STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	1	8	-	7	2	0	1

ST. JOSEPH'S INSTITUTION
YEAR 5 END OF YEAR EXAMINATION 2018

MATHEMATICS
HIGHER LEVEL
PAPER 1
Wednesday

26th September 2018
1 hr 30 mins
0800 – 0930 hrs

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the foolscap paper provided.
- The use of calculators is **not** permitted in this paper.
- A clean copy of the **Mathematics HL Formulae Booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers are to be given exactly.
- The maximum mark for this examination paper is *[**80** marks]*.
- This question paper consists of **10** printed pages including the Cover Sheet.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	TOTAL
										/80

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (40 marks)

Answer **all** questions in the **spaces** provided.

1 [Maximum mark: 4]

The complex numbers $u = 2 + 3i$ and $v = 3 + 2i$ satisfy the equation

$$\frac{1}{u} + \frac{1}{v} = \frac{10}{w}, \quad w \in \mathbb{C}.$$

Express w in the form $a + ib$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.

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2 **[Maximum: 5 marks]**

Let $g\colon \mathbb{N} \longrightarrow \mathbb{Z}^+$ be a piecewise function defined as

$$g(n) = \begin{cases} 1, & \text{if } n = 0 \\ g\left(\frac{1}{2}n\right), & \text{if } n \text{ is even} \\ 1 + g(n-1), & \text{if } n \text{ is odd} \end{cases}$$

(a) Find $g(3)$. [3]

(b) Does g have an inverse? Justify your answer. [2]

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3 **[Maximum mark: 8]**

(a) The term independent of x in the expansion $\left(\frac{3}{x^4}+2x^2\right)^6$ can be written in the form $2^p \times 3^q \times 5^r$, where $p, q, r \in \mathbb{N}$. Find the values of p, q and r. [4]

(b) Determine $\operatorname{Im}\left(\left(1-i\sqrt{2}\right)^5\right)$, where $i^2=-1$, leaving your answer in the form $a\sqrt{2}$, where $a \in \mathbb{Z}$. [4]

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4 **[Maximum mark: 5]**

Find the coordinates of the point of inflection of the function $\mathrm{f}\left(x\right)=xe^{x}$ where $x\in\mathbb{R}$, justifying your answer.

5 [Maximum mark: 5]

Determine the series of transformations that transform the circle with equation $x^2 - 4x + y^2 + 6y + 8 = 0$ into the ellipse with equation $x^2 + 4\left(y+3\right)^2 = 20$.

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6 **[Maximum: 4 marks]**

Solve for x : $\sin\left(\arcsin\left(\frac{1}{5}\right)+\arccos(x)\right)=1.$

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7 **[Maximum mark: 9]**

The function g is given by $\mathrm{g}\left(x\right)=\dfrac{x+1}{2x-1}, x\in\mathbb{R}, x\neq\dfrac{1}{2}$.

(a) Write down the equations of the horizontal and vertical asymptotes of g. [2]

(b) In the space below, sketch the graph of $y=\left|g\left(x\right)\right|+1$, indicating the asymptotes, critical point and point(s) of intersection with the axes. [6]

(c) State the range of $y=\left|g\left(x\right)\right|+1$. [1]

TURN OVER

*Do **NOT** write solutions on this page.*

SECTION B (40 marks)

*Answer **all** questions on the foolscap paper provided. **Please start each question on a new page.***

8 **[Maximum mark: 12]**

(a) **(i)** Express $4\cos 2x - 3\sin 2x$ in the form $R\cos(2x+\theta)$, where $R > 0$ and θ is acute, giving the exact value of R and θ. [2]

(ii) Hence, write down the greatest and least value of $\frac{2}{4\cos 2x - 3\sin 2x + 7}$. [2]

(b) A curve has equation $x - y = (x+y)^2$. Find $\frac{dy}{dx}$ in terms of x and y. [4]

(c) Consider another curve with equation $y = \log_3 x$. If $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{k}{x^2}$, find the value of k in the form $\frac{1-p}{p^2}$, where $p \in \mathbb{R}$. [4]

9 **[Maximum mark: 15]**

(a) **(i)** Evaluate

$$\sum_{r=0}^{\infty}\left[\sin^{2r}\left(\frac{5\pi}{6}\right)\right]$$

[3]

(ii) For what values of x, where $x \in (0,\ \pi)$ does the geometric series

$$\sin x + \sin 2x + 4\sin x\cos^2 x + \cdots \text{ exist?}$$ [5]

(b) A computer password is to be generated such that it consists of the following four digits 4, 4, 5 and 6, and the seven letters *a*, *e*, *E*, *X*, *I*, *t*, *T*.

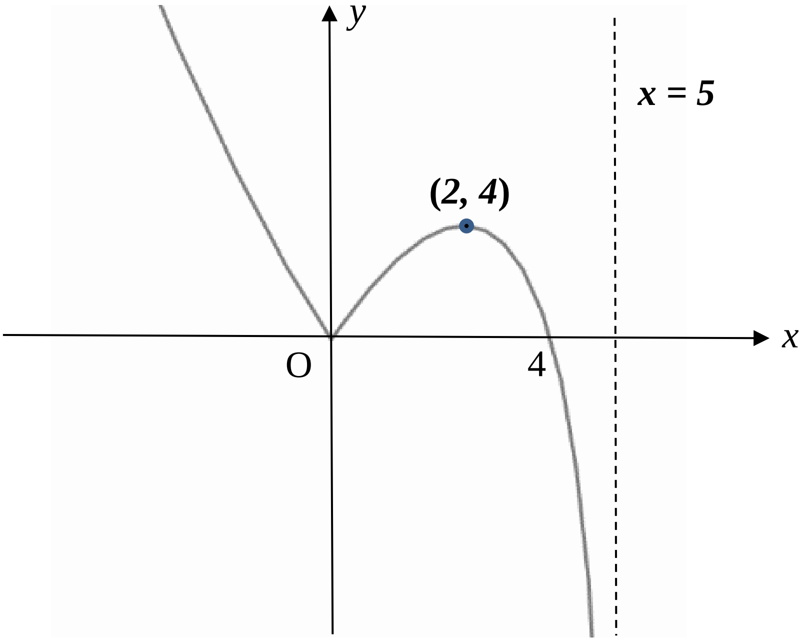
Find the number of ways of arranging the password such that the digits must always be separated and two of the same letters are to always be beside each other. Leave your answer in the form $m! \times n!$, where m and n are positive integers to be determined. [7]

TURN OVER

*Do **NOT** write solutions on this page.*

10 **[Maximum mark: 13]**

(a) The graph of the function $y = f(x)$ is shown below.



Sketch the graph of $y = \frac{1}{f(x)}$. [4]

(b) **(i)** Show that $\frac{d}{dx}(\arccos(x)) = -\frac{1}{\sqrt{1-x^2}}$. [4]

(ii) Hence, find $g'\left(\frac{1}{2}\right)$ if the function $g(x) = \frac{\arccos(x)}{\mathrm{h}(x)}$, where $h(x)$ is a non-zero function such that $h\left(\frac{1}{2}\right) = 1$, $h'\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{\pi}$. [5]

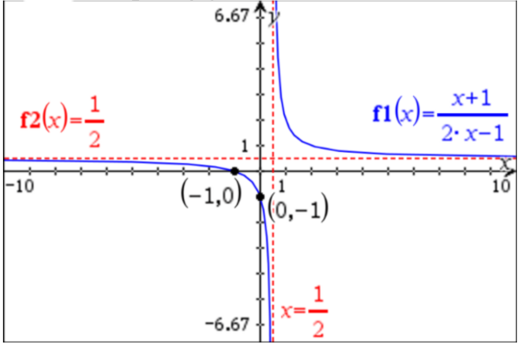
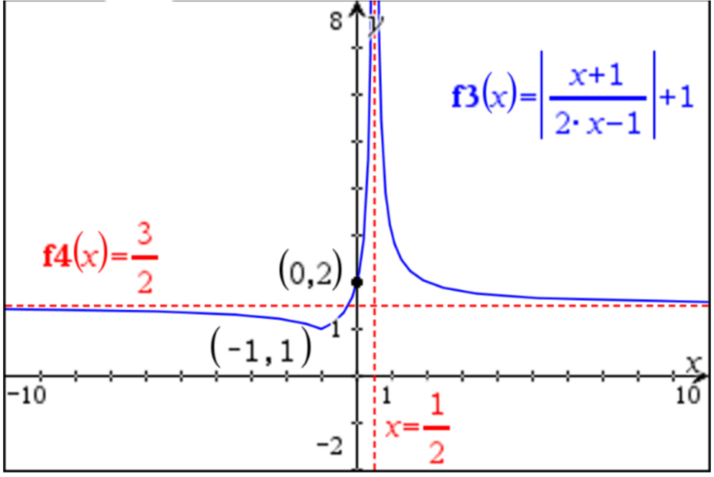
End of Paper

Year 5 HL Maths End of Year Exam 2018 – Paper 1 Markscheme

Qn	Suggested solution	Markscheme
Section A		
1	***Complex Number operations***	**[Max mark: 4]**
	Method 1	
	$\frac{10}{w}=\frac{1}{u}+\frac{1}{v}$	
	$=\frac{2-3i}{13}+\frac{3-2i}{13}$	**M1**-use of conjugate
	$=\frac{5-5i}{13}$	**A1**
	$w=\frac{26}{1-i}=13+13i$	**M1 A1**
	Method 2	
	$\frac{u+v}{uv}=\frac{10}{w}$	
	$w=\frac{10uv}{u+v}=\frac{10(2+3i)(3+2i)}{2+3i+3+2i}$	**M1**
	$=\frac{130i}{5+5i}$	**A1**
	$=\frac{26i}{1+i}\times\frac{1-i}{1-i}$	**M1**-use of conjugate
	$=13i(1-i)=13+13i$	**A1**
2	***Composite Functions***	**[Max mark: 5]**
(a)	$g(3)=1+g(2)$	**M1**
	$=1+g(1)$	**A1**
	$=1+1+g(0)$	
	$=3$	**A1**
(b)	Since $g(1)=g(2)$ but $1\neq 2$, g is not a one-one function.	**R1**
	Hence, g does not have an inverse.	**A1**-with justification
3	***Binomial Expansion***	**[Max mark: 8]**
(a)	General term $=\binom{6}{k}\left(3x^{-4}\right)^{k}\left(2x^{2}\right)^{6-k}=\binom{6}{k}\left(3^{k}2^{6-k}\right)x^{12-6k}$	**M1**
	For term independent of x, $k = 2$	
	Term indep of $x=\binom{6}{2}\left(3^{2}2^{4}\right)$	**A1**
	$=\left(\frac{6\times5}{2}\right)\left(3^{2}2^{4}\right)=2^{4}\times3^{3}\times5$	
	$p=4, q=3, r=1$	**A2,1,0** (A1 if answer left as $2^{4}\times3^{3}\times5$)

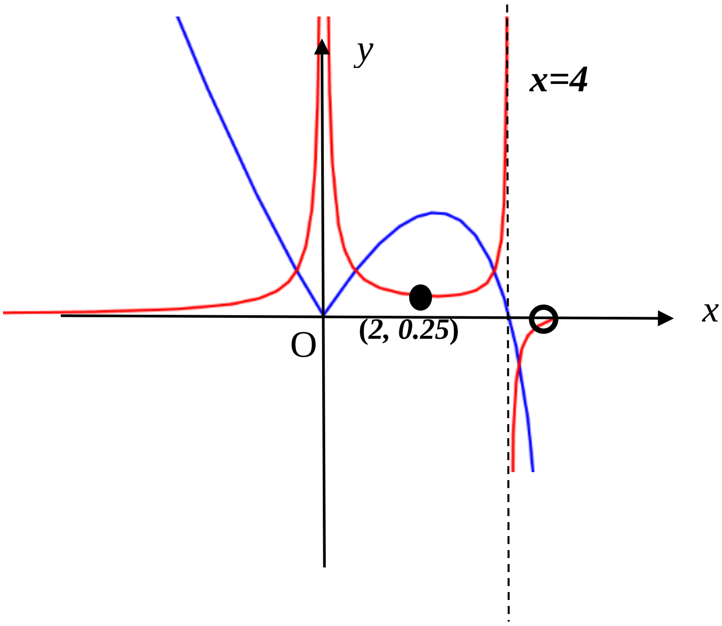
Qn	**Suggested solution**	**Markscheme**
(b)	General term of $\left(1-i\sqrt{2}\right)^5=\binom{5}{k}\left(-i\sqrt{2}\right)^k$	**M1** – general term or expansion
	For imaginary part, k = 1, 3, 5	
	$a\sqrt{2}i=\binom{5}{1}\left(-i\sqrt{2}\right)+\binom{5}{3}\left(-i\sqrt{2}\right)^3+\binom{5}{5}\left(-i\sqrt{2}\right)^5$	**M1** – only terms with odd powers
	$=-5\sqrt{2}i+10\left(2\sqrt{2}\right)i+4\sqrt{2}\left(-i\right)$	**A1** $\left(-i\right)^3=i$ and $\left(-i\right)^5=-i$
	$=11\sqrt{2}i$	
	$\therefore \operatorname{Im}\left(\left(1-i\sqrt{2}\right)^5\right)=11\sqrt{2}$	**A1** – no i
4	***Points of Inflection***	**[Max mark: 5]**
	$f'\left(x\right)=\left(1+x\right)e^x$	**A1**
	$f''\left(x\right)=e^x+e^x\left(1+x\right)=\left(2+x\right)e^x$	**A1**
	At points of inflection, $f''\left(x\right)=0$ (or $f''\left(x\right)$ is not defined) o.e.	**M1**
	Since $e^x>0\ \ \forall x,\ x=-2$	
	When $x<-2, f''\left(x\right)<0$ (concave downward)	**R1** – justify change in concavity
	When $x>-2, f''\left(x\right)>0$ (concave upward)	
	$\therefore\left(-2,-\frac{2}{e^2}\right)$ is a point of inflection.	**A1** – coordinates
5	***Graph Transformation with Completing the Square***	**[Max mark: 5]**
	Circle is:	
	$x^2-4x+y^2+6y+8=0$	
	$\left(x-2\right)^2-4+\left(y+3\right)^2-9+8=0$	**M1** – completing sq.
	$\left(x-2\right)^2+\left(y+3\right)^2=5$	**A1**
	<u>Method 1</u>	
	$4\left(x-2\right)^2+4\left(y+3\right)^2=20$	
	$\left(2x-4\right)^2+4\left(y+3\right)^2=20$	**(A1)**
	Horizontal scaling/stretch with factor 2 $\left(\because x\to 0.5x\right)$	**A1**
	followed by Translation by $\begin{pmatrix}-4\\0\end{pmatrix}$ o.e. $\left(\because x\to x+4\right)$	**A1**
	OR	
	Translation by $\begin{pmatrix}-2\\0\end{pmatrix}$ o.e. $\left(\because x\to x+2\right)$	
	followed by Horizontal scaling with factor 2 $\left(\because x\to 0.5x\right)$	

Qn	Suggested solution	Markscheme
	Method 2	
	Ellipse is:	
	$x^2+4\left(y+3\right)^2=20$	
	$\left(\frac{1}{2}x\right)^2+\left(y+3\right)^2=5$	**(A1)**
	Translation by $\begin{pmatrix}-2\\0\end{pmatrix}$ o.e.	**A1**
	followed by Horizontal scaling/stretch with factor 2	**A1**
	Method 3	
	Ellipse is:	
	$x^2+4\left(y+3\right)^2=20$	
	$x^2+\left(2y+6\right)^2=20$	**(A1)**
	Translation by $\begin{pmatrix}-2\\-3\end{pmatrix}$ o.e.	**A1**
	followed by Vertical scaling/stretch with factor $\frac{1}{2}$	**A1**
6	***Inverse Trigo Function***	**[Max mark: 4]**
	Let $A=\arcsin\left(\frac{1}{5}\right)$ and $B=\arccos\left(x\right)$	
	Then	
	$\begin{cases}\sin A=\frac{1}{5} & 0\le A\le\frac{\pi}{2} & \Rightarrow\cos A>0\\ \cos B=x & 0\le B\le\pi & \Rightarrow\sin B>0\end{cases}$	**R1**–seen anywhere to justify $\sin B=+\sqrt{1-x^2}$
	$\sin\left(A+B\right)=1$	
	$\sin A\cos B+\cos A\sin B=1$	**M1**–compound angle
	$\frac{1}{5}x+\frac{\sqrt{24}}{5}\left(\sqrt{1-x^2}\right)=1$	
	$24\left(1-x^2\right)=\left(5-x\right)^2$	
	$25x^2-10x+1=0$	**A1**
	$\left(5x-1\right)^2=0$	
	$x=\frac{1}{5}$	**A1**

Qn	Suggested solution	Markscheme
7	***Rational Functions with absolute value and range***	**[Max mark: 9]**
(a)	$x=\frac{1}{2}$ and $y=\frac{1}{2}$	**A1 A1**
(b)	*Note: Graph of g*  Required graph:  	**(G1)** **G1** – left branch **G1** – right branch **G1** – both asymptotes **G1** – (–1, 1) with correct graph behaviour **G1** – (0, 2)
(c)	$[1,\infty)$ **OR** $\{y\in\mathbb{R}: y\geq 1\}$ o.e.	**A1**

Qn	Suggested solution	Markscheme
Section B		
8	***Trigonometry (R-form) and Implicit differentiation***	**[Max mark: 12]**
ai	R = $\sqrt{9+16}=5$ $\theta=\arctan\frac{3}{4}$	**A1** **A1**
ii	Greatest = 1, Least = $\frac{1}{6}$	**A1A1**
b	$x-y=(x+y)^2$ $1-\frac{dy}{dx}=2(x+y)(1+\frac{dy}{dx})$ $\frac{dy}{dx}=\frac{1-2x-2y}{1+2x+2y}$	**M2,1,0** **A2,1,0**
c	$y=\log_3 x=\frac{lnx}{ln3}$ $\frac{dy}{dx}=\frac{1}{xln3}$ $\frac{d^2y}{dx^2}=-\frac{1}{x^2ln3}$ $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2$ $=-\frac{1}{x^2ln3}+\left(\frac{1}{xln3}\right)^2$ $=\frac{1-ln3}{x^2(ln3)^2}$ $k=\frac{1-ln3}{(ln3)^2}$	**M1** **M1** **M1** **A1**

Qn	Suggested solution	Markscheme
9	***Sigma Notation, GP, Combinatorics***	**[Max mark: 15]**
ai	$S_\infty = \frac{1}{1 - sin^2\left(\frac{5\pi}{6}\right)}$	**M1**
	$S_\infty = \frac{1}{1 - sin^2\left(\frac{\pi}{6}\right)}$	**M1**
	$S_\infty = \frac{1}{1 - \left(\frac{1}{2}\right)^2}$	
	$S_\infty = \frac{4}{3}$	**A1**
ii	$r = \frac{2sinxcosx}{sinx} = 2cosx$	**M1A1**
	For S_∞ to exist, $\lvert r \rvert < 1$ $-1 < 2cosx < 1$	**M1**
	$\frac{\pi}{3} < x < \frac{2\pi}{3}$	**A1A1**
b	Number of ways to arrange = $2!\,2!\,5!\frac{4!}{2!}\binom{6}{4}$	**M5 o.e.** (2! Perm. Ee, Tt) (5! Perm. letters) $\binom{6}{4}$ Perm. insertion of digits in 6 spaces btw letters) ($\frac{4!}{2!}$ Perm. digits with 44 repeat)
	$= 2!\,2!\,5!\frac{4!}{2!}\,\frac{6!}{2!4!}$	**M1**
	$= 5!6!$	**A1**

Qn	Suggested solution	Markscheme
10	***Logarithms, Inverse Trigo***	**[Max mark: 13]**
a		**G1 – left branches** **G1 – middle branch with correct local min pt (2, 0.25)** **G1 – right branch with hollow point at *x* = 5** **G1 – asymptote at *x* = 4, *y*=0 , *x*=0** ***G2 if student draws original graph and did not intersect at *y* = 1**
bi	Let $y = \arccos x$	
	$\cos y = x$	
	$-\sin y \frac{dy}{dx} = 1$	**M1**
	$\frac{dy}{dx} = -\frac{1}{\sin y}$	**M1**
	$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - cos^2 y}}$	**M1**
	$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$	**A1**
ii	$g'\left(\frac{1}{2}\right) = \frac{h\left(\frac{1}{2}\right) \cdot \left(-\frac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}}\right) - \arccos\left(\frac{1}{2}\right) \cdot h'\left(\frac{1}{2}\right)}{[h\left(\frac{1}{2}\right)]^2}$	**M2,1,0 – formula** **M1 - subn**
	$g'\left(\frac{1}{2}\right) = \frac{(1) \cdot \left(-\frac{1}{\sqrt{\frac{3}{4}}}\right) - \left(\frac{\pi}{3}\right) \cdot \left(\frac{\sqrt{3}}{\pi}\right)}{(1)^2}$	**M1 (arccos 0.5)**
	$g'\left(\frac{1}{2}\right) = -\sqrt{3}$	**A1**

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

STUDENT NAME: ______________________________

EXAMINATION CODE

8	8	1	8	-	7	2	0	2

TEACHER NAME: ______________________________

ST JOSEPH'S INSTITUTION
YEAR 5 END OF YEAR EXAMINATION 2018

MATHEMATICS
HIGHER LEVEL
PAPER 2
Monday

1st October 2018
1 hr 30 mins
0800 – 0930 hrs

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the foolscap paper provided
- The use of a scientific or examination graphical calculator is permitted in this paper.
- Ti-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted apps and be ram cleared.
- A clean copy of the **Mathematics HL Formulae Booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures.
- The maximum mark for this examination paper is *[**80** marks]*.
- Number of printed pages = 10.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Total/80

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to *show all working.*

SECTION A (40 marks)

Answer **all** questions in the **spaces** provided.

1 **[Maximum mark: 5]**

Find an expression for the inverse of

$$h(x) = \frac{3}{4 - 4\mathrm{e}^{-5x}}\ , x \in \mathbb{R}\backslash\{0\}$$

and state the range of $y = h(|x|)$.

