-\frac{{z_2}^2}{{z_1}^*}\right) &= \arg(-1) + \arg\left({z_2}^2\right) - \arg\left({z_1}^*\right) \\
&= \pi + 2\arg(z_2) + \arg(z_1) \\
&= \pi + 2\times\frac{\pi}{3} - \frac{\pi}{4} \\
&= \frac{17\pi}{12} \equiv -\frac{7\pi}{12}
\end{aligned}
$$

Therefore $z_3 = 2\mathrm{e}^{\mathrm{i}\left(-\frac{7\pi}{12}\right)}$.

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<u>Method 3: Using Cartesian Form</u>

$$
\begin{aligned}
z_3 &= -\frac{{z_2}^2}{z_1^*} \\
&= -\frac{\left(1+\sqrt{3}\mathrm{i}\right)^2}{\sqrt{2}+\sqrt{2}\mathrm{i}} \\
&= -\frac{1+2\sqrt{3}\mathrm{i}-3}{\sqrt{2}+\sqrt{2}\mathrm{i}} \\
&= \frac{2-2\sqrt{3}\mathrm{i}}{\sqrt{2}+\sqrt{2}\mathrm{i}} \\
&= \frac{4\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}}{2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}} \\
&= 2\mathrm{e}^{-\mathrm{i}\frac{7\pi}{12}}
\end{aligned}
$$

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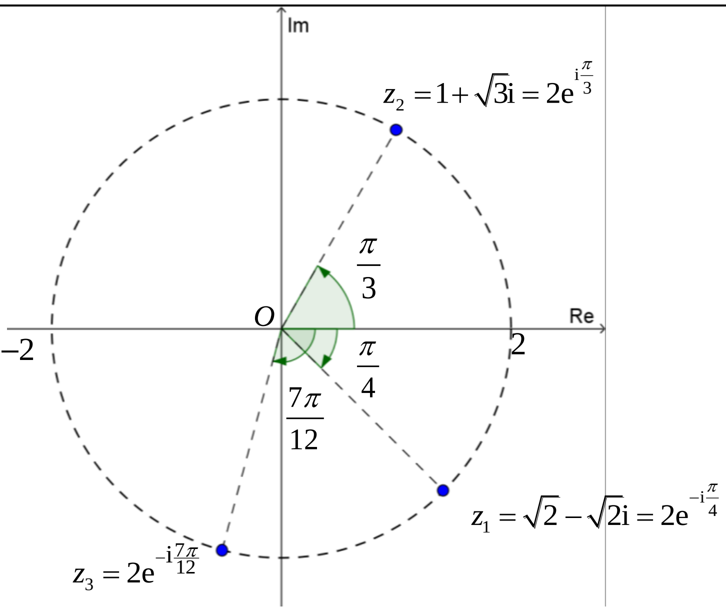
12 (a) (ii)

</td><td>



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12 (a) (iii)

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No, because…

…the difference in argument between any pair of adjacent complex numbers is not constant, **OR**

…the difference in argument between z_1 and z_2 is not $2\pi/3$ (or any other pair of the two complex numbers) **OR**

$$(z_1)^3 = \left(2\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\right)^3 = 8\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}} \text{ but}$$

$$(z_2)^3 = \left(2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right)^3 = 8\mathrm{e}^{\mathrm{i}\pi} \neq (z_1)^3$$

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12 (b)	**Method 1: Using exponential form**

$$\frac{1}{e^{i4\theta}-1}=\frac{1}{e^{i2\theta}\left(e^{i2\theta}-e^{-i2\theta}\right)}$$

$$=\left(\frac{1}{e^{i2\theta}-e^{-i2\theta}}\right)e^{-i2\theta}$$

$$=\frac{1}{2i\sin 2\theta}\left(\cos 2\theta-i\sin 2\theta\right)$$

$$=-\left(\frac{1}{2}\cot 2\theta\right)i-\frac{1}{2}$$

$$=-\frac{1}{2}-\left(\frac{1}{2}\cot 2\theta\right)i$$

Hence $\operatorname{Re}(w)=-\frac{1}{2}$ (shown)

and $\operatorname{Im}(w)=-\frac{1}{2}\cot 2\theta$

Method 2: Using polar form

$$\frac{1}{e^{i4\theta}-1}=\frac{1}{\cos 4\theta+i\sin 4\theta-1}\left(\frac{\cos 4\theta-1-i\sin 4\theta}{\cos 4\theta-1-i\sin 4\theta}\right)$$

$$=\frac{\cos 4\theta-1-i\sin 4\theta}{\left(\cos 4\theta-1\right)^2+\left(\sin 4\theta\right)^2}$$

$$=\frac{\cos 4\theta-1-i\sin 4\theta}{\cos^2 4\theta-2\cos 4\theta+1+\sin^2 4\theta}$$

$$=\frac{-\left(1-\cos 4\theta\right)-i\sin 4\theta}{2\left(1-\cos 4\theta\right)}$$

$$=-\frac{1}{2}-\frac{2\sin 2\theta\cos 2\theta}{2\left(1-\left(1-2\sin^2 2\theta\right)\right)}i$$

$$=-\frac{1}{2}-\frac{1}{2}i\cot 2\theta$$

Hence $\operatorname{Re}(w)=-\frac{1}{2}$ (shown)

and $\operatorname{Im}(w)=-\frac{1}{2}\cot 2\theta$