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2 **[Maximum mark: 5]**

Solve for $w \in \mathbb{C}$:

$$\begin{cases} 0 = (w+1)(w^*+1) + Im(w) \times Im(w^*) \\ -5 = \ Re(w^2) \end{cases}$$

3 [Maximum mark: 6]

A polynomial function, $P(x)$, satisfies the following conditions:

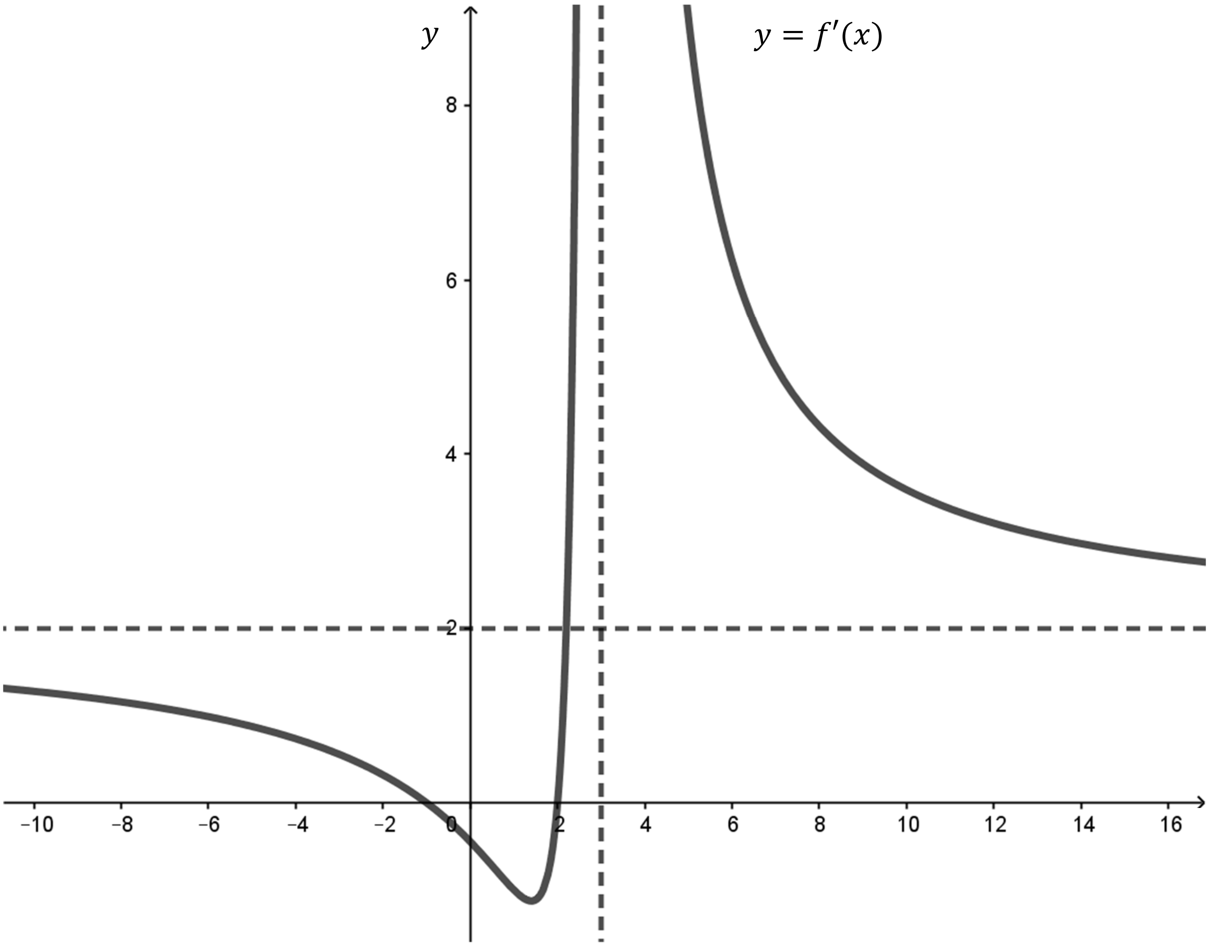
- $\deg P = 4$ with leading coefficient 1;
- $P(1) = 0$;
- $P(2) = -1$;
- sum of roots is 4; and,
- product of roots is -6.

Find the remainder when $P(x)$ is divided by $(x + 1)$.

4 [Maximum mark: 5]

The graph of $y = f'(x)$ is given below.

f' has two asymptotes $x = 3$ and $y = 2$; two zeros at $x = -1, 2$; and a minimum at $x = \sqrt{2}$.



Justify why $f(x) \to 2x + C$ for some constant C as $x \to \infty$ or as $x \to -\infty$.

On the same set of axes above, sketch a possible graph of $y = f(x)$.

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5 **[Maximum mark: 7]**

Let $f(x) = 2x^3 + 2x^2 - 5x + 1$ and $g(x) = \ln(x - 1)$.

Find **exactly** the sum of the roots of $f\big(g(x)\big) = 0$.

6 **[Maximum mark: 5]**

Let $f(x) = x\cos x\,, x \in D_f$, where $D_f = [a, b]$ is a maximal domain for which f^{-1} exists and a and b are positive constants to be determined.

By using the graph of $y = f(x)$, determine the smallest possible value of a and of b such that there exists a positive number $k \in D_f$ where $f(k) = f^{-1}(k)$.

Hence, find k.

7 **[Maximum mark: 7]**

How many different integers greater than 30 can be formed from the digits $\{1, 2, 2, 3, 3\}$ if no digit can be used more than once?

Do NOT write solutions on this page

SECTION B (40 marks)

Answer all questions on the foolscap paper provided. Please start each question on a new page.

8 [Maximum Mark: 9]

(i) John has to crack a 3-digit code. These three digits are the first three terms of a sequence u_n respectively.
Given that u_n is a quadratic polynomial in n, $n \in \mathbb{Z}^+$ and $u_4 = 14$, $u_6 = 40$ and $u_{10} = 140$, find u_n in terms of n and deduce the 3-digit code. **[6]**

(ii) A curve has equation $y = \tan^{-1}\left(3x^2 + x\right)$. Find the set of values of x for which the curve is increasing. **[3]**

9 [Maximum Mark: 12]

The terms of the sequence a_1, a_2, a_3, …, a_n are in arithmetic progression and $b_r = \left(\frac{1}{3}\right)^{a_r}$, for $r = 1, 2, 3, \ldots, n$.

(i) Show that the sequence b_1, b_2, b_3, …, b_n is geometric. **[3]**

Given that $b_1 = 9$ and the common ratio of the geometric progression is $\frac{1}{9}$,

(ii) find an expression for a_r and the value of the common difference of the arithmetic progression, **[4]**

(iii) find the least value of n such that the sum of the first n terms of the arithmetic progression exceeds 200, **[3]**

(iv) find $\sum_{r=1}^{n} \ln\left(b_r\right)$, leaving your answer in terms of n. **[2]**

TURN OVER

10 [Maximum Mark: 19]

(a) Using the formulae for $\sin(A \pm B)$ and $\cos(A \pm B)$, prove that

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B$$

and $\cos(A+B)+\cos(A-B)=2\cos A\cos B$ **[4]**

Hence, prove that $(\cos 8x+\cos 2x)^2+(\sin 8x+\sin 2x)^2=4\cos^2 3x$. **[4]**

(b) Given that $\tan 3x \neq 2$, find the exact solution(s) of the equation

$(\cos 8x+\cos 2x)^2+(\sin 8x+\sin 2x)^2=\sin 6x, \quad -\frac{\pi}{2}<x<\frac{\pi}{2}$. **[6]**

(c)(i) Express $\mathrm{f}(x)=(\cos 8x+\cos 2x)^2+(\sin 8x+\sin 2x)^2-2$ in the form $p\cos(qx)$. **[2]**

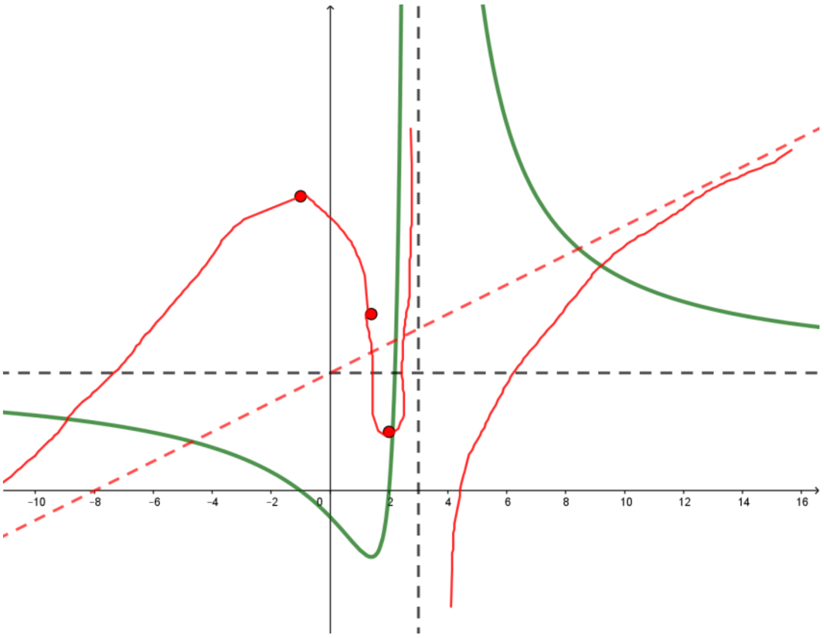
(ii) Find the period of $\mathrm{f}(x)$. **[1]**

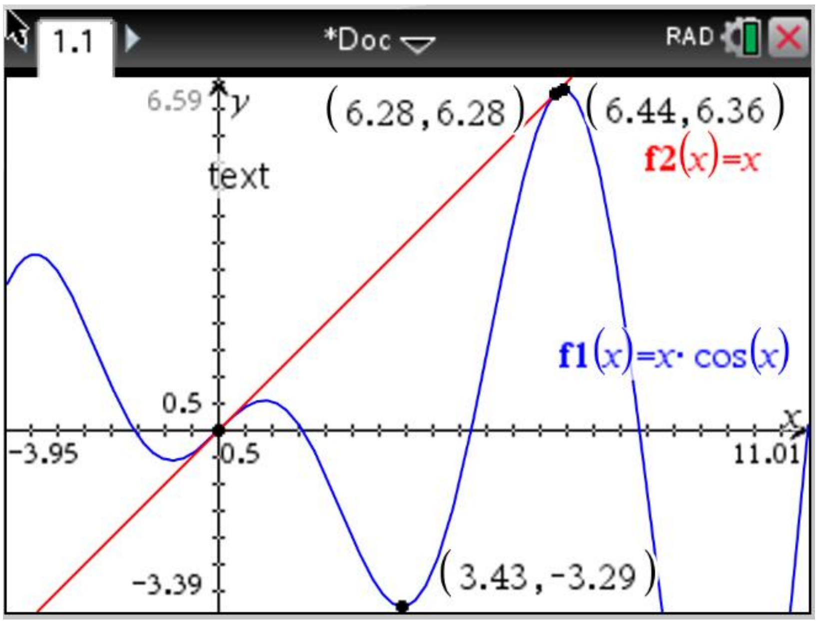
(iii) For which values of k is $|\mathrm{f}(x)-k|$ strictly above the x-axis? **[2]**

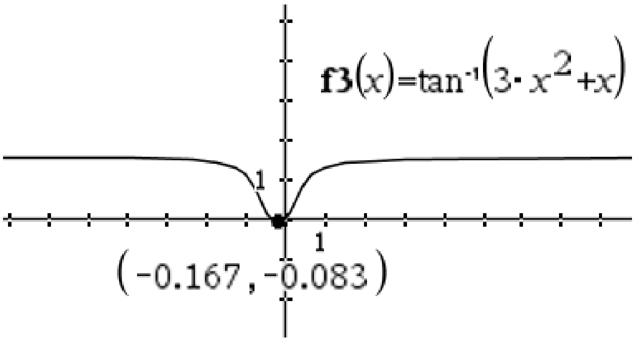
END OF PAPER

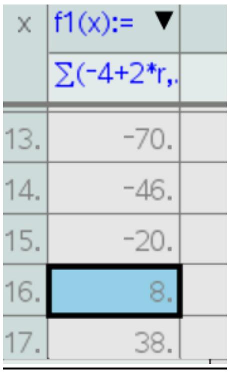
Year 5 HL Maths Promotional Exam 2018 – Paper 2 Mark Scheme

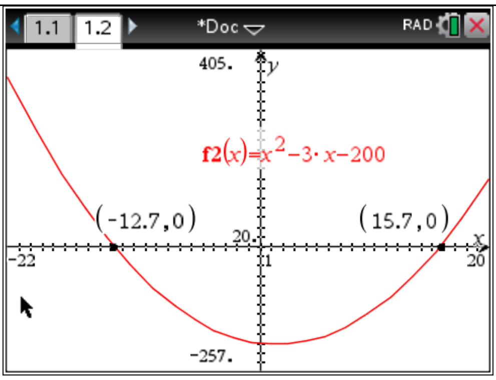
Qn	Suggested solution	Mark
1	**Inverse of Expo + Range**	**[Max mark: 5]**
	Let $y=\frac{3}{4-4e^{-5x}}$. Then $x=-\frac{1}{5}\ln\left(1-\frac{3}{4y}\right)$. Thus, $h^{-1}(x)=-\frac{1}{5}\ln\left(1-\frac{3}{4x}\right)$. Range of $h(\lvert x\rvert)=(0.75,+\infty)$.	**M1** **A1** **A1** **(M1 for -0.75)** **A1**
2	**Complex Equation**	**[Max mark: 5]**
	Let $w=a+ib, a, b\in\mathbb{R}$. $(w+1)(w^*+1)=-Im(w)\times Im(w^*)\Longrightarrow a^2+b^2+2a+1=b^2$ $\Longrightarrow (a+1)^2=0\Longrightarrow a=-1$ $\text{Re}(w^2)=-5\Longrightarrow 1-b^2=-5\Longrightarrow b=\pm\sqrt{6}$ (seen anywhere)	**M1**-attempt to simplify both expressions **A1** **M1**-attempt to get the real component of w^2 **A1A1**
3	**Polynomial Function + Vieta's & Remainder/Factor Theorem**	**[Max mark: 6]**
	$P(x)=x^4+px^3+qx^2+rx+s$ Product of roots = $-6\Longrightarrow s=-6$ Sum of roots = $4\Longrightarrow p=-4$ $P(1)=0$ and $P(2)=-1\Longrightarrow q=\frac{3}{2}$ and $r=\frac{15}{2}$ Remainder is $P(-1)=-7$ **OR** $P(x)=(x-1)(x^3+bx^2+cx+d)$ Product of roots = $-6\Longrightarrow d=6$ Sum of roots = $4\Longrightarrow b=-3$ $P(2)=-1\Longrightarrow(2-1)(2^3-3\times 2^2+2c+6)=-1\Longrightarrow c=-\frac{3}{2}$ Remainder is $P(-1)=-7$	**A1** **A1** **M1A1A1** **A1** **A1** **A1** **A1** **M1A1** **A1**

4	**Sketching f given the graph of f’**	**[Max mark: 5]**
	$f'(x) \to 2$ as $x \to \pm\infty \Longrightarrow f(x)$ approaches an oblique asymptote of gradient 2.	**R1**
		**G1 - $x < -1$** **G1 - $-1 < x < \sqrt{2}$** **G1 - $\sqrt{2} < x < 3$** **G1 - $x > 3$** **Award G1G1G1G0 for correct concavity and behavior of f but without oblique asymptote**
5	**Zeros, Composition, Logarithms**	**[Max mark: 7]**
	$f(1) = 0 \Longrightarrow (x+1)$ divides $f(x)$.	**A1**
	Using long/synthetic division or any other valid method, we get $$f(x) = (x-1)(2x^2+4x-1)$$	**M1A1**
	And so the other zeros of f are $-1 \pm \frac{1}{2}\sqrt{6}$.	**A1**
	Thus, the roots of $f\big(g(x)\big) = 0$ are $$e+1, e^{-1+\frac{1}{2}\sqrt{6}} + 1 \text{ and } e^{-1-\frac{1}{2}\sqrt{6}} + 1$$	**M1A1**
	Thus, the exact sum of the roots of $f\big(g(x)\big) = 0$ is $$3 + e + e^{-1+\frac{1}{2}\sqrt{6}} + e^{-1-\frac{1}{2}\sqrt{6}}$$	**A1**

6	**1-1 Functions**	**[Max mark: 5]**
	For $[a, b]$ to be maximal, $f(a)$ and $f(b)$ must be adjacent local extreme.	**(R1** – award for any pair a and b resulting to a maximal domain**)**
		**G1** – $y = x$
	$a = 3.43$	**A1**
	$b = 6.44$	**A1**
	$k = 6.28$	**A1**
7	**Combinatorics**	**[Max mark: 7]**
	Case 1. Two digits 31, 32, 33 ➔ 3 ways	**A1**
	Case 2. Three digits: XYZ or XXY 3! + 2C1*2C1*3 = 18 ways	**M1A1**
	Case 3. Four digits: XXYZ or XXYY 2C1*4!/2! + 4!/(2!2!) = 30 ways	**M1A1**
	Case 4. Five digits: XXYYZ 5!/(2!2!) = 30 ways	**A1**
	Total: 81 integers	**A1** *****Award M1 for correct cases IF there is still an available method mark

8	**Solving System of Linear Eq & Quadratics**	**[Max mark: 9]**
(i)	Let $u_n = an^2 + bn + c$, where a, b, c are constants.	
	$u_4 = 16a + 4b + c = 14$	**M1**
	$u_6 = 36a + 6b + c = 40$	**A1**
	$u_{10} = 100a + 10b + c = 140$	
	Using GDC, $a = 2$, $b = -7$ and $c = 10$	**M1A1**
	$\therefore u_n = 2n^2 - 7n + 10$	
	$u_1 = 2(1)^2 - 7(1) + 10 = 5$	
	$u_2 = 2(2)^2 - 7(2) + 10 = 4$	**M1**
	$u_3 = 2(3)^2 - 7(3) + 10 = 7$	
	$\therefore$ The 3-digit code is 547 .	**A1**
(ii)	**Method 1**	
		**G1** **A1 (x-ordinate of min pt)**
	Set of values of x is $(-0.167, \infty)$.	**A1**
	Method 2	
	$y = \tan^{-1}\left(3x^2 + x\right)$	
	$\frac{dy}{dx} = \frac{1}{1 + \left(3x^2 + x\right)^2}(6x + 1)$	
	$\frac{dy}{dx} > 0 \Rightarrow x > -\frac{1}{6}$	
	Set of values of x is $\left(-\frac{1}{6}, \infty\right)$.	

9	**AP and GP**	**[Max mark:12]**
(i)	$\frac{b_r}{b_{r-1}} = \frac{\left(\frac{1}{3}\right)^{a_r}}{\left(\frac{1}{3}\right)^{a_{r-1}}} = \left(\frac{1}{3}\right)^{a_r - a_{r-1}} = \left(\frac{1}{3}\right)^{d} \equiv$ a constant $\because$ d = common difference of the AP which is a constant, hence the sequence b_1, b_2, b_3, …, b_n is geometric.	**M1A1** **R1**
(ii)	Given $b_1 = 9$, $r = \frac{1}{9}$, $\Rightarrow b_r = 9\left(\frac{1}{9}\right)^{r-1} = \left(\frac{1}{9}\right)^{r-2} = \left(\frac{1}{3}\right)^{2r-4}$ $\therefore a_r = 2r - 4$ $d = a_r - a_{r-1} = 2r - 4 - [2(r-1) - 4] = 2$	**M1** **A1** **M1 A1**
(iii)	**Method 1** $\sum_{r=1}^{n}(2r-4) > 200$ $\sum_{r=1}^{n}(2r-4) - 200 > 0$ Using GDC, table,  x | f1(x):= Σ(-4+2*r,. 13. | -70. 14. | -46. 15. | -20. 16. | 8. 17. | 38. Least value of n is 16.	**A1** **M1** **A1**
	Method 2 $a_1 = 2(1) - 4 = -2$ $S_n > 200$ $\Rightarrow \frac{n}{2}[2(-2)+(n-1)(2)] > 200$ $\Rightarrow \frac{n}{2}(2n-6) > 200$ $\Rightarrow n^2 - 3n - 200 > 0$	**M1**

	 $n<-12.7$ (rejected) *or* $n>15.7$ Hence, the least value of n is 16.	A1 A1
(iv)	$\sum_{r=1}^{n}\ln(b_r)=\sum_{r=1}^{n}\ln\left(\frac{1}{3}\right)^{2r-4}$ $=\ln\left(\frac{1}{3}\right)\sum_{r=1}^{n}(2r-4)$ $=\ln\left(\frac{1}{3}\right)\left[n(n-3)\right]$	M1 A1
10	**Trigo**	**[Max mark: 19]**
(a)	$\sin(A+B)+\sin(A-B)$ $=\sin A\cos B+\sin B\cos A+\sin A\cos B-\sin B\cos A$ $=2\sin A\cos B$ (proven) $\cos(A+B)+\cos(A-B)$ $=\cos A\cos B-\sin A\sin B+\cos A\cos B+\sin A\sin B$ $=2\cos A\cos B$ (proven)	M1A1 M1A1
	$(\cos 8x+\cos 2x)^2+(\sin 8x+\sin 2x)^2$ $=(2\cos 5x\cos 3x)^2+(2\sin 5x\cos 3x)^2$ $=4\cos^2 3x(\cos^2 5x+\sin^2 5x)$ $=4\cos^2 3x$ (proven)	M1A1 M1R1

10(b)	$\left(\cos 8x+\cos 2x\right)^2+\left(\sin 8x+\sin 2x\right)^2=\sin 6x$, $-\frac{\pi}{2}<x<\frac{\pi}{2}$ $4\cos^2 3x=\sin 6x$ $4\cos^2 3x=2\sin 3x\cos 3x$ $2\cos 3x\left(2\cos 3x-\sin 3x\right)=0$ $\cos 3x=0$ or $2\cos 3x-\sin 3x=0$ $\Rightarrow 3x=\pm\frac{\pi}{2}$ or $\tan 3x=2$ (rejected as $\tan 3x\neq 2$) $\Rightarrow x=\pm\frac{\pi}{6}$	**M1A1** **A1** **M1** **A1A1**
(c)(i)	$\mathrm{f}(x)=\left(\cos 8x+\cos 2x\right)^2+\left(\sin 8x+\sin 2x\right)^2-2$ $=4\cos^2 3x-2$ $=2\left(2\cos^2 3x-1\right)$ $=2\cos 6x$	**M1A1**
(ii)	Period of $\mathrm{f}(x)=\frac{2\pi}{6}=\frac{\pi}{3}$	**A1**
(iii)	$k<-2$ or $k>2$	**A1A1**

STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	1	9	-	7	2	0	1

ST. JOSEPH'S INSTITUTION
YEAR 5 END OF YEAR EXAMINATION 2019

MATHEMATICS
HIGHER LEVEL
PAPER 1
Thursday

17 October 2019
1 hr 30 mins
0800 – 0930 hrs

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the foolscap paper provided.
- The use of calculators is **not** permitted in this paper.
- A clean copy of the **Mathematics HL Formulae Booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers are to be given exactly.
- The maximum mark for this examination paper is *[**80** marks]*.
- This question paper consists of **10** printed pages including the Cover Sheet.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	TOTAL
										/80

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (40 marks)

Answer **all** questions in the **spaces** provided.

1 [Maximum mark: 7]

The function f is defined as $f: x \mapsto \ln\sqrt{x},\ x \in (0, \infty)$.

Some values are given in the table below for the function g and its derivative g'.

x	2	e	3	5
$g(x)$	-3e	$-\text{e}^2$	$-\text{e}$	4e^2
$g'(x)$	$-\text{e}^3$	$-\text{e}$	$\frac{\text{e}}{2}$	e^2

(a) Show that $f'(\text{e}) = \frac{1}{2\text{e}}$. [2]

(b) It is given that $h(x) = f(x) \times g(2\ln x + 3)$.

Find the value of $h'(\text{e})$. [5]

2 **[Maximum mark: 4]**

Prove that $\tan\left(\pi + \arccos x\right) = \frac{-\sqrt{1-x^2}}{x}$, where $x \in [-1, 0)$.

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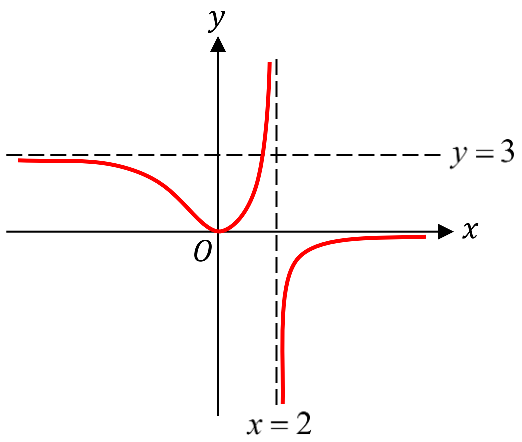
TURN OVER

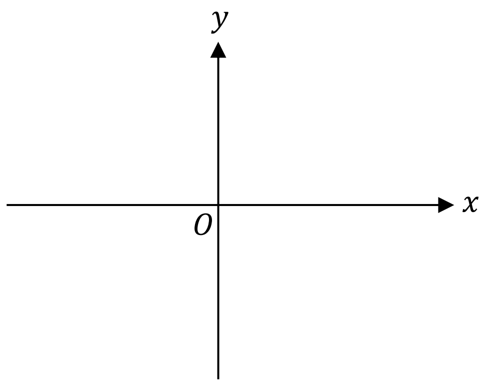
3 **[Maximum mark: 7]**

The graph of the function f has asymptotes at $x = 2$, $y = 0$ and $y = 3$ and a minimum point at the origin, as shown in the figure below.

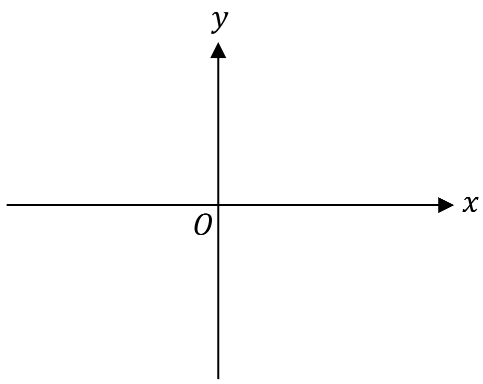


Sketch the following graphs on the axes below, indicating clearly any axial intercepts, turning points and the equations of all asymptotes:

(a) $y = \dfrac{1}{f(x)}$, [3]



(b) $y = f'(x)$. [4]



4 **[Maximum mark: 5]**

Solve the equations $\left(1+\mathrm{i}\right)w+\mathrm{i}z=2\mathrm{i}$ and $\left(1-\mathrm{i}\right)z-2w=2$, given z and w are in the form $a+b\mathrm{i}$ where $a, b\in\mathbb{R}$ and $\mathrm{i}^2=-1$.

TURN OVER

5 **[Maximum mark: 6]**

It is given that the curve C is defined by $\mathrm{e}^{x\sin y}+2x+\ln y=5$.

Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a function of both x and y. [6]

TURN OVER

6 **[Maximum mark: 5]**

Expand and simplify the expression $(z+1)(z^*+1)$, where z^* is the conjugate of z.

Hence, or otherwise, prove that $\frac{z-1}{z+1}$ is a purely imaginary number if $|z|=1$.

TURN OVER

7 **[Maximum mark: 6]**

In the triangle ABC, it is given that $AC = 8\,\text{cm}$, $BC = a$ cm, $\angle BAC = 30°$ and $\angle ABC = \theta°$.

(a) If $a = 4\sqrt{2}$, find the possible values of θ. [3]

(b) Find the set of all the values of a for which there is a unique value for the length of AB. [3]

TURN OVER

*Do **NOT** write solutions on this page.*

SECTION B (40 marks)

*Answer **all** questions on the foolscap paper provided.* ***Please start each question on a new page.***

8 [Maximum mark: 15]

Let $P(x) = 3kx^2 - kx + 1$ for some real number k such that $P(x) > 0$ for all $x \in \mathbb{R}$.

(a) Determine the range of possible values of k. **[4]**

(b) Determine the value of k so that P is tangent to $y = kx$. **[4]**

Let $Q(x)$ be a polynomial such that for $k = -1$, $Q(x) = (x + 2) \times P(x) - (x + 2)$, $x \in \mathbb{R}$.

(c) Find the remainder when $Q(x)$ is divided by $(x - 1)$. **[3]**

(d) Find the remainder when $Q(x)$ is divided by $(x - 1)(x + 1)$. **[4]**

9 [Maximum mark: 13]

(a) Determine the coefficient of x^8 in the expansion of $(2 - x)^{10}$. **[4]**

(b) In the expansion of $(2 + x^2)^8$ written in ascending powers of x, two consecutive terms share the same coefficient.

Determine the two powers of x having the same coefficients. **[5]**

(c) Suppose

$$(2 - x)^{10}(2 + x^2)^8 = 2^n - 1310720x + \cdots + Mx^{25} + x^{26}.$$

Find the value of n and of M. **[4]**

TURN OVER

*Do **NOT** write solutions on this page.*

10 [Maximum mark: 12]

(a) Find the solutions to the equation, leaving your answer in the Cartesian form: **[3]**

$$z - 1 = \mathrm{i}\left(\frac{z-1}{z}\right), \quad z \in \mathbb{C} \text{ and } \mathrm{i}^2 = -1.$$

(b) Let $f_n(x) = (x - u_1)(x - u_2)(x - u_3) \cdots (x - u_n)$, $x \in \mathbb{R}$, for some positive integer n, where $u_1, u_2, \ldots, u_n$ form a geometric sequence.

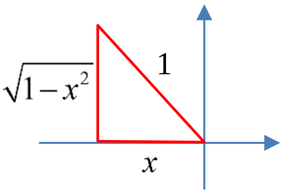
Further, suppose u_1 and u_3 satisfy the equation in (a) such that $\arg u_1 < \arg u_3$.

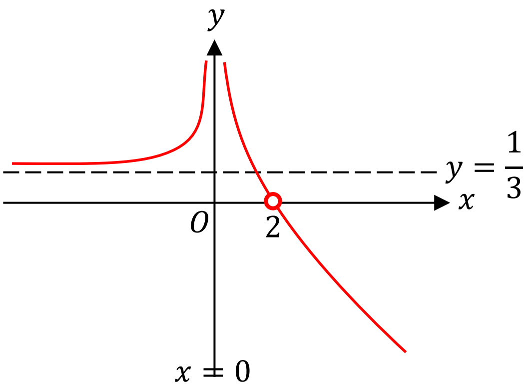
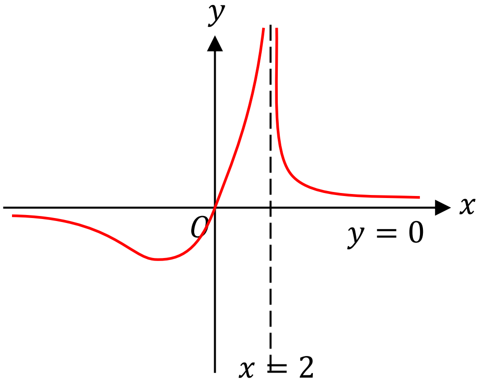
i. Find the possible values of the common ratio, leaving your answer in the Cartesian form. **[5]**

ii. Evaluate $f_9(0)$, leaving your answer in the Cartesian form. **[4]**

End of Paper

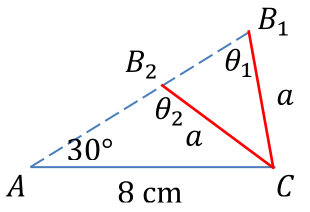
Year 5 HL Maths End of Year Exam 2019 – Paper 1 Mark Scheme

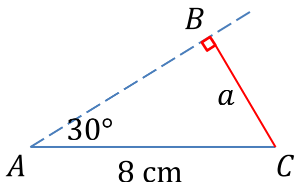
Qn	**Suggested solution**	**Markscheme**
	Section A	
1	***Differentiation, Chain Rule, Product Rule***	**Max mark: 7**
(a)	$f(x)=\ln\sqrt{x}=\frac{1}{2}\ln x$ $f'(x)=\frac{1}{2x}$ $\therefore f'(e)=\frac{1}{2e}$	M1 A1 AG
(b)	$h(x)=f(x)g(2\ln x+3)$ $h'(x)=f(x)g'(2\ln x+3)\cdot\frac{2}{x}+f'(x)g(2\ln x+3)$ $h'(e)=f(e)g'(2(1)+3)\cdot\frac{2}{e}+f'(e)g(2(1)+3)$ $=\frac{2f(e)g'(5)}{e}+f'(e)g(5)$ $=\frac{2\left(\frac{1}{2}\right)e^2}{e}+\left(\frac{1}{2e}\right)4e^2$ $=e+2e$ $=3e$	M1 A1 (Product rule) M1 A1 A1
2	***Trigonometry, Arctrigo***	**Max mark: 4**
	$\tan(\pi+\arccos x)=-\frac{\sqrt{1-x^2}}{x}$, where $x\in[-1,\ 0]$ Let $A=\arccos x$. Then, $\cos A=x$. Since $x\in[-1,\ 0]$, then $\frac{\pi}{2}\le A\le\pi$ **Method 1:** Then, $\sin A=\sqrt{1-\cos^2 A}=\sqrt{1-x^2}$ Therefore, $\tan A=\frac{\sin A}{\cos A}=\frac{\sqrt{1-x^2}}{x}$ **Method 2:**  $\cos A=\frac{x}{1}$ By Pythagora's Theorem, 1 oppo. side $=\sqrt{1-x^2}$ $\therefore\tan A=\frac{\sqrt{1-x^2}}{x}$	A1 M1 A1 M1 AG

Qn	Suggested solution	Markscheme
	Continuing from either method: Hence, $LHS = \tan\left(\pi - A\right)$ $= -\tan A$ (property: $\tan(\pi - A) = -\tan A$ for any A) $= -\frac{\sqrt{1-x^2}}{x}$ **Alternative reasoning:** $\tan\left(\pi - A\right) = \frac{\tan\pi - \tan A}{1 - \tan\pi\tan A}$ $= \frac{0 - \tan A}{1 - 0}$ $= -\tan A$	(R1) AG
3	***Functions/Differentiation, Graphs of reciprocal and derivative***	**Max mark: 7**
(a)		A1 shape A1 horizontal asymptote A1 vertical asymptote
(b)		A1 shape A1 horizontal asymptote A1 vertical asymptote A1 min point in negative x region.
4	***Complex Numbers, Solving, Rationalisation***	**Max mark: 5**
	$\left(1+i\right)w + zi = 2i$ ………… (1) $\left(1-i\right)z - 2w = 2$ ………… (2) **Method 1: By elimination v1** $(1)\times(1-i)$, we have $\left(1+i\right)(1-i)w + zi(1-i) = 2i(1-i)$ $(1+i)z + 2w = 2 + 2i$ ……….. (3)	M1 (Equalizing the coefficients)

Qn	Suggested solution	Markscheme
	(2)+(3), we have $2z = 4 + 2i$	M1 (Eliminating one variable)
	$z = 2 + i$	A1
	Subst $z = 2 + i$ into (2), we have $(1-i)(2+i) - 2w = 2$	M1
	$2w = (3-i) - 2$ $w = \frac{1-i}{2}$	A1
	Method 2: By elimination v2 $(1) \times (-1-i)$, we have	
	$(1+i)(-1-i)w + zi(-1-i) = 2i(-1-i)$ $(1-i)z - 2iw = 2 - 2i$ .... (3)	M1 (Equalizing the coefficients)
	(2)-(3), we have $(-2+2i)w = 2i$ $w = \frac{2i}{-2+2i} = \frac{i}{-1+i}$	M1 (Eliminating one variable)
	$= \frac{i}{-1+i} \times \frac{-1-i}{-1-i}$	M1 (Rationalising)
	$= \frac{1-i}{2}$	A1
	Subst $w = \frac{1-i}{2}$ into (2), we have $(1-i)z - 2\left(\frac{1-i}{2}\right) = 2$ $(1-i)z = 3 - i$ $z = \frac{3-i}{1-i} \times \frac{1+i}{1+i}$ $= \frac{4+2i}{2}$	
	$= 2 + i$	A1
	Method 3: By substitution From (2),	
	$(1-i)z - 2 = 2w$ $w = \frac{(1-i)z - 2}{2}$ .... (3)	M1 (Subject)
	Subst (3) into (1),	M1 (Eliminating one variable)

Qn	**Suggested solution**	**Markscheme**
	$(1+i)\cdot\frac{(1-i)z-2}{2}=(2-z)i$	
	$\frac{(1+i)(1-i)z-2(1+i)}{2}=(2-z)i$	M1 (Rationalisation)
	$\frac{2z-2(1+i)}{2}=2i-zi$	A1
	$z-(1+i)=2i-zi$	
	$z+zi=1+3i$	
	$z=\frac{1+3i}{1+i}\cdot\frac{1-i}{1-i}$	
	$=\frac{(1+3)+(-1+3)i}{2}$	A1
	$=2+i$	
	Subst $z=2+i$ into (3), we have	
	$w=\frac{(1-i)(2+i)-2}{2}$	
	$=\frac{3-i-2}{2}$	
	$=\frac{1-i}{2}$	
	Ans: $z=2+i$, $w=\frac{1-i}{2}$	
5	***Implicit Differentiation***	**Max mark: 6**
	$e^{x\sin y}+2x+\ln y=5$	
	Differentiating w.r.t. x,	
	$\frac{d}{dx}\left(e^{x\sin y}+2x+\ln y\right)=\frac{d}{dx}(5)$	M1 (Implicit differentiation)
	$e^{x\sin y}\left(\sin y+x\cos y\frac{dy}{dx}\right)+2+\frac{1}{y}\frac{dy}{dx}=0$	M1 (chain) M1 (product) A1
	$\left(e^{x\sin y}x\cos y+\frac{1}{y}\right)\frac{dy}{dx}=-\left(2+e^{x\sin y}\sin y\right)$	
	$\left(e^{x\sin y}xy\cos y+1\right)\frac{dy}{dx}=-y\left(2+e^{x\sin y}\sin y\right)$	M1 ($\frac{dy}{dx}$ subject)
	$\frac{dy}{dx}=-\frac{y\left(2+e^{x\sin y}\sin y\right)}{e^{x\sin y}xy\cos y+1}$ (o.e.)	A1
6	***Complex Numbers, Properties of conjugate***	**Max mark: 5**
	(i) $(z+1)(z^*+1)$	
	$=zz^*+(z+z^*)+1$	A1
	$=\lvert z\rvert^2+2\mathrm{Re}(z)+1$ (Optional)	

Qn	Suggested solution	Markscheme
	(ii) **Method 1: Hence**	
	$\frac{z-1}{z+1}$	
	$=\frac{z-1}{z+1}\cdot\frac{z^*+1}{z^*+1}$	M1
	$=\frac{zz^*+(z-z^*)-1}{zz^*+(z+z^*)+1}$	A1
	$=\frac{\lvert z\rvert^2+2\operatorname{Im}(z)-1}{\lvert z\rvert^2+2\operatorname{Re}(z)+1}$	
	$=\frac{1+2\operatorname{Im}(z)-1}{1+2\operatorname{Re}(z)+1}$	A1
	$=\frac{\operatorname{Im}(z)}{1+\operatorname{Re}(z)}$, which is a purely imaginary number	R1
	Method 2: Otherwise	
	Let $z=a+ib$, where $a,b\in\mathbb{R}$	
	Then, $\frac{z-1}{z+1}=\frac{ai+b-1}{ai+b+1}$	
	$=\frac{a+ib-1}{a+ib+1}$	
	$=\frac{(a-1)+ib}{(a+1)+ib}\times\frac{(a+1)-ib}{(a+1)-ib}$	M1
	$=\frac{\left[(a-1)(a+1)+b^2\right]+i\left[b(a+1)-b(a-1)\right]}{(a+1)^2+b^2}$	
	$=\frac{\left[a^2-1+b^2\right]+i\left[ab+b-ab+b\right]}{(a+1)^2+b^2}$	A1
	$=\frac{\left[(a^2+b^2)-1\right]+i\left[2b\right]}{(a+1)^2+b^2}$, $\left(\lvert z\rvert=1\Rightarrow a^2+b^2=1\right)$	A1
	$=i\left[\frac{2b}{(a+1)^2+b^2}\right]$, which is a purely imaginary number	R1
7	***Trigonometry, Ambiguous case of the Sine Rule***	**Max mark: 6**
(a)		
	Given $a=4\sqrt{2}$,	
	By sine rule, $\frac{\sin\theta}{8}=\frac{\sin 30°}{4\sqrt{2}}$	M1
	Simplifying, $\sin\theta=\frac{8\left(\frac{1}{2}\right)}{4\sqrt{2}}=\frac{1}{\sqrt{2}}$	
	Solving, $\theta=45°$ or $135°$	A1 A1

Qn	**Suggested solution**	**Markscheme**
(b)	Unique length of AB means that the triangle is either (1) a right-angled triangle ($\angle ABC = 90°$), or (2) length of BC exceeds the length of AC.	R1
	For (1), $\sin 30° = \frac{a}{8} \Rightarrow a = 8 \sin 30° \Rightarrow a = 4$	A1
		
	For (2), $a \geq 8$ Therefore, $a = 4$ or $a \geq 8$.	A1

Qn	**Suggested solution**	**Markscheme**
	Section B	
8	***Quadratic (Discriminants) & Polynomial Functions***	**[Max mark: 15]**
(a)	Observe that if $k = 0$, $P(x) = 1 > 0$ for all $x \in \mathbb{R}$. Suppose, $k \neq 0$. Then $P(x) > 0$ for all $x \in \mathbb{R}$ only if $(-k)^2 - 4(3k)(1) < 0$ $\Rightarrow k^2 - 12k < 0$ $\Rightarrow k(k - 12) < 0$ $\Rightarrow 0 < k < 12$ Thus, $0 \leq k < 12$.	**A1** **M1** **M1** **A1**
(b)	$3kx^2 - kx + 1 = kx \Rightarrow 3kx^2 - 2kx + 1 = 0$ must have zero discriminant, i.e., $(-2k)^2 - 4(3k)(1) = 0$. $\Rightarrow 4k^2 - 12k = 4k(k - 3) = 0$ $\Rightarrow k = 3$ or $k = 0$ Reject $k = 0$ as the resulting functions, e.g., $P(x) = 1$ and $y = 0$, are not tangent.	**A1** **M1** **A1** – both **A1** – reasoning not needed
(c)	$Q(x) = (x + 2)(-3x^2 + x + 1) - (x + 2)$ $= (x + 2)(-3x^2 + x)$ Remainder $= Q(1) = (1 + 2)(-3 + 1) = -6$	**A1 (seen anywhere)** **M1A1**
(d)	**Method 1** Let $Q(x) = (x^2 - 1)S(x) + (Ax + B)$ for some $S(x)$ and real numbers A and B. $Q(1) = A + B = -6$ $Q(-1) = -A + B = (1)(-3 - 1) = -4$ $\Rightarrow A = -1$ and $B = -5$. Thus, the remainder is $-x - 5$.	**A1** **A1** **M1** **A1**

Qn	Suggested solution	Markscheme
	Method 2	
	By long division, $\begin{array}{r l} & -3x \quad -5 \\ x^2-1\,\vert & \overline{-3x^2-5x+2x} \\ & -3x^2+3x \\ \hline & \quad -5x^2-x \\ & \quad -5x^2+5 \\ \hline & \quad\quad -x-5 \end{array}$	**A1A1 – correct quotient** **M1 – correct division**
	Thus, the remainder is $-x-5$.	**A1**
	Method 3	
	$Q(x) = (x+2)(-3x^2+x)$ $\quad = -3x^3-5x^2+2x$	**A1**
	$\quad = -3x^3-5x^2+(3x-x)+(5-5)$ $\quad = (-3x^3+3x)+(-5x^2+5)+(-x-5)$ $\quad = -3x(x^2-1)-5(x^2-1)+(-x-5)$	**A1**
	$\quad = (x^2-1)(-3x-5)+(-x-5)$	**M1**
	Thus, the remainder is $-x-5$	**A1**
9	***Complex Numbers & Geometric Sequence***	**[Max mark: 12]**
(a)	$z-1 = \mathrm{i}\left(\frac{z-1}{z}\right) \Longrightarrow (z-1)-\mathrm{i}\left(\frac{z-1}{z}\right) = 0$ $\Longrightarrow (z-1)\left(1-\frac{\mathrm{i}}{z}\right) = (z-1)(z-\mathrm{i}) = 0$	**M1**
	Thus, $z = 1$ or $z = \mathrm{i}$.	**A1A1**
(b)	Let $u_1 = 1$ and $u_3 = \mathrm{i}$.	**A1 – seen anywhere**
	(i)	
	Let $r = a+\mathrm{i}b$ for some real numbers a and b. $\Longrightarrow \frac{u_3}{u_1} = \mathrm{i} = r^2 = a^2-b^2+2abi$ $\Longrightarrow a^2-b^2 = 0$ and $2ab = 1 \Longleftrightarrow b = \frac{1}{2a}$.	**A1**
	$a^2-b^2 = 0 \Longrightarrow a = \pm b \Longrightarrow 2a^2 = \pm 1 \Longrightarrow 2a^2 = 1$.	**M1**
	Thus, $a = \pm\frac{1}{\sqrt{2}}$ and $b = \pm\frac{1}{\sqrt{2}}$.	
	Therefore, $r = \pm\frac{1}{\sqrt{2}}(1+\mathrm{i})$.	**A1A1**
	(ii)	
	$f_9(0) = (-u_1)(-u_2)\cdots(-u_9)$	

Qn	Suggested solution	Markscheme
	$= -(u_1u_2\cdots u_9)$	**A1**
	$= -(r^0r^1\cdots r^8)$	
	$= -r^{0+1+\cdots+8}$	**M1**
	$= -r^{36}$	
	$= -(r^2)^{18}$	**M1**
	$= -\mathrm{i}^{18}$	
	$= 1$	**A1**
10	***Binomial Theorem***	**[Max mark: 13]**
(a)	The general term in the expansion of $(2-x)^{10}$ is given by $\binom{10}{k}2^{10-k}(-x)^k$ or $\binom{10}{m}2^m(-x)^{10-m}$	**A1**
	Letting $k=8$ or $m=2$, gives us	**A1**
	$\frac{10!}{8!\,2!}2^2 = 5\times9\times4 = 180.$	**M1A1**
(b)	The general term in the expansion of $(2+x^2)^8$ is given by $\binom{8}{k}2^kx^{2(8-k)}$ or $\binom{8}{m}2^{8-m}x^{2m}$	**A1**
	Since two consecutive terms in the expansion have the same coefficients:	
	$\binom{8}{k}2^k = \binom{8}{k+1}2^{k+1}$ or $\binom{8}{m}2^{8-m} = \binom{8}{m+1}2^{7-m}$	**A1**
	$\binom{8}{k} = \binom{8}{k+1}2$ or $\binom{8}{m}2 = \binom{8}{m+1}$	
	$\frac{8!}{k!(8-k)!} = \frac{8!}{(k+1)!(8-(k+1))!}2$ or $\frac{8!}{m!(8-m)!}2 = \frac{8!}{(m+1)!(8-(m+1))!}$	
	$\frac{1}{k!(8-k)\cdot(7-k)!} = \frac{2}{(k+1)\cdot k!(7-k)!}$ or $\frac{2}{m!(8-m)(7-m)!} = \frac{1}{(m+1)m!(7-m)!}$	**M1**
	$\frac{1}{(8-k)} = \frac{2}{(k+1)}$ or $\frac{2}{(8-m)} = \frac{1}{(m+1)}$	
	$k+1 = 16-2k$ or $2m+2 = 8-m$	
	$k = 5$ or $m = 2$	**A1**
	Thus, x^6 and x^4 have the same coefficients.	**A1**
(c)	By inspection, it follows that $n = 10+8 = 18$.	**A1**
	Method 1	
	Also, $M = -\sum$ roots $= 2\times10$.	**R1**
	Therefore, $M = -20$.	**M1A1**
	Method 2	
	Observe that there is only one way of getting x^{25}, i.e., by considering x^9 in the expansion of $(2-x)^{10}$ and x^{16} in the expansion of $(2+x^2)^8$.	**R1**

Qn	Suggested solution	Markscheme
	Thus, $Mx^{25} = \binom{10}{9}2(-x)^9 \times x^{16} = -20x^{25}$ Therefore, $M = -20.$	**M1** **A1**

STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	1	9	-	7	2	0	2

ST JOSEPH'S INSTITUTION
YEAR 5 END OF YEAR EXAMINATION 2019

MATHEMATICS
HIGHER LEVEL
PAPER 2
Monday

21 October 2019
1 hr 30 mins
0800 – 0930 hrs

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the foolscap paper provided
- The use of a scientific or examination graphical calculator is permitted in this paper.
- Ti-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted apps and be ram cleared.
- A clean copy of the **Mathematics HL Formulae Booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures.
- The maximum mark for this examination paper is *[**80** marks]*.
- Number of printed pages = 9.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Total
/ 5	/ 6	/ 5	/ 8	/ 8	/ 8	/ 9	/ 16	/ 15	/ 80

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to *show all working.*

SECTION A (40 marks)

Answer **all** questions in the **spaces** provided.

1 **[Maximum mark: 5]**

The diagram shows a right-angled triangle ABC with AC = 2 cm and BD = x cm . CD is an arc of a circle with centre A and radius 2 cm.

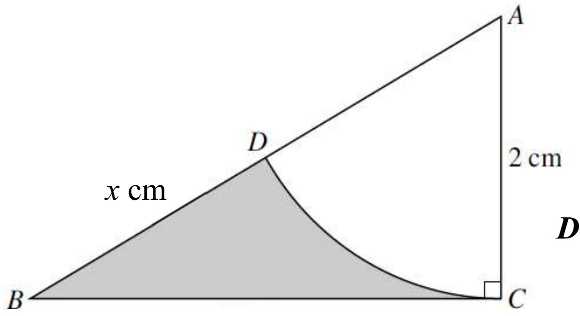


Diagram not to scale

(a) Show that the area of the shaded region is $\left(\sqrt{x^2+4x}-2\arccos\left(\dfrac{2}{x+2}\right)\right)\text{cm}^2$. [3]

(b) Find the value of x when the area of the shaded region is 5 cm^2 . [2]

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2 **[Maximum mark: 6]**

A curve has equation $y = \dfrac{x^2 - x}{\mathrm{e}^x}$ defined for all real values of x.

(a) Find the range of values of x for which the curve is strictly increasing. [3]

(b) Find the range of values of x for which the curve is concave upwards. [3]

TURN OVER

3 **[Maximum mark: 5]**

The roots of a cubic polynomial equation with real coefficients, $p(x) = 0$, are the consecutive terms of an arithmetic sequence and one of the roots is $2 - \mathrm{i}$.
Find a possible $p(x)$, in the form $ax^3 + bx^2 + cx + d$, where a, b, c and d are values to be determined.

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4 **[Maximum mark: 8]**

The function f is defined by $f(x)=\frac{2x-3}{x-1}, x \in \mathbb{R}, x \neq 1, x \neq 2$ and f^{n} is denoted as $\underbrace{(f \circ f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x)$.

(a) Find the function f^{2}. [2]

(b) Find, f^{-1}, the inverse function of f, stating its domain. [3]

(c) Hence, or otherwise, define the function f^{2019} for $n \in \mathbb{Z}^{+}$. [3]

TURN OVER

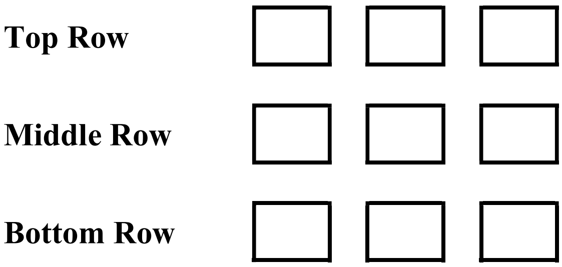
5 **[Maximum mark: 8]**

There are 8 cards printed with a different single digit from 1 to 8 on each card.

(a) Alex selects 4 cards at random without replacement.
Find the number of possible selections such that there is an equal number of odd-digit cards and even-digit cards. [2]

(b) Ben uses 5 of the cards without repetition to form a five-digit number that begins and ends with an even digit.
How many numbers did he form? [3]

(c) Cherry places all the 8 cards into a 3 by 3 grid such that there are exactly two cards on the top row and three cards each in in the middle row and bottom row.
Find the number of arrangements such that both cards on the top row are odd-digit cards. [3]



TURN OVER

6 **[Maximum mark: 8]**

Solve the simultaneous equations

$$\log_x y = \log_y x\,,$$
$$\log_x (x - y) = \log_y (x + y)\,.$$

TURN OVER

Do NOT write solutions on this page

SECTION B (40 marks)

Answer all questions on the foolscap paper provided. ***Please start each question on a new page.***

7 **[Maximum Mark: 9]**

(a) Alex, Bryan and Caleb work for GoGrab! taxi company. The company pays them different rates for different periods of time in a day. During off-peak periods, the firm pays \$$x$ for each km driven. For peak and super-peak periods, the firm pays \$$y$ more and \$$z$ more per km respectively. The table below shows the distance (in km) covered by Alex, Byran and Caleb during the three different periods, together with the amount they were paid, on a particular day.

	Off-peak	**Peak**	**Super-peak**	**Total**
Alex	63	26	4	\$146.70
Bryan	59	34	12	\$170.30
Caleb	30	52	28	\$189.40

Write down and solve equations to find the values of x, y and z. [4]

(b) Find the equations of the asymptotes of the graph of $y = \frac{2 - e^x}{2e^x - 1}$. [5]

8 **[Maximum Mark: 16]**

(a) Given

$$\begin{aligned} 4\sin B + 3\sin C = 6 \\ 3\cos C + 4\cos B = 1 \end{aligned},$$

find the value of $\cos(B - C)$. [6]

(b) Let $h(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right), x \neq 0$.

(i) Find $h'(x)$ in simplified form. [4]

(ii) Hence show that $h(x) = \frac{\pi}{2}$ for $x > 0$. [3]

(iii) Show that h is an odd function. Hence state the value of $h(x)$ for $x < 0$. [3]

TURN OVER

Do NOT write solutions on this page

9 [Maximum Mark: 15]

Let $f(x)=|x|-2$.

(a) (i) Sketch the graph of $y=f(x)$. [1]

(ii) State the zeros of f. [1]

(b) (i) Sketch the graph of $y=\left(f\circ f\right)(x)$. [2]

(ii) State the zeros of $f\circ f$. [1]

We denote f^{n} as $\underbrace{f\circ f\circ f\circ\ldots\circ f}_{n\text{ times}}$.

(c) (i) Find the zeros of f^{3}. [1]

(ii) Find the zeros of f^{4}. [1]

(iii) Deduce the zeros of f^{8}. [1]

(d) The zeros of f^{2n} are $a_1, a_2, a_3, \ldots, a_N$.

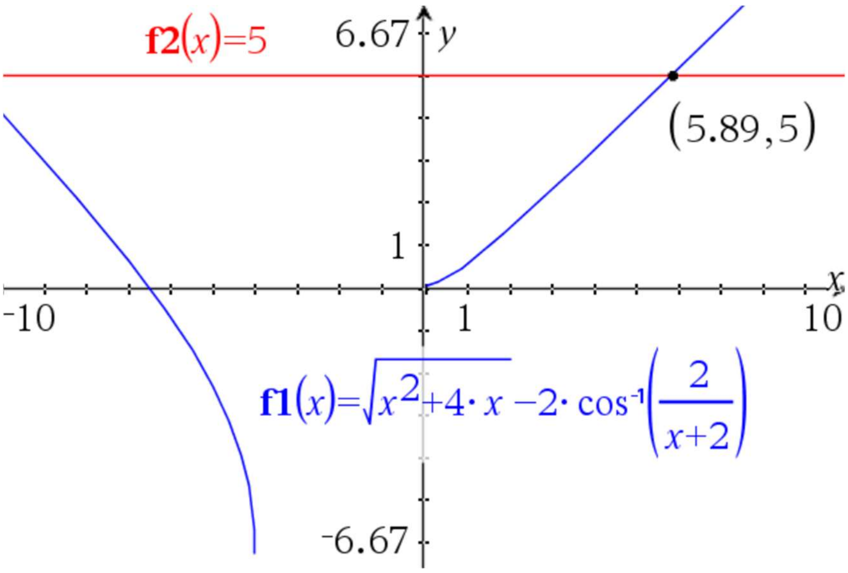
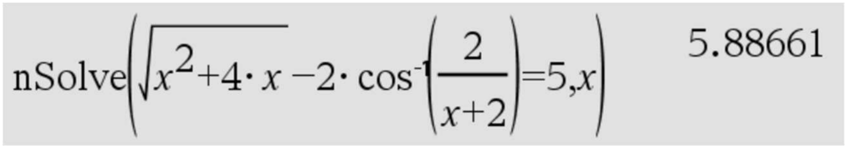
(i) State the relation between n and N. [1]

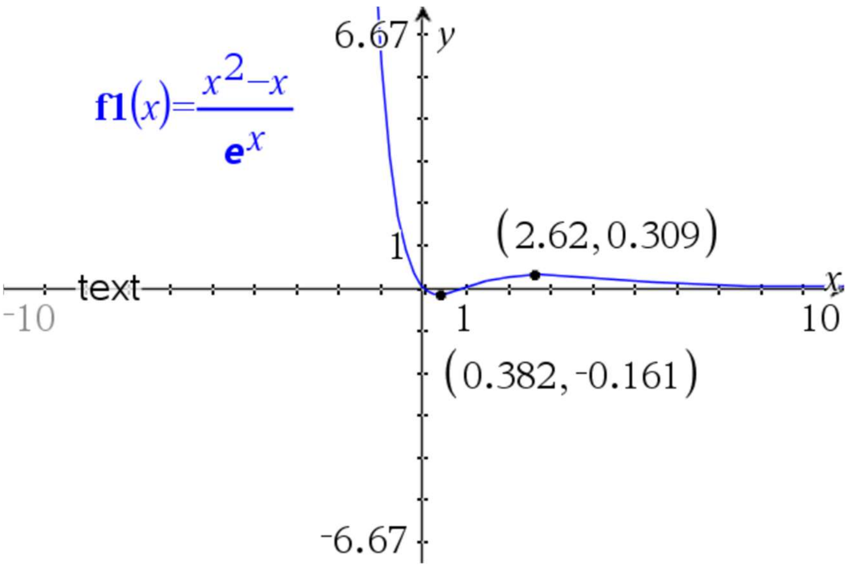
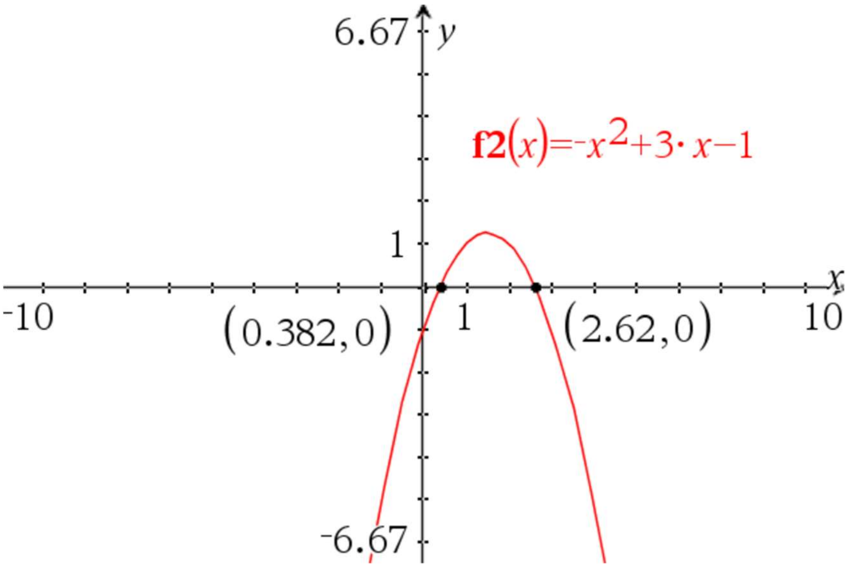
(ii) Find, and simplify, an expression for $\sum_{r=1}^{N}|a_r|$ in terms of n. [3]

(iii) Find, and simplify, an expression for the product $a_1\cdot a_2\cdot a_3\cdot\ldots\cdot a_N$, in terms of n, where $a_i\neq 0$ for $i=1,2,\ldots,N$. [3]

End of Paper

Year 5 HL Maths End of Year Exam 2019 – Paper 2 Mark Scheme

Qn	Suggested solution	Markscheme
	Section A	
1	***Trigonometry – Circular Measure***	**Max mark: 5**
(a)	$\cos A=\dfrac{2}{x+2}$ gives $A=\arccos\left(\dfrac{2}{x+2}\right)$ Area of shaded region = Area of Triangle ABC – Area of Sector ADC $=\frac{1}{2}(2)\left(\sqrt{(x+2)^2-2^2}\right)-\frac{1}{2}\left(2^2\right)\arccos\left(\frac{2}{x+2}\right)$ $=\left(\sqrt{x^2+4x}-2\arccos\left(\frac{2}{x+2}\right)\right)\text{cm}^2$	A1 M1 A1 for Area of sector ADC AG
(b)	Solving the equation $\sqrt{x^2+4x}-2\arccos\left(\frac{2}{x+2}\right)=5$ by GDC using either Graphing App or nSolve gives x = 5.88661… = 5.89 (3 sf)   $\text{nSolve}\left(\sqrt{x^2+4\cdot x}-2\cdot\cos^{-1}\left(\frac{2}{x+2}\right)=5,x\right)$ 5.88661	M1 A1

Qn	**Suggested solution**	**Markscheme**
2	***Differentiation – Strictly Increasing Function and Concavity***	**Max mark: 6**
(a)	**Either** Using GDC to sketch the graph of $y=\frac{x^2-x}{\mathrm{e}^x}$ and finding the coordinates of the maximum and minimum points.	M1
		
	Or $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{e}^x(2x-1)-\mathrm{e}^x(x^2-x)}{(\mathrm{e}^x)^2}=\frac{-x^2+3x-1}{\mathrm{e}^x}$	
	Curve is strictly increasing means $\frac{\mathrm{d}y}{\mathrm{d}x}>0$ i.e. $-x^2+3x-1>0$	M1
	$0.382<x<2.62$ (3 sf)	A1 for correct end values A1 for correct inequality
		

<table>
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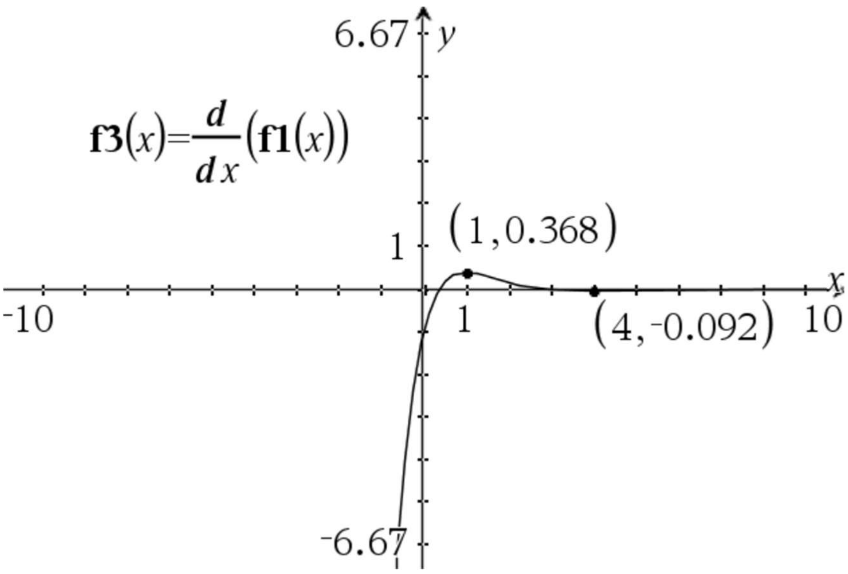
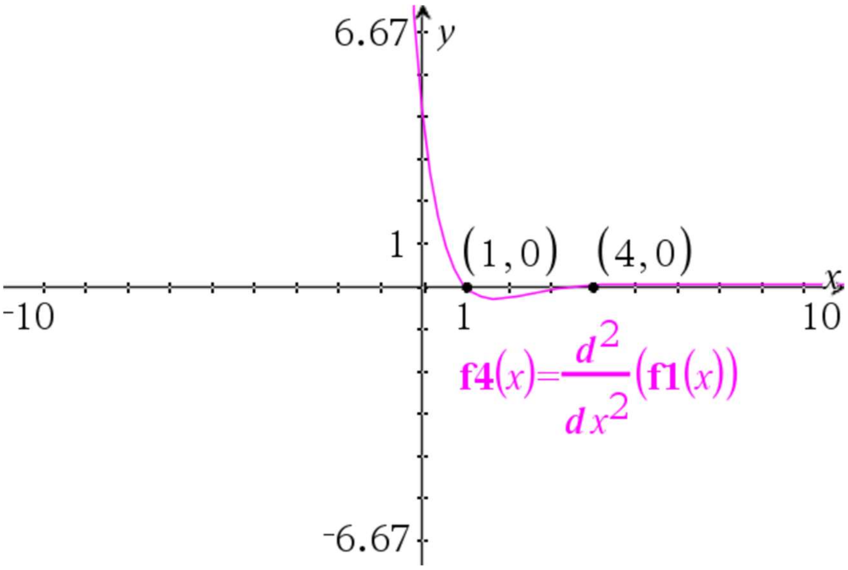
Either

Using GDC to sketch the graph of $y = \frac{\mathrm{d}y}{\mathrm{d}x}$ and finding the coordinates of the maximum and minimum points.



Or

Using GDC to sketch the graph of $y = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and finding the zeros.



Or

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{e}^x(-2x+3) - \mathrm{e}^x(-x^2+3x-1)}{(\mathrm{e}^x)^2} = \frac{x^2-5x+4}{\mathrm{e}^x} = \frac{(x-1)(x-4)}{\mathrm{e}^x}$$

Curve is concave upwards means $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ i.e. $(x-1)(x-4) > 0$

$x < 1$ or $x > 4$

</td>
<td>

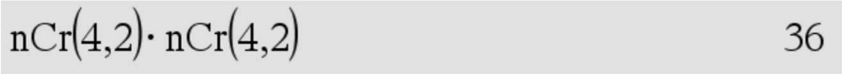
M1

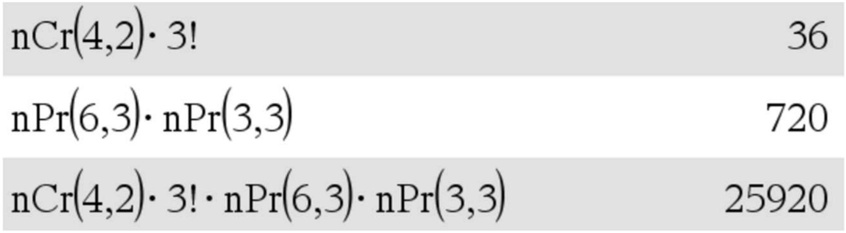
M1

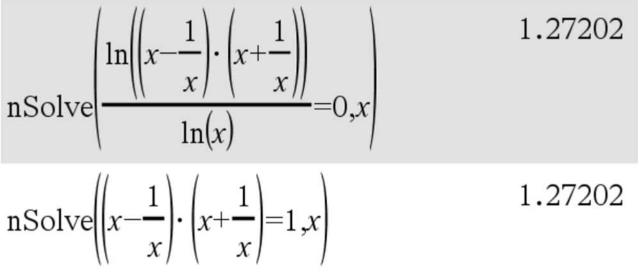
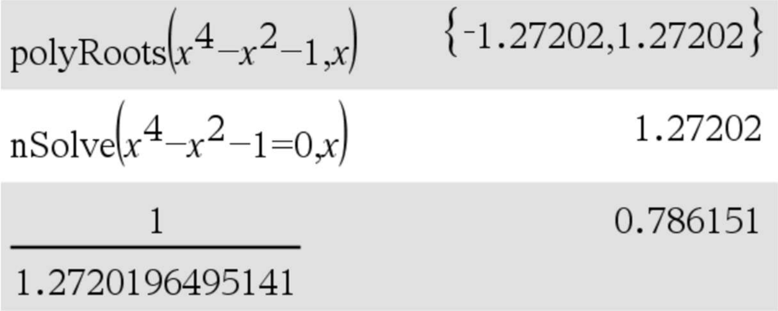
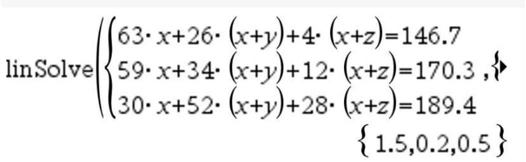
M1

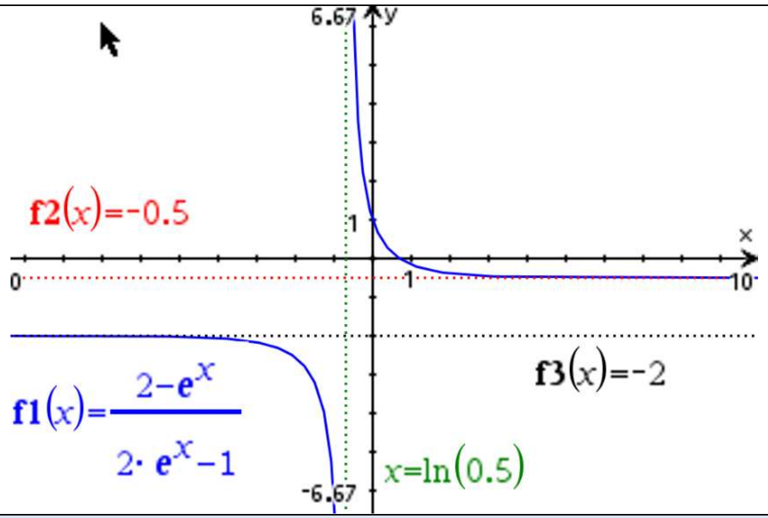
A1 A1

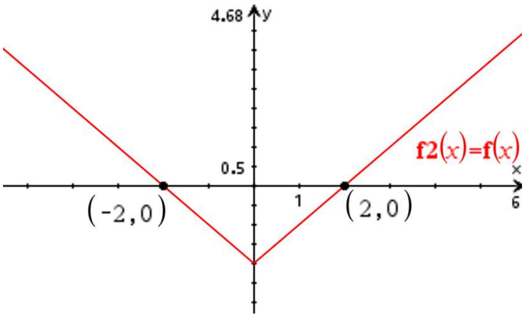
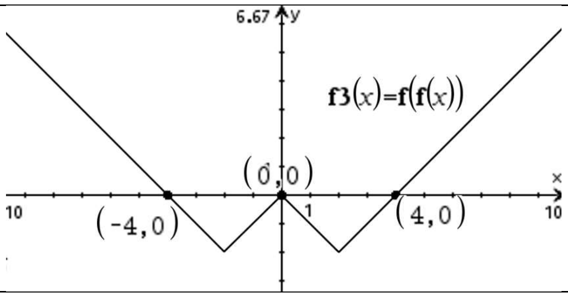
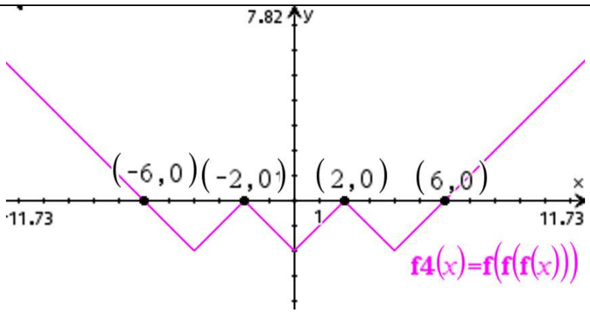
</td>
</tr>
</table>

3	***Polynomials – Fundamental Theorem of Algebra, Sum and Product of Roots with Arithmetic Sequence***	**Max mark: 5**
	Given one complex root = 2 – i By the Fundamental Theorem of Algebra, Complex conjugate root = 2 + i Third real root = 2 $p(x) = (x-2)(x-2-i)(x-2+i)$ $= (x-2)(x^2-4x+5)$ $= x^3-6x^2+13x-10$ Note that GDC can be used to evaluate the sum and product of the complex roots  $2+i+2-i$ 4 $(2+i)\cdot(2-i)$ 5 $2+i+2-i+2$ 6 $(2+i)\cdot(2-i)\cdot 2$ 10	R1 Reason Must be stated A1 A1 M1 A1
4	***Functions – Inverse and Composite Functions (non GDC)***	**Max mark: 8**
(a)	$f^2(x)=f\left(\frac{2x-3}{x-1}\right)=\frac{2\left(\frac{2x-3}{x-1}\right)-3}{\left(\frac{2x-3}{x-1}\right)-1}$ $f^2(x)=\frac{x-3}{x-2}$	M1 A1
(b)	Any valid method of finding inverse $f^{-1}(x)=\frac{x-3}{x-2},\ x\in\mathbb{R},\ x\neq 1,\ x\neq 2$	M1 A1 For Rule A1 For Domain
(c)	From (a) and (b), $f^2(x)=f^{-1}(x)$ is equivalent to $f^3(x)=x$ 2019 is divisible by 3 $f^{2019}(x)=x,\ x\in\mathbb{R},\ x\neq 1,\ x\neq 2$	M1 M1 A1
5	***Permutations and Combinations***	**Max mark: 8**
(a)	Number of possible selections = ${}^4C_2\times{}^4C_2$ or $\binom{4}{2}\times\binom{4}{2}$ $= 36$  nCr(4,2)· nCr(4,2) 36	M1 A1

(b)	Number of numbers formed = ${}^{4}P_{2}\times{}^{6}P_{3}$ or $(4\times3)\times(6\times5\times4)$ $= 1440$  nPr(4,2)· nPr(6,3) 1440	M1 M1 A1
(c)	Number of odd-digit cards for top row $= {}^{4}C_{2}\times3!$ or $\binom{4}{2}\times3!=36$ Number of arrangements for the middle and bottom row $= {}^{6}P_{3}\times{}^{3}P_{3}$ or $6!\ = 720$ Total number of arrangements = $36\times720=25920$  nCr(4,2)· 3! 36 nPr(6,3)· nPr(3,3) 720 nCr(4,2)· 3! · nPr(6,3)· nPr(3,3) 25920	A1 A1 A1
6	***Simultaneous Logarithmic Equations***	**Max mark: 8**
	$\log_x y=\log_y x$ Applying change of base on either side of the equation $\log_x y=\dfrac{\log_x x}{\log_x y}$ gives $(\log_x y)=\pm1$ $y=\dfrac{1}{x}$ or $y=x$ (reject since $x-y>0$) Substituting $y=\dfrac{1}{x}$ into $\log_x(x-y)=\log_y(x+y)$ gives $\log_x\left(x-\dfrac{1}{x}\right)=\log_{\frac{1}{x}}\left(x+\dfrac{1}{x}\right)$ Therefore, $\log_x\left(x-\dfrac{1}{x}\right)=-\log_x\left(x+\dfrac{1}{x}\right)$ $\log_x\left(x-\dfrac{1}{x}\right)+\log_x\left(x+\dfrac{1}{x}\right)=0$ gives $\log_x\left(x-\dfrac{1}{x}\right)\left(x+\dfrac{1}{x}\right)=0$ or $\left(x-\dfrac{1}{x}\right)\left(x+\dfrac{1}{x}\right)=1$ **Either** Solving $\log_x\left(x-\dfrac{1}{x}\right)\left(x+\dfrac{1}{x}\right)=0$ or $\left(x-\dfrac{1}{x}\right)\left(x+\dfrac{1}{x}\right)=1$ by GDC using nSolve gives x = 1.27201… = 1.27 (3 sf) and y = 0.78615… = 0.786 (3 sf)	 M1 A1 Must reject $y=x$ M1 A1 A1 M1 A1 A1

	 $\text{nSolve}\left(\frac{\ln\left(\left(x-\frac{1}{x}\right)\cdot\left(x+\frac{1}{x}\right)\right)}{\ln(x)}=0,x\right)$ 1.27202 $\text{nSolve}\left(\left(x-\frac{1}{x}\right)\cdot\left(x+\frac{1}{x}\right)=1,x\right)$ 1.27202 **Or** Solving $x^4-x^2-1=0$ gives $x^2=\frac{\sqrt{5}+1}{2}$ since $x^2>0$ $x=\sqrt{\frac{\sqrt{5}+1}{2}}$= 1.27 (3sf) since $x>0$ $y=\sqrt{\frac{2}{\sqrt{5}+1}}=0.786$ (3sf) or $y=\sqrt{\frac{\sqrt{5}-1}{2}}=0.786$ (3sf)  $\text{polyRoots}(x^4-x^2-1,x)$ $\{-1.27202,1.27202\}$ $\text{nSolve}(x^4-x^2-1=0,x)$ 1.27202 $\frac{1}{1.2720196495141}$ 0.786151	M1 A1 A1
	Section B	
7	***System of equations/Asymptotes***	**Max mark: 9**
(a)	$63x+26(x+y)+4(x+z)=165.10$ $59x+34(x+y)+12(x+z)=176.15$ $30x+52(x+y)+28(x+z)=205.70$  $\text{linSolve}\left(\begin{cases}63\cdot x+26\cdot(x+y)+4\cdot(x+z)=146.7\\59\cdot x+34\cdot(x+y)+12\cdot(x+z)=170.3\\30\cdot x+52\cdot(x+y)+28\cdot(x+z)=189.4\end{cases},\{\right.$ $\{1.5,0.2,0.5\}$ By GDC, $x=\$1.50,\ y=\$0.20, z=\$0.50$.	M1A1 M1 A1
(b)	$y=\frac{2-e^x}{2e^x-1}=-\frac{1}{2}+\frac{1.5}{2e^x-1}$ $2e^x-1=0\Rightarrow x=\ln\left(\frac{1}{2}\right)$ As $x\to\ln\left(\frac{1}{2}\right), y\to\pm\infty\Rightarrow x=\ln\left(\frac{1}{2}\right)$ is an asymptote. As $x\to\infty, y\to-\frac{1}{2}\Rightarrow y=-\frac{1}{2}$ is an asymptote. As $x\to-\infty, y\to-2\Rightarrow y=-2$ is an asymptote.	M1A1 M1 A1 A1

		
8	***Trigo(Addition Formula), Differentiation, Odd Function***	**Max mark: 16**
(a)	$16\sin^2 B+9\sin^2 C+24\sin B\sin C=36 \;\; -(1)$ $9\cos^2 C+16\cos^2 B+24\cos B\cos C=1 \;\; -(2)$ $(1)+(2),\ 25+24\cos(B-C)=37$ $\Rightarrow \cos(B-C)=\frac{1}{2}$	M1 A1 A1 M1A1 A1
(b) (i)	$h(x)=\arctan(x)+\arctan\left(\frac{1}{x}\right), x\neq 0$ $h'(x)=\frac{1}{1+x^2}+\frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\frac{-1}{x^2}$ $=\frac{1}{1+x^2}+\frac{x^2}{1+x^2}\cdot\frac{-1}{x^2}$ $=0$	M1A1 A1 A1
(ii)	Since $h'(x)=0$ for all $x\neq 0, h(x)=c,$ where c is a constant. $h(x)=h(1)$ $=\arctan(1)+\arctan(1)$ $=\frac{\pi}{4}+\frac{\pi}{4}$ $=\frac{\pi}{2}$	R1 M1 A1
(iii)	$h(x)=\arctan(x)+\arctan\left(\frac{1}{x}\right)$ $h(-x)=\arctan(-x)+\arctan\left(-\frac{1}{x}\right)$ $=-\arctan(x)-\arctan\left(\frac{1}{x}\right)$ $(\because \arctan(x)$ is an odd function.$)$ $=-h(x)$	M1 R1
	For $x<0,\ h(x)=-\frac{\pi}{2}$.	A1

9	***Function, Composite functions, AP, Factorial***	**Max mark: 15**
(a) (i)	$f(x)=\lvert x\rvert-2$ 	G1
(ii)	Zeros of $f=\pm 2$	A1
(b) (i)		G2
(ii)	Zeros of $f\circ f$ are $\pm 4,\ 0$.	A1
(c) (i)	 Zeros of f^3 are $\pm 6, \pm 2$.	A1
(ii)	Zeros of f^4 are $\pm 8, \pm 4, 0$	A1
(iii)	Zeros of f^8 are $\pm 16, \pm 12, \pm 8, \pm 4, 0$	A1
(d) (i)	$N=2n+1$	A1
(ii)	$\sum_{r=1}^{N}\lvert a_r\rvert=2(4+8+12+16+...+4n)$ $=8(1+2+3+...+n)$ $=8\cdot\frac{n}{2}(1+n)$ $=4n(1+n)$	A1 M1(AP sum) A1
(iii)	$a_1\cdot a_2\cdot a_3\cdot\ldots\cdot a_N=(-1)^n(4\cdot 8\cdot 12\cdot 16\cdots 4n)^2$ $=(-1)^n 4(1\cdot 2\cdot 3\cdot 4\cdots n)^2$ $=(-1)^n 4(n!)^2$	M1 A1 A1

STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	2	0	-	7	2	0	1

ST. JOSEPH'S INSTITUTION
YEAR 5 END OF YEAR EXAMINATION 2020

MATHEMATICS: ANALYSIS AND APPROACHES
HIGHER LEVEL
PAPER 1
Thursday

8 October 2020
1 hr 30 mins
0800 – 0930 hrs

INSTRUCTIONS TO CANDIDATES

- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the foolscap paper provided.
- The use of calculators is **not** permitted in this paper.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures.
- The maximum mark for this examination paper is [85 marks].
- This question paper consists of **9** printed pages including the Cover Sheet.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	TOTAL
									/85

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (40 marks)

Answer ***all*** *question in the* ***spaces*** *provided.*

1. [**Maximum mark: 5**]

(a) Solve the inequality $x^2 + x - 20 \geq 0$. [**2**]

(b) Hence, by means of a suitable substitution, solve the inequality $x^2 + |x| - 20 \geq 0$. [**3**]

Solution:

(a)

$$x^2 + x - 20 \geq 0$$

$$(x+5)(x-4) \geq 0 \qquad \textbf{M1}$$

$$x \leq -5 \text{ or } x \geq 4 \qquad \textbf{A1}$$

(b) Replace x by $|x|$ to get **A1**

$$|x| \leq -5 \text{ or } |x| \geq 4 \qquad \textbf{M1}$$

$$\text{rejected} \qquad x \geq 4 \text{ or } x \leq -4 \qquad \textbf{A1}$$

2. [**Maximum mark: 4**]

A function f is defined by

$$f(x) = x^2 + x, \quad x \leq -\frac{1}{2}.$$

(a) Find an expression for f^{-1}. [**3**]

(b) Find the range of f^{-1}. [**1**]

Solution:

(a)

$$\begin{aligned} y &= x^2 + x \\ y &= \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} \\ \left(x + \frac{1}{2}\right)^2 &= y + \frac{1}{4} \\ x + \frac{1}{2} &= \pm\sqrt{y + \frac{1}{4}} \\ x &= -\frac{1}{2} \pm \sqrt{y + \frac{1}{4}} \end{aligned}$$

M1

Since $x \leq -\frac{1}{2}$, we reject the positive square root. **R1**

Hence $f^{-1} = -\frac{1}{2} - \sqrt{x + \frac{1}{4}}$ **A1**

(b) $\mathrm{R}_{f^{-1}} = \mathrm{D}_f = \left(-\infty, -\frac{1}{2}\right]$. **A1**

3. [**Maximum mark: 6**]

(a) Expand $\dfrac{1}{(1-3x)^2}$ in ascending powers of x up to the term in x^3. [**2**]

(b) Hence, expand $\left(\dfrac{1-x}{1-3x}\right)^2$ in ascending powers of x up to the term in x^2. [**3**]

(c) State the range of values of x such that the expansions above are valid. [**1**]

Solution:

(a) Using Binomial expansion,

$$\begin{aligned}(1-3x)^{-2} &= 1+(-2)(-3x)+\frac{(-2)(-3)}{2!}(-3x)^2+\frac{(-2)(-3)(-4)}{3!}(-3x)^3+\cdots && \textbf{M1}\\ &= 1+6x+27x^2+108x^3+\cdots && \textbf{A1}\end{aligned}$$

(b) Hence

$$\begin{aligned}\left(\frac{1-x}{1-3x}\right)^2 &= (1-x)^2(1-3x)^{-2} && \textbf{M1}\\ &= (1-2x+x^2)(1+6x+27x^2+\cdots) && \textbf{A1 for } 1-2x+x^2\\ &= 1+4x+16x^2+\cdots && \textbf{A1}\end{aligned}$$

(c) The expansion is valid for $|x| < \frac{1}{3}$. **A1**

4. [**Maximum mark: 8**]

A function $g(x)$ is given by the rule $g(x) = \dfrac{x^2+6}{3-x^2}$.

(a) State the maximal domain of g. [**2**]

(b) Find the equations of the asymptotes of $y = g(x)$. [**4**]

(c) Describe a single transformation that maps the graph of $y = g(x)$ onto the graph of $y = h(x)$, where $h(x) = \dfrac{x^2-2x+7}{2+2x-x^2}$. [**2**]

Solution:

(a) For $g(x) = \dfrac{x^2+6}{3-x^2}$ to be defined, we need $3 - x^2 \neq 0$. **(M1)**

Hence the maximal domain of g is $\{x \in \mathbb{R} \mid x \neq \pm\sqrt{3}\}$ or $\mathbb{R} \setminus \{\pm\sqrt{3}\}$. **A1**

(b) $g(x) = \dfrac{x^2+6}{3-x^2} = -1 + \dfrac{9}{3-x^2}$ **(M1)**

The asymptotes are
$y = -1,$ **A1**
$x = \sqrt{3},$ **A1**
$x = -\sqrt{3}.$ **A1**

(c) $h(x) = \dfrac{x^2-2x+7}{2+2x-x^2} = \dfrac{(x-1)^2+6}{3-(x-1)^2}$ **M1**

Hence the transformation needed is a translation by $\begin{pmatrix}1\\0\end{pmatrix}$. **A1**

5. [**Maximum mark: 8**]

The r^{th} term of a series is given by the expression $U_r = 2^{r+2} - r(r-2)$, where $r \in \mathbb{Z}^+$.

Given that

$$\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1),$$

show that

$$\sum_{r=1}^{n} U_r = 8(2^n - 1) - \frac{n}{6}(n+1)(2n-5).$$

Hence or otherwise, find

$$\sum_{r=1}^{n} \left(2^r - \left(\frac{r}{2}\right)\left(\frac{r}{2} - 1\right)\right)$$

giving your answer in terms of n.

Solution:

$$S_n = \sum_{r=1}^{n} \left(2^{r+2} - r(r-2)\right)$$

$$= \sum_{r=1}^{n} 2^{r+2} - \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 2r$$

M1 - split $\sum$

$$= \frac{2^3(1-2^n)}{1-2} - \frac{n}{6}(n+1)(2n+1) + \frac{n}{2}(2+2n)$$

M1 - use of AP, GP or given sum formula

A1 - correct sum of GP, **A1** - correct sum of AP

$$= 8(-1+2^n) - \frac{n(n+1)}{6}(2n+1-6)$$

A1

$$= 8(2^n - 1) - \frac{n(n+1)}{6}(2n-5) \quad \text{(shown.)}$$

AG

$$\sum_{r=1}^{n} \left(2^r - \left(\frac{r}{2}\right)\left(\frac{r}{2} - 1\right)\right) = \sum_{r=1}^{n} \left[\left(\frac{1}{4}\right)(2^{r+2}) - \left(\frac{1}{2} \cdot r\right)\left(\frac{1}{2}\right)(r-2)\right]$$

M1

$$= \frac{1}{4}\sum_{r=1}^{n} U_r$$

A1 or the next line directly

$$= \left(\frac{1}{4}\right)\left(8(2^n - 1) - \frac{n}{6}(n+1)(2n-5)\right)$$

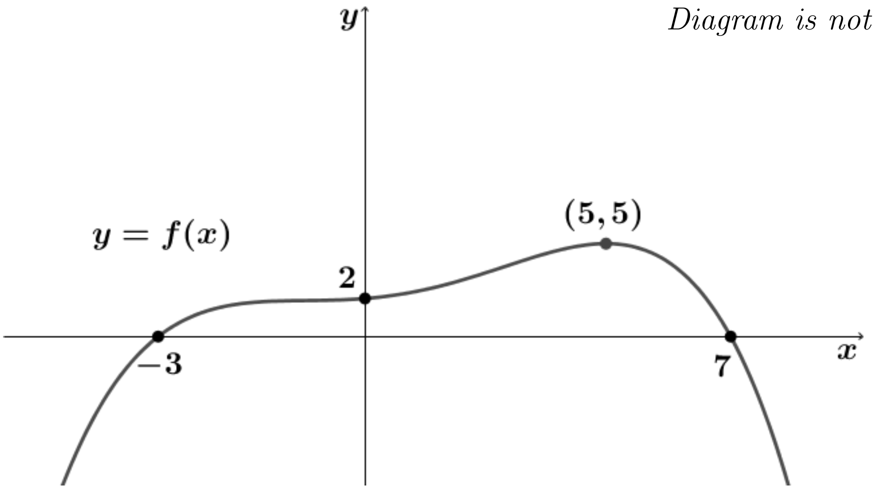
$$= 2(2^n - 1) - \frac{n}{24}(n+1)(2n-5)$$

A1

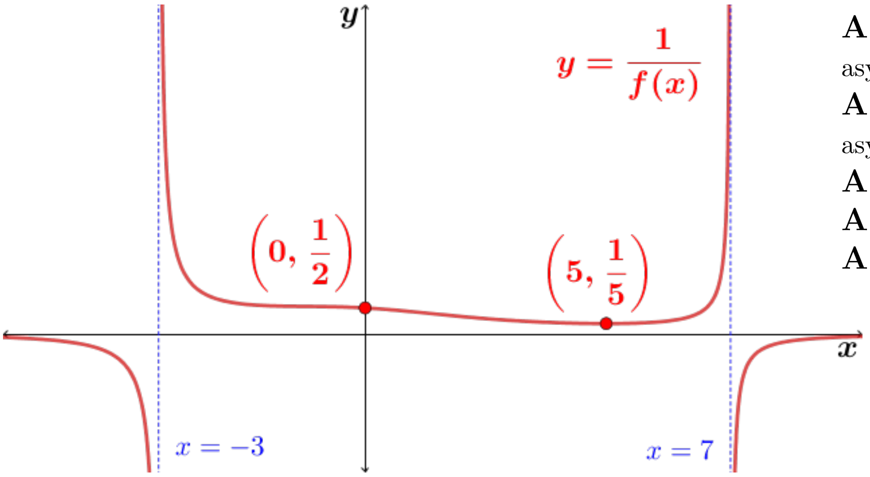
6. [**Maximum mark: 9**]

The graph of a polynomial function $y = f(x)$ is shown below. The curve cuts the $x-$axis at $x = -3$ and $x = 7$, cuts the $y-$axis at $y = 2$, and has a maximum point at $(5, 5)$.

Diagram is not drawn to scale

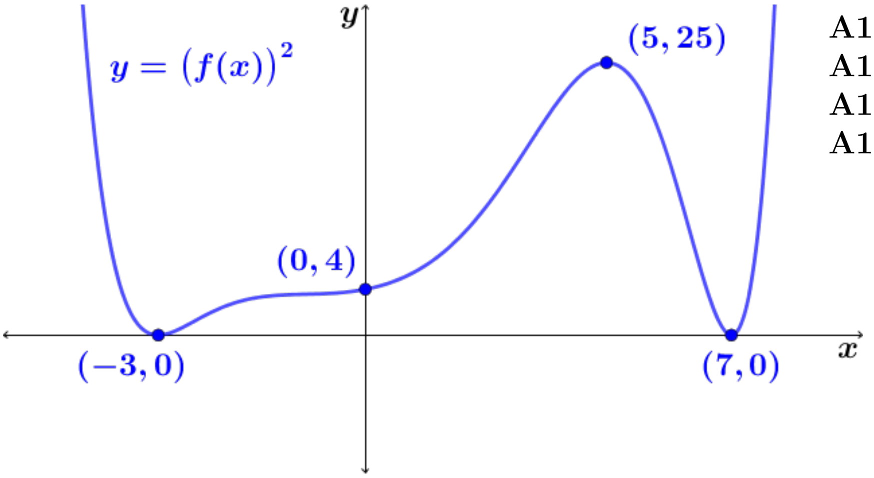


(a) On the set of axes below, sketch the graph of $y = \dfrac{1}{f(x)}$, labelling its asymptotes, turning points, and axial intercepts clearly. [**5**]



A1 - vertical asymptotes
A1 - horizontal asymptote
A1 - $y-$intercept
A1 - point $(5, \frac{1}{5})$
A1 - shape

(b) On the set of axes below, sketch the graph of $y = (f(x))^2$, labelling its turning points and axial intercepts clearly. [**4**]



A1 - shape
A1 - x-intercepts
A1 - point $(0, 4)$
A1 - point $(5, 25)$

Turn Over

*Do **NOT** write solutions on this page.*

SECTION B (45 marks)

*Answer **all** questions on the foolscap paper provided. **Please start each question on a new page.***

7. [**Maximum mark: 10**]

(a) A boy claims to have found a function that only produces prime numbers. He then writes the function as

$$\pi(n) = n^2 + n + 5, \quad n \in \mathbb{Z}^+.$$

Prove or disprove his claim. [**3**]

(b) A girl claims to have discovered a formula that reads

$$\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}, \quad n \in \mathbb{Z}^+.$$

Prove her discovery using mathematical induction. [**7**]

8. [**Maximum mark: 18**]

(a) Show that $5x^2 + nx - 1 = 0$ has 2 real roots for all real values of n. [**2**]

Let $P(x) = (x^2 - 6x + m^2)(5x^2 + nx - 1)$ for some real constants m and n.

(b) Find the range of values of m for which $P(x) = 0$ yields only real roots. [**4**]

(c) If the sum of the roots of $P(x) = 0$ is 10 and one of the roots is $3 - \sqrt{7}\,\mathrm{i}$, where $\mathrm{i}^2 = -1$,

(i) find the value of n; and,

(ii) find the possible values of m. [**6**]

(d) Find the remainder when $P(x)$ is divided by $2x - 1$, leaving your answer in terms of m and n. [**2**]

(e) Solve the inequality $P(x) < 0$ for $m = n = 3$. [**4**]

Do ***NOT*** *write solutions on this page.*

9. [**Maximum mark: 17**]

Suppose $z = \dfrac{4\sqrt{2}\,\mathrm{i}}{\sqrt{6}+\sqrt{2}\,\mathrm{i}}$, where $\mathrm{i}^2 = -1$.

(a) Show that $z = 1 + \sqrt{3}\,\mathrm{i}$. [**2**]

(b) Hence, find in terms of n, [**4**]

(i) $\arg(z^n)$

(ii) $|z^n|$

(c) If $w^2 = z$, find the possible values of w, leaving your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r > 0$ and $\theta \in (-\pi, \pi]$. [**4**]

Let $z^n = r\big(f(n) + \mathrm{i}\,g(n)\big)$ for all $n \in \mathbb{Z}^+$, where f and g are trigonometric functions of n and $r = |z^n|$.

(d) Find $f(1), f(2)$ and $f(3)$. [**4**]

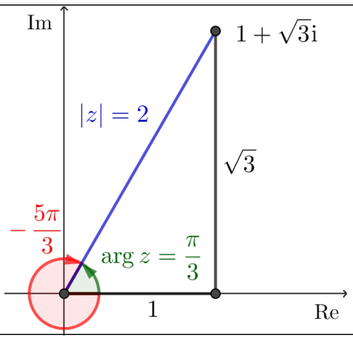
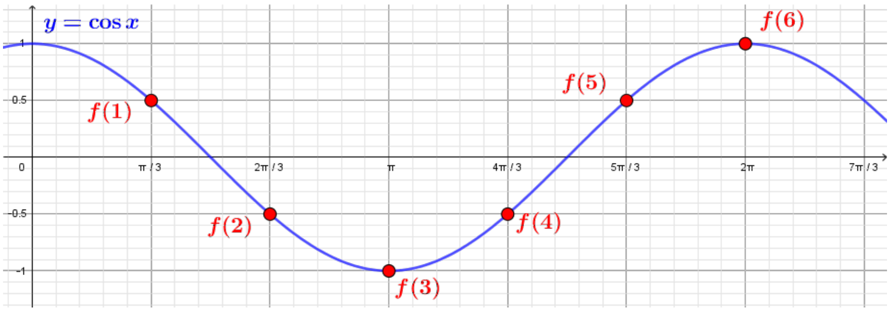
(e) Calculate the product of the first 12 values of $f(n)$, that is, evaluate [**3**]

$$f(1) \times f(2) \times f(3) \times \cdots \times f(11) \times f(12).$$

End of Paper

7. Proof by Counterexample, Mathematical Induction	Maximum mark: 10
(a)	
$\pi(4) = 4^2 + 4 + 5 = 25$ is clearly not prime.*	M1R1 - substitution to disprove claim
Therefore, the claim is not true.	A1
*Any valid counterexample is acceptable.	
(b)	
Let $P(n)$ be the statement $\displaystyle\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$ for any $n \in \mathbb{Z}^+$.	
For $n = 1$: $\displaystyle \text{LHS} = \sum_{r=1}^{1} \frac{r}{(r+1)!} = \frac{1}{(1+1)!} = \frac{1}{2}$ $\displaystyle \text{RHS} = 1 - \frac{1}{(1+1)!} = 1 - \frac{1}{2} = \frac{1}{2}$	A1 (A0 for bad presentation)
Therefore, $P(1)$ is true.	
Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$, i.e., $\displaystyle\sum_{r=1}^{k} \frac{r}{(r+1)!} = 1 - \frac{1}{(k+1)!}$	M1
For $n = k+1$: $\displaystyle\sum_{r=1}^{k+1} \frac{r}{(r+1)!} = \sum_{r=1}^{k} \frac{r}{(r+1)!} + \frac{k+1}{\big((k+1)+1\big)!}$	M1 - correct sum
$\displaystyle = 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$	A1 - use of $P(k)$
$\displaystyle = 1 - \frac{k+2}{(k+2)!} + \frac{k+1}{(k+2)!}$	
$\displaystyle = 1 - \frac{k+2-(k+1)}{(k+2)!}$	M1 - single fraction
$\displaystyle = 1 - \frac{1}{(k+2)!}$	A1
$\displaystyle = 1 - \frac{1}{\big((k+1)+1\big)!}$	
Therefore, $P(k+1)$ is true.	
Since $P(1)$ is true and $P(k+1)$ is also true whenever $P(k)$ is true for any $k \in \mathbb{Z}^+$, then by mathematical induction $P(n)$ is true for all $n \in \mathbb{Z}^+$.	A1 - only if all are correct

8. Polynomials, Sum and Product, Remainder Theorem	Maximum mark: 18
(a)	
$y = 5x^2 + nx - 1$ has discriminant $\Delta_n = n^2 - 4(5)(-1) = n^2 + 20$	A1 - Δ
which is always positive no matter what n is.	R1 (tolerate $n^2 > 0$)
Thus, $5x^2 + nx - 1 = 0$ always yields 2 real roots whatever n is.	AG
(b)	
Let $\Delta_m = 36 - 4m^2$ be the discriminant of the factor $x^2 - 6x + m^2$.	A1
P yields no complex roots if and only if $\Delta_m \geq 0$. (as $\Delta_n > 0$ for all n.)	R1
$\Delta_m = 36 - 4m^2 \geq 0 \Longrightarrow m^2 \leq 9 \Longrightarrow -3 \leq m \leq 3$	M1A1
(c)(i)	
The sum of the roots is 10 means $6 - \dfrac{n}{5} = 10 \Longrightarrow n = -20$.	M1A1
(c)(ii)	
Since $3 - \sqrt{7}\,\mathrm{i}$ is a root, so is $3 + \sqrt{7}\,\mathrm{i}$ as $\underline{\text{all coefficients are real}}$.	A1$\underline{\text{R1}}$ - real coeff.
Thus, $m^2 = (3 - \sqrt{7}\,\mathrm{i})(3 + \sqrt{7}\,\mathrm{i}) = 16$	M1
Therefore, $m = \pm 4$.	A1
(d)	
The remainder is given by $P\left(\frac{1}{2}\right)$.	M1 - seen anywhere
$P\left(\frac{1}{2}\right) = \left(\left(\frac{1}{2}\right)^2 - 6\left(\frac{1}{2}\right) + m^2\right)\left(5\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)n - 1\right)$	
$= \left(m^2 - \dfrac{11}{4}\right)\left(\dfrac{n}{2} + \dfrac{1}{4}\right)$	A1
(e)	
For $m = n = 3$,	
$P(x) = (x^2 - 6x + 9)(5x^2 + 3x - 1) = (x - 3)^2(5x^2 + 3x - 1)$.	M1
$P(x) < 0 \Longrightarrow \underline{5x^2 + 3x - 1 < 0}$ and $x \neq 3$	$\underline{\text{A1}}$ - award even if no “$x \neq 3$”
$\Longrightarrow \dfrac{-3 - \sqrt{29}}{10} < x < \dfrac{-3 + \sqrt{29}}{10}$	M1A1

9. Complex Numbers		Maximum mark: 17
(a) $z=\dfrac{4\sqrt{2}\,\mathrm{i}}{\sqrt{6}+\sqrt{2}\,\mathrm{i}}\times\dfrac{\sqrt{6}-\sqrt{2}\,\mathrm{i}}{\sqrt{6}-\sqrt{2}\,\mathrm{i}}=\dfrac{8\sqrt{3}\,\mathrm{i}+8}{6+2}=1+\sqrt{3}\,\mathrm{i}$		M1A1
(b) (i) $\arg z^n=n\times\arg z=\dfrac{n\pi}{3}$		(M1)A1
(ii) $\lvert z^n\rvert=\lvert z\rvert^n=2^n$		(M1)A1
(c) Method 1	(c) Method 2	
Let $w=r\mathrm{e}^{\mathrm{i}\,\theta}$ so that $w^2=r^2\mathrm{e}^{\mathrm{i}\,2\theta}$.	Let $w=a+ib$ so that	
As $w^2=z\implies r^2\mathrm{e}^{\mathrm{i}\,2\theta}=1+\sqrt{3}\,\mathrm{i}$	$w^2=a^2-b^2+2ab\,\mathrm{i}=1+\sqrt{3}\,\mathrm{i}$	
$=2\mathrm{e}^{\mathrm{i}\,\pi/3}=2\mathrm{e}^{-\,\mathrm{i}\,5\pi/3}$ (as above)	$a^2-b^2=1$ and $2ab=\sqrt{3}$	M1
Thus, $r^2=2$ and $2\theta=\dfrac{\pi}{3}$ or $-\dfrac{5\pi}{3}$	$\implies 4b^4+4b^2-3=0$	
So $r=\sqrt{2}$ and $\theta=\dfrac{\pi}{6}$ or $-\dfrac{5\pi}{6}$	$\implies (2b^2-1)(2b^2+3)=0$	A1
Therefore, $w=\sqrt{2}\mathrm{e}^{\mathrm{i}\,\pi/6}$	$\implies b=\pm\dfrac{1}{\sqrt{2}}$ and $a=\pm\dfrac{\sqrt{3}}{\sqrt{2}}$	A1
or $w=\sqrt{2}\mathrm{e}^{-\,\mathrm{i}\,5\pi/6}$.	$w=\sqrt{2}\mathrm{e}^{\mathrm{i}\,\pi/6}$ or $\sqrt{2}\mathrm{e}^{-\,\mathrm{i}\,5\pi/6}$	A1
(d) $z^n=2^n\mathrm{cis}\left(\dfrac{n\pi}{3}\right)=r\left(\cos\left(\dfrac{n\pi}{3}\right)+\mathrm{i}\sin\left(\dfrac{n\pi}{3}\right)\right)$		(M1A1) - de Moivre's
		
$f(1)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$		A1
$f(2)=\cos\left(\frac{2\pi}{3}\right)=-\frac{1}{2}$ and $f(3)=\cos\left(\frac{3\pi}{3}\right)=-1$.		A1
(e) The first six values of f are $\frac{1}{2}$, $-\frac{1}{2}$, -1, $-\frac{1}{2}$, $\frac{1}{2}$, 1. (See graph above.)		M1
The product of every six terms is $-\frac{1}{16}$.		A1 - seen anywhere
Thus, $f(1)\times\cdots\times f(12)=\big(f(1)\times\cdots\times f(6)\big)^2=\dfrac{1}{(-16)^2}=\dfrac{1}{256}$.		A1

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

STUDENT NAME: ____________________

EXAMINATION CODE

8	8	2	0	-	7	2	0	1

TEACHER NAME: ____________________

ST. JOSEPH'S INSTITUTION
YEAR 5 END OF YEAR EXAMINATION 2020

MATHEMATICS: ANALYSIS AND APPROACHES
HIGHER LEVEL
PAPER 1
Thursday

8 October 2020
1 hr 30 mins
0800 – 0930 hrs

INSTRUCTIONS TO CANDIDATES

- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the foolscap paper provided.
- The use of calculators is **not** permitted in this paper.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures.
- The maximum mark for this examination paper is [85 marks].
- This question paper consists of **9** printed pages including the Cover Sheet.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	TOTAL
									/85

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (40 marks)

Answer ***all*** *question in the* ***spaces*** *provided.*

1. [**Maximum mark: 5**]

(a) Solve the inequality $x^2 + x - 20 \geq 0$. [**2**]

(b) Hence, by means of a suitable substitution, solve the inequality $x^2 + |x| - 20 \geq 0$. [**3**]

2. [**Maximum mark: 4**]

A function f is defined by

$$f(x) = x^2 + x, \quad x \leq -\frac{1}{2}.$$

(a) Find an expression for f^{-1}. [**3**]

(b) Find the range of f^{-1}. [**1**]

3. [**Maximum mark: 6**]

(a) Expand $\dfrac{1}{(1-3x)^2}$ in ascending powers of x up to the term in x^3. [**2**]

(b) Hence, expand $\left(\dfrac{1-x}{1-3x}\right)^2$ in ascending powers of x up to the term in x^2. [**3**]

(c) State the range of values of x such that the expansions above are valid. [**1**]

4. [**Maximum mark: 8**]

A function $g(x)$ is given by the rule $g(x) = \dfrac{x^2+6}{3-x^2}$.

(a) State the maximal domain of g. [**2**]

(b) Find the equations of the asymptotes of $y = g(x)$. [**4**]

(c) Describe a single transformation that maps the graph of $y = g(x)$ onto the graph of $y = h(x)$, where $h(x) = \dfrac{x^2-2x+7}{2+2x-x^2}$. [**2**]

5. **[Maximum mark: 8]**

The r^{th} term of a series is given by the expression $U_r = 2^{r+2} - r(r-2)$, where $r \in \mathbb{Z}^+$.

Given that

$$\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1),$$

show that

$$\sum_{r=1}^{n} U_r = 8(2^n - 1) - \frac{n}{6}(n+1)(2n-5).$$

Hence or otherwise, find

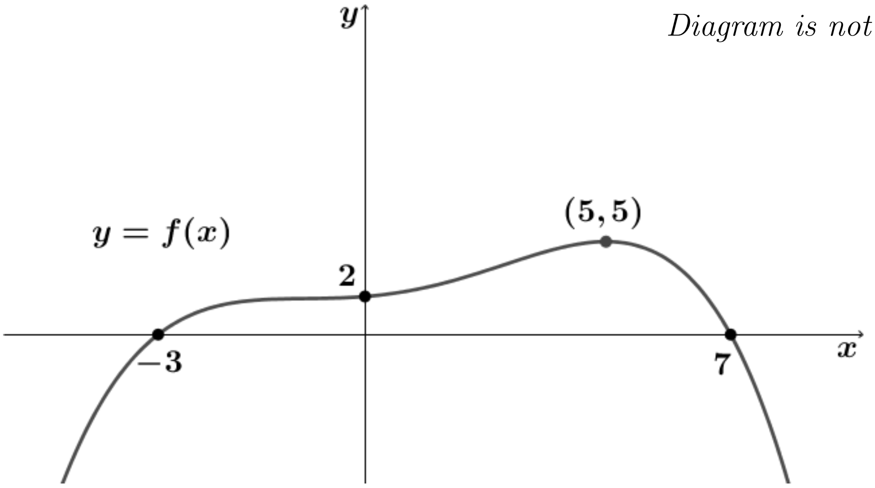
$$\sum_{r=1}^{n} \left(2^r - \left(\frac{r}{2}\right)\left(\frac{r}{2} - 1\right)\right)$$

giving your answer in terms of n.

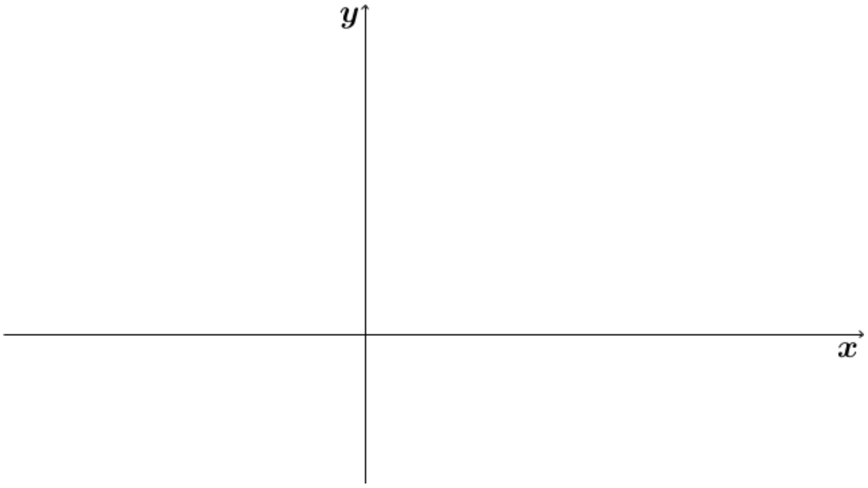
6. [**Maximum mark: 9**]

The graph of a polynomial function $y = f(x)$ is shown below. The curve cuts the x−axis at $x = -3$ and $x = 7$, cuts the y−axis at $y = 2$, and has a maximum point at $(5, 5)$.

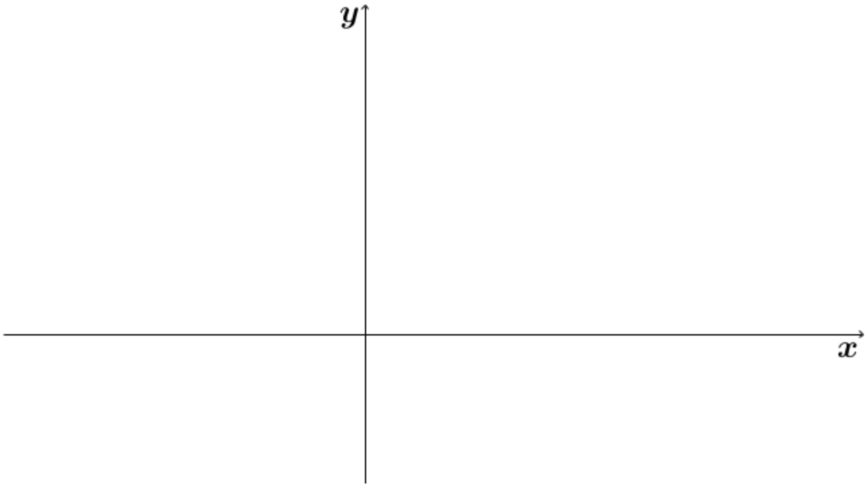
Diagram is not drawn to scale



(a) On the set of axes below, sketch the graph of $y = \dfrac{1}{f(x)}$, labelling its asymptotes, turning points, and axial intercepts clearly. [**5**]



(b) On the set of axes below, sketch the graph of $y = (f(x))^2$, labelling its turning points and axial intercepts clearly. [**4**]



*Do **NOT** write solutions on this page.*

SECTION B (45 marks)

*Answer **all** questions on the foolscap paper provided.* ***Please start each question on a new page.***

7. [**Maximum mark: 10**]

(a) A boy claims to have found a function that only produces prime numbers. He then writes the function as

$$\pi(n) = n^2 + n + 5, \quad n \in \mathbb{Z}^+.$$

Prove or disprove his claim. [**3**]

(b) A girl claims to have discovered a formula that reads

$$\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}, \quad n \in \mathbb{Z}^+.$$

Prove her discovery using mathematical induction. [**7**]

8. [**Maximum mark: 18**]

(a) Show that $5x^2 + nx - 1 = 0$ has 2 real roots for all real values of n. [**2**]

Let $P(x) = (x^2 - 6x + m^2)(5x^2 + nx - 1)$ for some real constants m and n.

(b) Find the range of values of m for which $P(x) = 0$ yields only real roots. [**4**]

(c) If the sum of the roots of $P(x) = 0$ is 10 and one of the roots is $3 - \sqrt{7}\,\mathrm{i}$, where $\mathrm{i}^2 = -1$,

(i) find the value of n; and,

(ii) find the possible values of m. [**6**]

(d) Find the remainder when $P(x)$ is divided by $2x - 1$, leaving your answer in terms of m and n. [**2**]

(e) Solve the inequality $P(x) < 0$ for $m = n = 3$. [**4**]

Do ***NOT*** *write solutions on this page.*

9. [**Maximum mark: 17**]

Suppose $z = \dfrac{4\sqrt{2}\,\mathrm{i}}{\sqrt{6}+\sqrt{2}\,\mathrm{i}}$, where $\mathrm{i}^2 = -1$.

(a) Show that $z = 1 + \sqrt{3}\,\mathrm{i}$. [**2**]

(b) Hence, find in terms of n, [**4**]

(i) $\arg(z^n)$

(ii) $|z^n|$

(c) If $w^2 = z$, find the possible values of w, leaving your answer in the form $re^{\mathrm{i}\theta}$ where $r > 0$ and $\theta \in (-\pi, \pi]$. [**4**]

Let $z^n = r\big(f(n) + \mathrm{i}\,g(n)\big)$ for all $n \in \mathbb{Z}^+$, where f and g are trigonometric functions of n and $r = |z^n|$.

(d) Find $f(1), f(2)$ and $f(3)$. [**4**]

(e) Calculate the product of the first 12 values of $f(n)$, that is, evaluate [**3**]

$$f(1) \times f(2) \times f(3) \times \cdots \times f(11) \times f(12).$$

End of Paper

STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	2	0	-	7	2	0	2

ORA et LABORA

ST JOSEPH'S INSTITUTION
YEAR 5 END OF YEAR EXAMINATION 2020

MATHEMATICS: ANALYSIS AND APPROACHES
HIGHER LEVEL
PAPER 2
Thursday

15 October 2020
1 hr 30 mins
0800 – 0930 hrs

INSTRUCTIONS TO CANDIDATES

- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the writing paper provided
- The use of a scientific or examination graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted apps and be RAM cleared.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is [85 marks].
- This question paper consists of **11** printed pages including the Cover Sheet.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	TOTAL
									/85

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to *show all working.*

SECTION A (40 marks)

Answer **all** questions in the **spaces** provided.

1 [Maximum mark: 4]

Find exactly, the range of values of k, $k \in \mathbb{R}$, for which the following equation has real roots

$$kx^2 + (k-1)x + (k-1) = 0\,.$$

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2 **[Maximum mark: 6]**

Solve the following system of equations

$$\log_{x-1} y = 3,$$
$$\log_{y+1} x = \frac{1}{2}.$$

TURN OVER

3 **[Maximum mark: 7]**

Farmer Adam has a fenced rectangular field, measuring 10 metres by 4 metres, for his goat, Billy, to graze. To keep Billy from running away, Adam attaches a rope to a wooden post at one corner of his field, and the other end of the rope to Billy. The length of Billy's rope, x metres, where $4 < x < 10$, is such that Billy can graze an area that is exactly $\frac{3}{4}$ of Adam's field.

(a) Show that x satisfies the equation

$$x^2 \arcsin\left(\frac{4}{x}\right) + 4\sqrt{x^2 - 16} = 60\,.$$ [5]

(b) Find the value of x. [2]

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4 [Maximum mark: 7]

Let f and g be the following functions

$$f(x) = x^4 + 0.2x^3 - 5.8x^2 - x + 4, \quad 0 \le x \le p,$$

$$g(x) = -3 + 2\sin(x-1), \quad -\frac{\pi}{2} + 1 \le x \le \frac{\pi}{2} + 1.$$

(a) Find the largest value of p in 3 significant figures for which f has an inverse. [2]

(b) Solve the equation $f^{-1}(x) = 1$. [2]

(c) Hence, solve $(f^{-1} \circ g)(x) = 1$. [3]

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5 [Maximum mark: 7]

The arithmetic sequence $\{u_n\}, n \in \mathbb{Z}^+$ has first term $u_1 = 1.6$ and common difference $d = 1.5$. The geometric sequence $\{v_n\}, n \in \mathbb{Z}^+$ has first term $v_1 = 3$ and common ratio $r = 1.2$.

(a) Find an expression for $u_n - v_n$ in terms of *n*. [2]

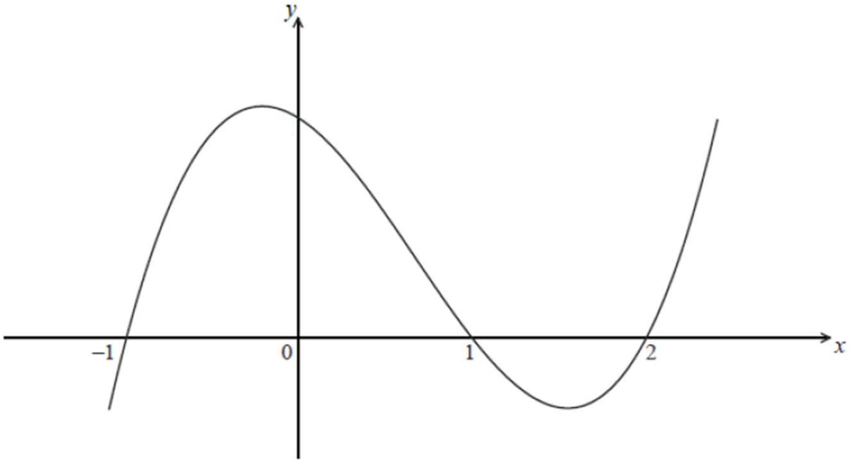
(b) List all the possible values of *n* for which $u_n > v_n$. [3]

(c) Find the greatest value of $u_n - v_n$, giving your answer to 4 significant figures. [2]

TURN OVER

6 [Maximum mark: 9]

The diagram shows the graph of $y = f\left(x\right) = x^3 + ax^2 + bx + c,\ x \in \mathbb{R}$, where a, b, and c are real constants. The graph cuts the x-axis at the points $\left(-1,0\right)$, $\left(1,0\right)$ and $\left(2,0\right)$.



(a) Find the value of a, of b, and of c. [3]

(b) The graph of $y = f\left(x\right)$ is translated by $\begin{pmatrix} m \\ 0 \end{pmatrix}$ to the graph of $y = g\left(x\right)$ such that the y-intercept of $y = g\left(x\right)$ is also its **minimum** turning point. Find the value of m. [2]

(c) Using the value of m to 3 significant figures, sketch the graph of $y = g\left(-\left|x\right|\right)$, labelling its x-intercepts. [4]

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MORE SPACE IS AVAILABLE ON THE NEXT PAGE

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Do NOT write solutions on this page.

SECTION B (45 marks)

Answer all questions on the writing paper provided. ***Please start each question on a new page.***

7 **[Maximum Mark: 17]**

(a) Prove that $\frac{\sec 2\theta - 1}{\sec 2\theta + 1} \equiv \tan^2 \theta$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. [4]

(b) Hence, or otherwise, find the values of θ for which $3(\sec 2\theta - 1) = \sec 2\theta + 1$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. [3]

It is given that the first three terms of a geometric sequence are $\sec x$, $\operatorname{cosec} 2x$ and $\frac{1}{2}\operatorname{cosec} x \operatorname{cosec} 2x$ respectively, where $0 < x < \frac{\pi}{2}$.

(c) Find the common ratio of the sequence. [2]

(d) Find the range of values of x for which the geometric series $\sec x + \operatorname{cosec} 2x + \frac{1}{2}\operatorname{cosec} x \operatorname{cosec} 2x + \ldots$ converges. [4]

(e) Given that $x = \arccos\left(\frac{1}{2}\right)$, show that the sum to infinity of the series is $3 + \sqrt{3}$. [4]

Do NOT write solutions on this page.

8 **[Maximum Mark: 16]**

The figure shows a solid in which HIJKLM and PQRSTU are regular hexagons of sides 4 cm and 10 cm respectively. It is given that $\text{HP} = \text{IQ} = \text{JR} = \text{KS} = \text{LT} = \text{MU} = 15\,\text{cm}$.

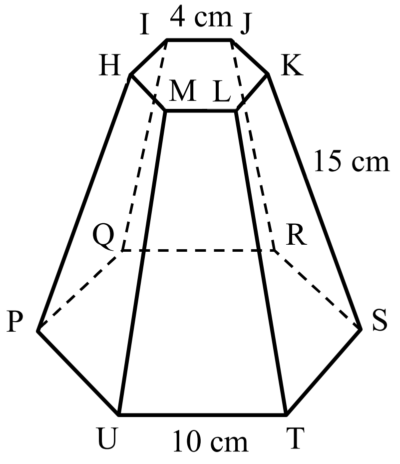


Diagram not drawn to scale

(a) Show that $\cos \text{M}\hat{\text{U}}\text{T} = \frac{1}{5}$. [1]

(b) Find the exact value of the perpendicular distance from T to MU. [3]

(c) Show that the distance between the planes HIJKLM and PQRSTU is $3\sqrt{21}$ cm. [4]

(d) Find the length of HS. [3]

(e) Find $\text{K}\hat{\text{H}}\text{S}$ and $\text{H}\hat{\text{K}}\text{S}$. [5]

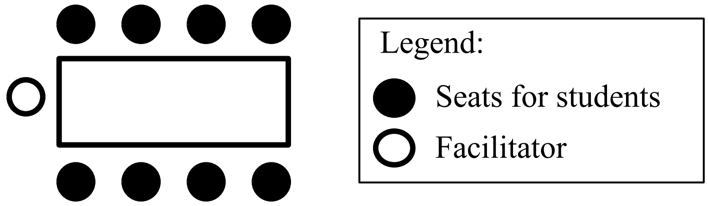
Do NOT write solutions on this page.

9 **[Maximum Mark: 12]**

A total of 8 students are randomly selected from three tutorial groups (TGs) to attend a focused group discussion. There are 14 students in each TG.

(a) Find the number of ways in which the students can be selected, if there are at least 2 students from each TG. [5]

Xiaoming and Ali are among the 8 students selected. The room for the focused group discussion has a large table in the middle with 4 seats on either side of the facilitator, as shown below.



(b) Find the number of ways in which the 8 students can be seated if

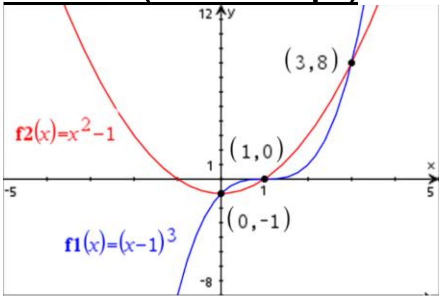
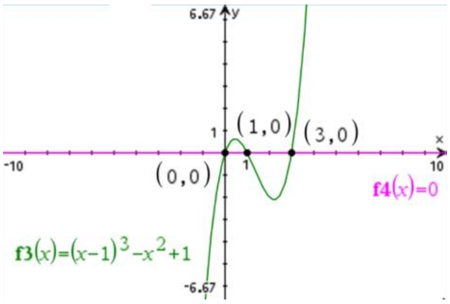
(i) Xiaoming refuses to sit on any of the 4 corner seats and Ali insists on sitting next to him,

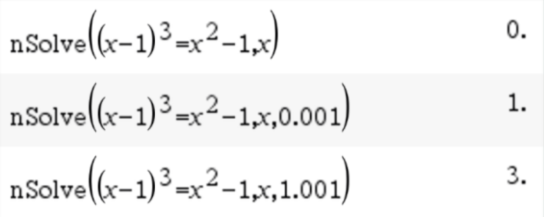
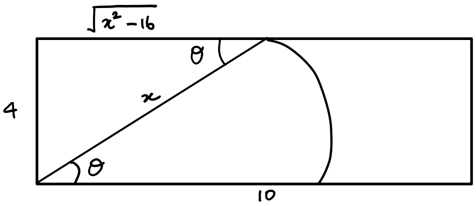
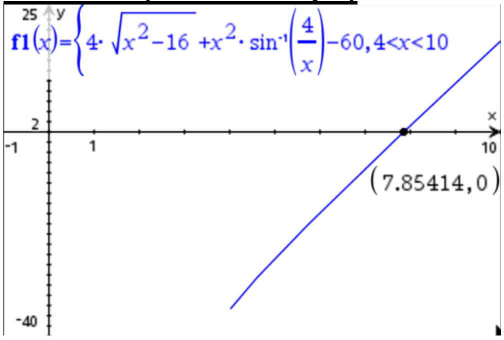
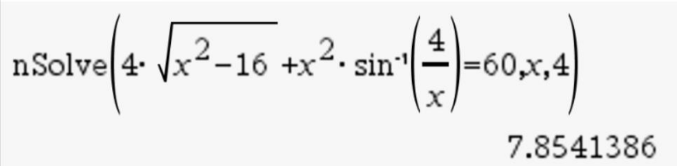
(ii) Xiaoming refuses to sit on any of the 4 corner seats and Ali insists on sitting next to him but not on the seat adjacent to the facilitator. [7]

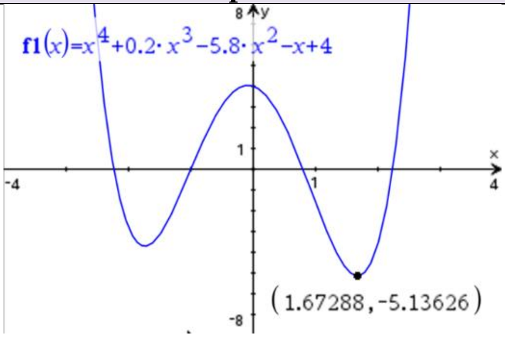
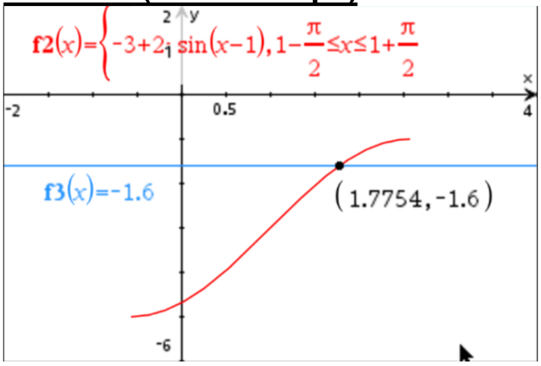
End of Paper

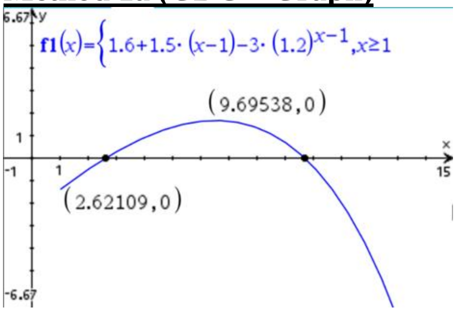
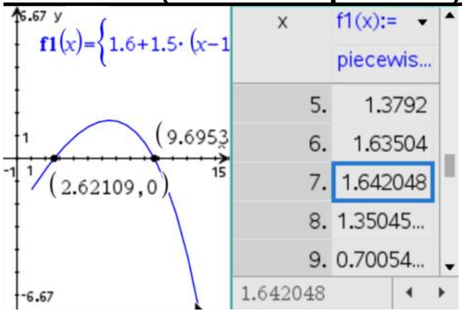
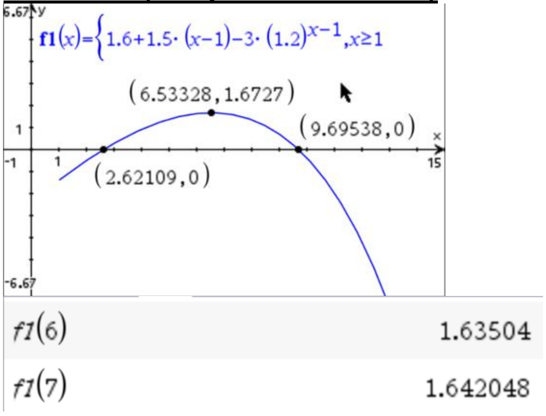
Year 5 MAA HL Maths End of Year Examination 2020 Paper 2 (Markscheme)

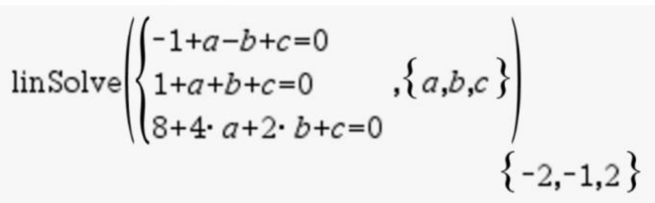
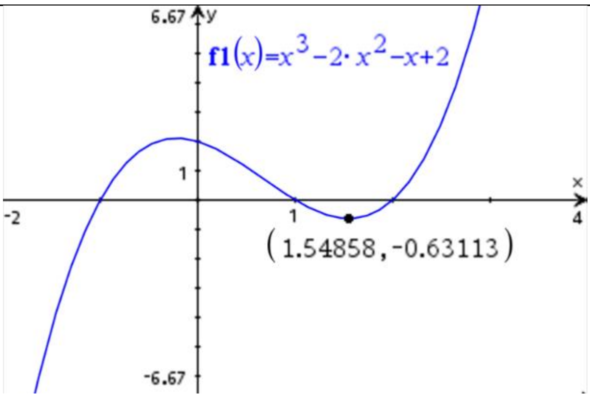
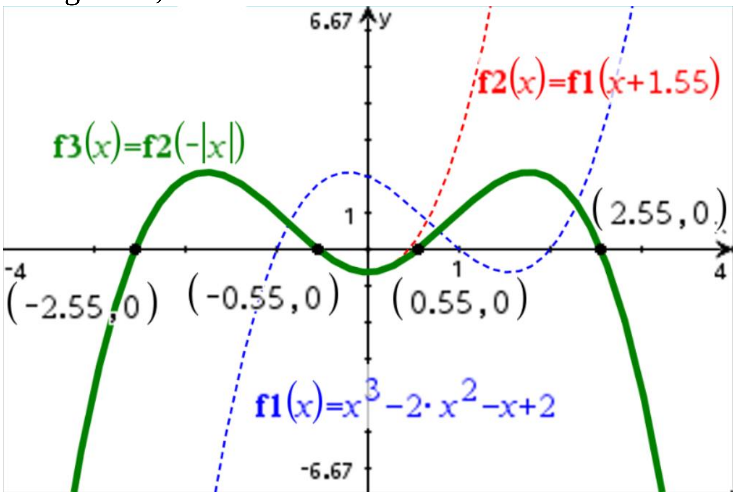
Section A

Qn	Suggested solution	Markscheme
1	***Quadratic Discriminant***	**[Marks: 4]**
	$kx^2+(k-1)x+(k-1)=0$ is a quadratic equation in *x*. For real roots, discriminant ≥ 0 $(k-1)^2-4k(k-1)\geq 0$ $(k-1)(k-1-4k)\geq 0$ OR $3k^2-2k-1\leq 0$ $(k-1)(3k+1)\leq 0$ $-\frac{1}{3}\leq k\leq 1$	**M1** – for setting $\Delta\geq 0$ **M1** – solving quad ineq **A1**-values **A1**-ineq sign
2	***Logarithmic Equations***	**[Marks: 6]**
	$\log_{x-1}y=3\Rightarrow y=(x-1)^3$ $\log_{y+1}x=\frac{1}{2}\Rightarrow x=(y+1)^{1/2}\Rightarrow y=x^2-1$ $\therefore (x-1)^3=x^2-1$ or $x=\left[(x-1)^3+1\right]^{1/2}$ **<u>Method 1</u>** $(x-1)^3=x^2-1=(x-1)(x+1)$ $\Rightarrow (x-1)\left[(x-1)^2-(x+1)\right]=0$ $\Rightarrow (x-1)(x^2-3x)=0\Rightarrow x(x-1)(x-3)=0$ $x=0,1$ (rej $\because$ $x>1$) or $x=3$ $\therefore y=8$ **<u>Method 2 (GDC – Graph)</u>**  **OR**  By GDC, $x=0,1,$ or 3 Since $x>1,\ x=3$ $y=8$	**M1** – change to exp form **A1** **A1** **M1** – solving cubic eqn. **A1** – only if $x=0,1$ rej. OR justify $x>1$ **A1** – for *y* **M1** – solving $(x-1)^3=x^2-1$ or $x=\left[(x-1)^3+1\right]^{1/2}$ o.e. **A1** – only if $x=0,1$ rej. OR justify $x>1$ **A1**–for *y*

Qn	Suggested solution	Markscheme
	Method 3 (GDC – nSolve)  $\text{nSolve}\left((x-1)^3=x^2-1,x\right)$ 0. $\text{nSolve}\left((x-1)^3=x^2-1,x,0.001\right)$ 1. $\text{nSolve}\left((x-1)^3=x^2-1,x,1.001\right)$ 3. By GDC, $x=0,1,$ or 3 Since $x>1$, $x=3$ $y=8$	**M1** – solving $(x+1)^3=x^5+1$ o.e. (**Awarded only if all 3 values of *x* are given before rejecting 0 and 1**) **A1** – only if $x=0,1$ rej. OR justify $x>1$ **A1** – for *y*
3	***Area of Sector and Triangle***	**[Marks: 7]**
(a)	 Area of triangle $=\frac{1}{2}(4)\left(\sqrt{x^2-16}\right)$ $\sin\theta=\frac{4}{x}\Rightarrow\theta=\arcsin\left(\frac{4}{x}\right)$ Area of sector $=\frac{1}{2}x^2\theta=\frac{1}{2}x^2\arcsin\left(\frac{4}{x}\right)$ Total area $=2\sqrt{x^2-16}+\frac{1}{2}x^2\arcsin\left(\frac{4}{x}\right)=\frac{3}{4}(40)=30$ $\therefore 4\sqrt{x^2-16}+x^2\arcsin\left(\frac{4}{x}\right)=60$	**M1** – for $\sqrt{x^2-16}$ **A1** – for area of triangle **A1** – for expression for θ **A1** – for $0.5x^2\theta$ **A1** – for correct sum **AG**
(b)	**Method 1 (GDC – Graph)**  By GDC, $x=7.85$ (3 sf) **Method 2 (GDC – nSolve)**  $\text{nSolve}\left(4\cdot\sqrt{x^2-16}+x^2\cdot\sin^{-1}\left(\frac{4}{x}\right)=60,x,4\right)$ 7.8541386 By GDC (nSolve), $x=7.85$ (3 sf)	 **(M1)** **A1** **(M1)** **A1**

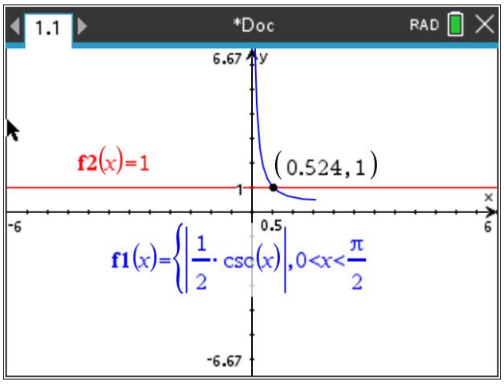
Qn	**Suggested solution**	**Markscheme**
4	***Inverse and Composite Functions***	**[Marks: 7]**
(a)	 By GDC, $p=1.67$ (3 sf.)	**(M1)** – finding min. pt. **A1** – must be 3 sf.
(b)	$f^{-1}(x)=1$ $\Rightarrow f(1)=x$ (Note that $1\in[0,1.67]$) $\therefore x=1+0.2-5.8-1+4=-1.6$ (exact)	**M1** **A1**
(c)	$(f^{-1}\circ g)(x)=1$ $\Rightarrow g(x)=f(1)$ $\Rightarrow g(x)=-1.6$ from (b) **<u>Method 1 (GDC – Graph)</u>**  By GDC, $x=1.78$ (3 sf.) **<u>Method 2</u>** $-3+2\sin(x-1)=-1.6$ $\sin(x-1)=0.7$ $x-1=0.77540$ (5 sf.) $x=1.78$ (3 sf.)	**A1** f.t. value of *x* in (b) **(M1)** **A1** f.t. value of *x* in (b) **M1** – solving trigo eqn **A1** f.t. value of *x* in (b)

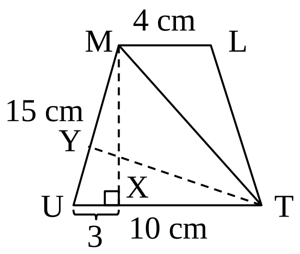
Qn	Suggested solution	Markscheme
5	***Arithmetic and Geometric Sequence***	**[Marks: 7]**
(a)	$u_n - v_n = \left[1.6+1.5(n-1)\right]-3(1.2)^{n-1}$	**A1** – for u_n **A1** – for v_n
(b)	**Method 1a (GDC – Graph)** 	**(M1)**
	By GDC, $2.62<n<9.70$	**(A1)**
	Since $n\in\mathbb{Z}^+, n=3,4,5,6,7,8,9$ (Accept $\{n\in\mathbb{Z}^+ \mid 3\le n\le 9\}$)	**A1**
	Method 1b (GDC – Graph Table)  x f1(x):= piecewis... 5. 1.3792 6. 1.63504 7. 1.642048 8. 1.35045... 9. 0.70054... 1.642048	**(M1 A1)**
	By GDC, $n=3,4,5,6,7,8,9$ (Accept $\{n\in\mathbb{Z}^+ \mid 3\le n\le 9\}$)	**A1**
(c)	**Method 1 (GDC – Graph Table)** By GDC, graph table (see (b) Method 1b), $u_6-v_6=1.635$ (4 sf)	
	$u_7-v_7=1.642$ (4 sf)	**M1**
	Greatest value of $u_n-v_n=1.642$ (4 sf)	**A1** – must be 4 sf.
	Method 2 (Graph – Max. Point)  f1(6) 1.63504 f1(7) 1.642048	**M1** – justification is needed if max. pt. is used, as n (or x) $\in\mathbb{Z}^+$
	Greatest value of $u_n-v_n=1.642$ (4 sf)	**A1** – must be 4 sf.

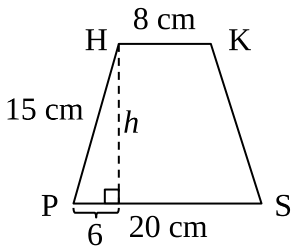
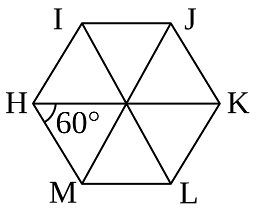
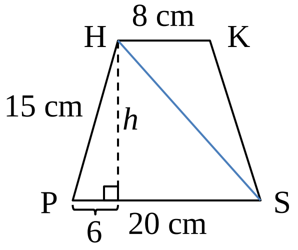
Qn	**Suggested solution**	**Markscheme**
6	***LinSolve and Graph Transformations***	**[Marks: 9]**
(a)	**Method 1 (GDC – linSolve)** Substitute $(-1,0),(1,0),(2,0)$  $\text{linSolve}\left(\begin{cases}-1+a-b+c=0\\1+a+b+c=0\\8+4\cdot a+2\cdot b+c=0\end{cases},\{a,b,c\}\right)$ $\{-2,-1,2\}$ $f(x)=x^3-2x^2-x+2$ $a=-2,b=-1,c=2$ **Method 2 (Algebraic expansion)** $f(x)=(x+1)(x-1)(x+2)$ $=x^3-2x^2-x+2$ $a=-2,b=-1,c=2$	**M1** **A2, 1, 0** – A2 for all correct, A1 for 2 out of 3 **M1** – factors from zeros **A2, 1, 0** – A2 for all correct, A1 for 2 out of 3
(b)	 $m=-1.55$ (3 sf)	**(M1)** – find min. pt. **A1** (f.t. *a*, *b*, *c* only if roots are –1, 1, 2)
(c)	$g(x)=f(x+1.55)$ Using GDC, 	**(M1** – reflection in *y*-axis, then $\|x\|$ **)** **A1** – shape, must be symmetrical about *y*-axis **A1** $(-2.55,0)$ & $(2.55,0)$ **A1** $(-0.55,0)$ & $(0.55,0)$ (f.t. *m* only if roots are $\pm(-1+m)$ and $\pm(1+m)$)

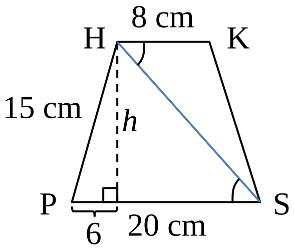
Section B

Qn	Suggested solution	Markscheme
7	***Trigonometry – Arctrigo, Further Trigo Functions***	**[Marks: 17]**
(a)	To prove $\frac{\sec 2\theta - 1}{\sec 2\theta + 1} \equiv \tan^2\theta$ $LHS = \frac{\sec 2\theta - 1}{\sec 2\theta + 1}$ $= \frac{\frac{1}{\cos 2\theta} - 1}{\frac{1}{\cos 2\theta} + 1}$ $= \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$ $= \frac{1 - \left(1 - 2\sin^2\theta\right)}{1 + (2\cos^2\theta - 1)}$ $= \frac{2\sin^2\theta}{2\cos^2\theta}$ $= \tan^2\theta = RHS$ (proven)	**M1** **A1** **M1 A1** **AG**
(b)	$3\left(\sec 2\theta - 1\right) = \sec 2\theta + 1$ $\frac{\sec 2\theta - 1}{\sec 2\theta + 1} = \frac{1}{3}$ Using (b), $\tan^2\theta = \frac{1}{3}$ $\tan\theta = \pm\frac{1}{\sqrt{3}}$ $\theta = \pm\frac{\pi}{6}$ (accept ± 0.524, but must be in radians)	**M1** **A1 A1**
(c)	GP: $\sec x + \csc 2x + \frac{1}{2}\csc x \csc 2x$ Common ratio, $r = \frac{U_2}{U_1} = \frac{U_3}{U_2}$ $\therefore r = \frac{\frac{1}{2}\csc x \csc 2x}{\csc 2x} = \frac{1}{2}\csc x = \frac{1}{2\sin x}$	**M1 A1** (Either form)
(d)	Series converges when $\lvert r \rvert < 1$ $\therefore \left\lvert \frac{1}{2}\csc x \right\rvert < 1$	**A1**

Qn	Suggested solution	Markscheme
	<u>Method 1</u>: by GDC  From the graph, the solution for $\left\|\frac{1}{2}\csc x\right\|<1$ is $0.524<x<\frac{\pi}{2}$ **<u>Method 2</u>: Analytical** $\left\|\frac{1}{2\sin x}\right\|<1$ $\|\sin x\|>\frac{1}{2}$ We know also that $\|\sin x\|<1$ for $0<x<\frac{\pi}{2}$ $\frac{1}{2}<\|\sin x\|<1$ $\frac{1}{2}<\sin x<1 \quad \left(\because \sin x>0 \quad \text{for } 0<x<\frac{\pi}{2}\right)$ Solving, by graph or reasoning, we have $\frac{\pi}{6}<x<\frac{\pi}{2}$	**<u>Method 1</u>** **M1 Sketch by GDC** **A1 A1** **<u>Method 2</u>** **M1** **A1 A1**
(e)	$S_\infty=\frac{\frac{1}{\cos x}}{1-\frac{1}{2\sin x}}$ $=\frac{2\sin x}{\cos x(2\sin x-1)}$ .... (*) At $x=\arccos\left(\frac{1}{2}\right)$, $\Rightarrow \cos x=\frac{1}{2} \quad (\text{i.e. } \sec x=2)$ $\Rightarrow \sin x=\frac{\sqrt{3}}{2} \quad \left(\text{for } 0<x<\frac{\pi}{2}\right)$	**M1** (S_∞ formula applied) **A1**

Qn	Suggested solution	Markscheme
	From (*), $S_\infty = \dfrac{2\left(\frac{\sqrt{3}}{2}\right)}{\frac{1}{2}\left(2\left(\frac{\sqrt{3}}{2}\right)-1\right)}$	
	$= \dfrac{2\sqrt{3}}{\sqrt{3}-1}$	**A1** (o.e.)
	$= \dfrac{2\sqrt{3}}{\sqrt{3}-1} \cdot \dfrac{\sqrt{3}+1}{\sqrt{3}+1}$	**A1**
	$= \sqrt{3}\left(\sqrt{3}+1\right) = 3+\sqrt{3}$ (shown)	**AG**
8	***Trigonometry – Solutions of Triangles, 3D***	**[Marks: 16]**
(a)	Consider the face MLTU. 	
	Let X be the foot of M on UT. Then, MXU is a rt angle triangle, with $\text{MU} = 15$, and $\text{UX} = 3$.	**A1** ($UX = 3$ cm)
	$\cos \text{M}\hat{\text{U}}\text{T} = \dfrac{3}{15}$	
	$= \dfrac{1}{5}$ (shown)	**AG**
(b)	Let Y be the foot of the perpendicular of T on MU.	
	<u>Method 1</u>: $\cos \text{Y}\hat{\text{U}}\text{T} = \cos \text{M}\hat{\text{U}}\text{X} = \dfrac{1}{5}$ (same angle)	**(M1)**
	$\text{YT} = 10 \sin \text{Y}\hat{\text{U}}\text{T}$	
	$= 10\sqrt{1-\cos^2 \text{Y}\hat{\text{U}}\text{T}}$ $\left(\sin \text{YUT} > 0, \text{ since } \text{Y}\hat{\text{U}}\text{T} \text{ is acute}\right)$	
	$= 10\sqrt{1-\left(\dfrac{1}{5}\right)^2} = 10\dfrac{\sqrt{24}}{5}$	**A1**
	$= 2\sqrt{24} = 4\sqrt{6}$	**A1** (o.e., exact)
	<u>Method 2</u>: By Equating Area of ΔMUT	
	$\dfrac{1}{2} \times 15 \times \text{YT} = \dfrac{1}{2} \times 10 \times \sqrt{15^2 - 3^2}$	**M1**
	$\text{YT} = \dfrac{10}{15} \times \sqrt{216} = \dfrac{2}{3} \times \sqrt{9 \times 24} = 2\sqrt{24} = 4\sqrt{6}$	**A1** ($\sqrt{216}$) **A1** (ans)

Qn	Suggested solution	Markscheme
(c)	  A **regular** hexagon is composed of 6 equilateral triangles.	
	$\text{HK} = 2\text{IJ} = 8$ cm	**A1**
	Similarly, $\text{PS} = 2\text{UT} = 20$ cm	**A1**
	Let *h* be the height of the solid.	
	By Pythagoras' Theorem, $h^2 + 6^2 = 15^2$	**M1**
	$h^2 = 225 - 36$	**A1**
	$h = \sqrt{189}$	
	$= 3\sqrt{21}$ (shown)	**AG**
(d)		
	<u>Method 1</u>: (Pythagora's Theorem on rt. angle below HP)	
	$\text{HS}^2 = h^2 + (20-6)^2$	**M1 A1** (Pyth. on the rt. angle Δ, with base 14)
	$= \left(3\sqrt{21}\right)^2 + 14^2$	
	$= 385$	
	$\text{HS} = \sqrt{385} = 19.621$	
	$= 19.6$ cm (to 3 s.f.)	**A1**
	<u>Method 2</u>: (Cosine rule on ΔHPS)	
	$\cos \text{H}\hat{\text{P}}\text{S} = \frac{6}{15}$	**A1**
	$= \frac{2}{5}$	
	By cosine rule, $\text{HS}^2 = 15^2 + 20^2 - 2(15)(20)\cos \text{H}\hat{\text{P}}\text{S}$	**M1**
	$\text{HS}^2 = 15^2 + 20^2 - 2(15)(20)\left(\frac{2}{5}\right)$	
	$\therefore \text{HS} = \sqrt{385} = 19.621$	
	$= 19.6$ cm (to 3 s.f.)	**A1**

Qn	**Suggested solution**	**Markscheme**
(e)		
	$K\hat{H}S = H\hat{S}P$ (alternate angles, HK // PS) $\tan H\hat{S}P = \frac{h}{14} = \frac{3\sqrt{21}}{14}$	**M1**
	$H\hat{S}P = 44.479°$ $\therefore K\hat{H}S = 44.479°$ $= 44.5°$ (to 3 s.f.) (accept 0.776 rad)	**A1**
	By sine rule, $\frac{\sin H\hat{K}S}{HS} = \frac{\sin K\hat{H}S}{15}$	**M1**
	$\sin H\hat{K}S = \frac{HS \sin K\hat{H}S}{15}$ $= \frac{19.621 \sin 44.479°}{15}$ $= 0.91651$	**A1 f.t. from (b)**
	$H\hat{K}S = \sin^{-1}(0.91651)$ $= 66.421°$ (rej., acute angle) or $113.57°$ $= 114°$ (to 3 s.f.)	**A1**
	(Equivalently, in radians, reject acute angle 1.16 and accept the obtuse angle 1.98 rad.) Alternate methods for finding $H\hat{K}S$ includes using cosine or by geometrical reasoning.	
9	***Permutations and Combinations***	**[Marks: 12]**
(a)	Identifying the two sets of cases: (2, 3, 3) and (2, 2, 4) Case 1: All possible combinations of (2, 3, 3) $= {}^{14}C_2 \times {}^{14}C_3 \times {}^{14}C_3 \times \frac{3!}{2!}$ $= 36\ 171\ 408$	**M1 A1**
	Case 2: Possible combinations of choosing 2, 2, 4 $= {}^{14}C_2 \times {}^{14}C_2 \times {}^{14}C_4 \times \frac{3!}{2!}$ $= 24\ 867\ 843$	**M1 A1**
	Total cases $= 36\ 171\ 408 + 24\ 867\ 843$ $= 61\ 039\ 251$	**A1**

Qn	Suggested solution	Markscheme
(b)	(i) Wherever XM sits, Ali has 2 choices beside him. No. of ways $= {}^4C_1 \times {}^2C_1 \times 6!$ $= 5760$	**M1 M1** **A1**
	(ii)	
	<u>Method 1</u>: by Complement	
	No. of ways with Ali at corner seats adjacent to the Facilitator $= {}^2C_1 \times 1 \times 6!$ $= 1440$	**M1** **A1**
	Required no. of ways $= 5760 - 1440$ $= 4320$	**M1** (complement) **A1**
	<u>Method 2</u>: Case by Case	
	Case 1: XM sits 1 seat away from Facilitator, and Ali has only 1 choice. No. of ways $= {}^2C_1 \times 1 \times 6!$ $= 1440$	**M1** (cases) **A1**
	Case 2: XM sits 2 seats away from Facilitator, and Ali has two choices. No. of ways $= {}^2C_1 \times {}^2C_1 \times 6!$ $= 2880$	**A1**
	Total cases $1440 + 2880$ $= 4320$	**A1**

STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

ST JOSEPH'S INSTITUTION
YEAR 5 END OF YEAR EXAMINATION 2021

MATHEMATICS: ANALYSIS AND APPROACHES

HIGHER LEVEL

PAPER 1

Thursday

7 October 2021

1.5 hours

0800 – 0930 hrs

INSTRUCTIONS TO CANDIDATES

- Write your name and your teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the writing paper provided.
- The use of calculators is **not** permitted in this paper.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- Unless otherwise stated in the question, all numerical answers are to be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is *[**85** marks]*.
- This question paper consists of **10** printed pages including the Cover Sheet.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	TOTAL
								/85

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (40 marks)

Answer **all** questions in the spaces provided.

1 [Maximum mark: 5]

Define a function f , including its maximal domain, such that its inverse exists and is given by

$$f^{-1}(x) = \frac{\mathrm{e}^x}{\mathrm{e}^x - 1},\ x \neq 0,\ x \in \mathbb{R}.$$

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2 **[Maximum mark: 6]**

The sum of the first n terms of a sequence is

$$\sum_{r=1}^{n} u_r = n^2 + n - 1,$$

for $n \in \mathbb{Z}^+$.

(a) **(i)** Find u_1.

(ii) Find an expression for u_n in terms of n for any positive integer $n \geq 2$. [4]

(b) **Hence**, state with reason whether the sequence is arithmetic or not. [2]

TURN OVER

3 **[Maximum mark: 9]**

Consider the function $f(x) = 2(x-1)^2 - 2$.

(a) State the minimum value of f. [1]

(b) Find the zeros of $y = f(x)$. [2]

(c) State the coordinates of the turning point of $y = \dfrac{1}{f(x)}$. [2]

(d) Sketch the graph of $y = \dfrac{1}{f(|x|)}$ in the given axes on the next page, clearly indicating the asymptotes and turning points. [4]

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MORE SPACE IS AVAILABLE ON THE NEXT PAGE

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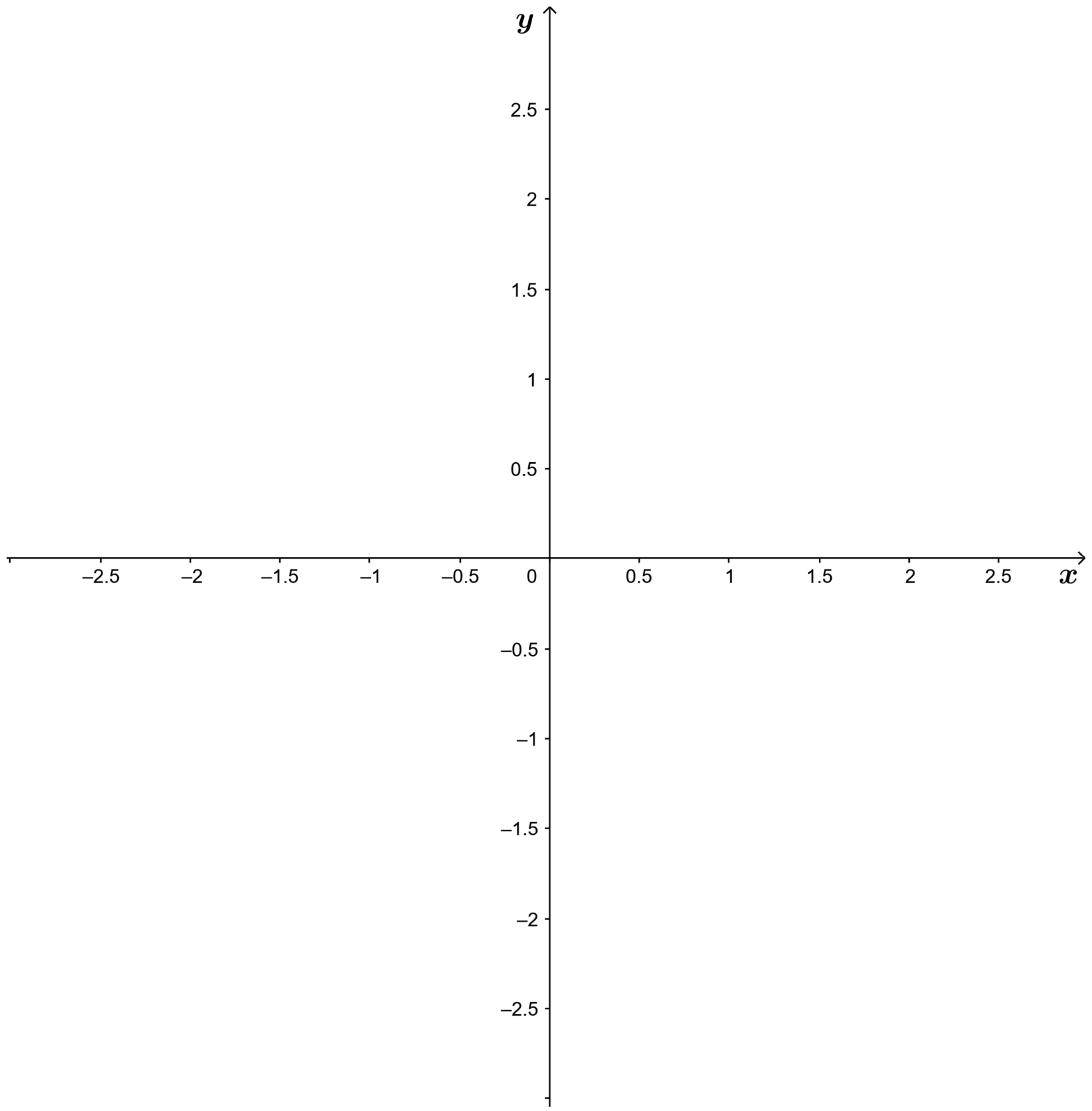


TURN OVER

4 **[Maximum mark: 8]**

Let $g(x) = x + \sin x, -\pi \le x \le \pi$.

(a) Find the values of x for which $g'(x) \ge 0$. [2]

(b) Hence or otherwise, show that g is one-one. [2]

(c) Solve for x such that $g(x) = g^{-1}(x)$. [4]

TURN OVER

5 **[Maximum mark: 8]**

For this question, leave all answers in factorial form, in terms of m and/or n.

m boys and n girls are to be seated in a row, where m and n are positive.

Find the number of ways this can be done in each of the following cases:

(a) there are no restrictions; [1]

(b) the n girls are seated together; [2]

(c) a particular boy and a particular girl must be adjacent; [2]

(d) no boys are adjacent given that there are equal numbers of boys and girls. Leave your answer in terms of n. [3]

TURN OVER

6 **[Maximum mark: 4]**

Mr Ang invested half of his savings in a bond that paid **simple** interest at r % per annum and received a total of \$550 in interests after 2 years.

He invested the remaining in another bond that paid interest at r % per annum compounded annually and received a total of \$605 in interests after 2 years.

Find the value of r .

*Do **NOT** write solutions on this page.*

SECTION B (45 marks)

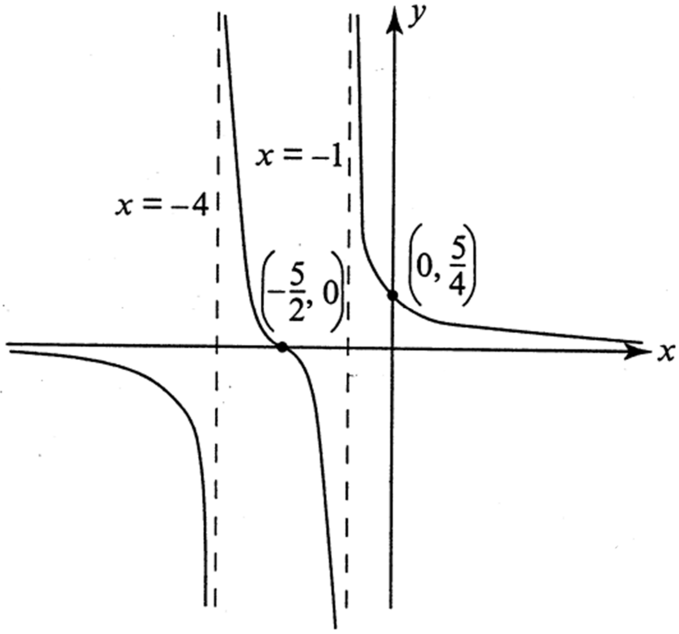
*Answer **all** questions on the writing paper provided.* ***Please start each question on a new page.***

7 [Maximum mark: 18]

The diagram shows the graph of the function $y = f(x)$ where

$$f(x) = \frac{ax+b}{x^2+cx+d}$$

with $f\left(-\frac{5}{2}\right) = 0,\ f(0) = \frac{5}{4}$. The lines $x = -4,\ x = -1$ and the x-axis are all asymptotes.



(a) Using the information given, find the value of c and of d and show that $a = 2,\ b = 5$.

[4]

(b) Using the values of *a, b, c,* and *d* found in **(a)**, find the value of *A* and of *B* for which

$f(x) = \frac{A}{x+4} + \frac{B}{x+1}$. [3]

(c) Show that $\left(-\frac{5}{2}, 0\right)$ is a point of inflexion. [3]

(d) State the set of values of *x* for which $f'(x) < 0$. [2]

(e) State the set of values of *x* for which $f''(x) > 0$. [2]

(f) Sketch the graph of $y = f'(x)$, clearly indicating any intercepts with the axes, the coordinates of any local maximum or minimum points and the equations of the asymptotes (if any). [4]

TURN OVER

*Do **NOT** write solutions on this page.*

8 [Maximum mark: 27]

(a) Consider the quadratic equation $2z^2 - (2-2\mathrm{i})z - 5\mathrm{i} = 0$.

(i) Show that $(2-2\mathrm{i})^2 = -8\mathrm{i}$. [1]

(ii) Write down $(2+2\mathrm{i})^2$ in Cartesian form. [1]

(iii) Using the quadratic formula, show that the roots of the quadratic equation are given by $\frac{1}{2}(1-\mathrm{i}) \pm \sqrt{2\mathrm{i}}$. [3]

(iv) Using the result in **(a)(ii)**, express each of the roots in the form $a + \mathrm{i}b$, where a and b are real numbers. [4]

Let $z = \cos\theta + \mathrm{i}\sin\theta$.

(b) **(i)** Find $|z|$. [2]

(ii) Deduce that $\frac{1}{z} = z^*$, where z^* is the conjugate of z. [1]

(iii) Find $z + \frac{1}{z}$. [2]

(iv) Show that $z^2 + \frac{1}{z^2} = 2\cos 2\theta$. [3]

(c) It is given that each of the four roots of the equation $5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$ has modulus equal to 1.

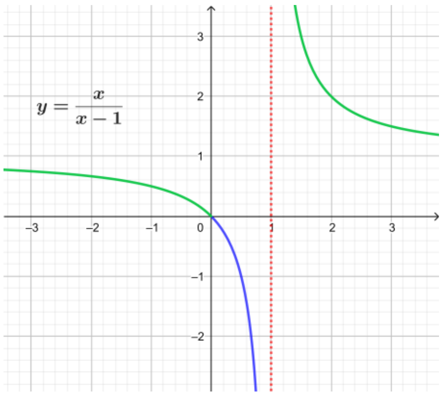
(i) Using the results in **(b)**, show that

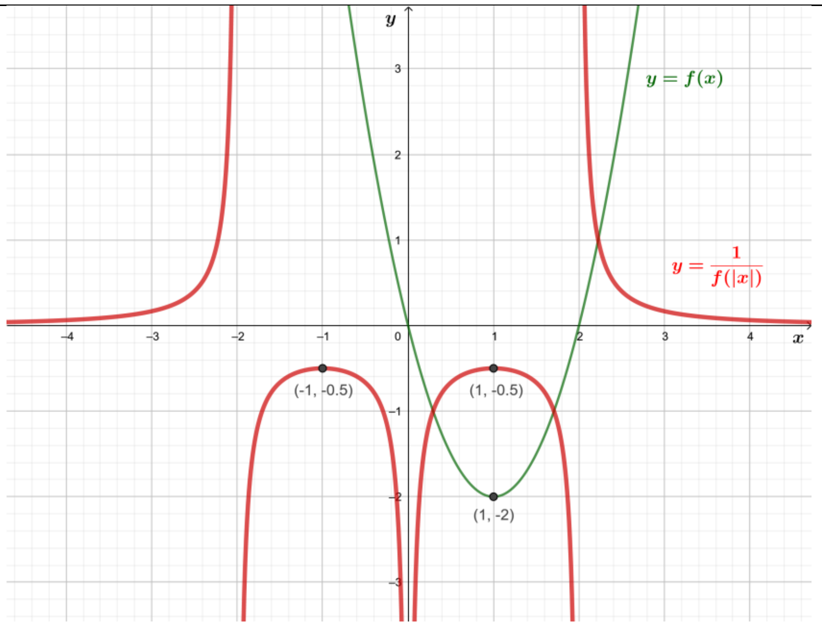
$$10\cos^2\theta - 11\cos\theta + 3 = 0.$$ [3]

(ii) Hence find these roots. [7]

End of Paper

Year 5 HL MAA End of Year Examination 2021 Paper 1 (Mark Scheme)

Qn	Suggested Solutions	Marks
1	Inverse Function	**[Maximum mark: 5]**
	Let $y=\frac{\mathrm{e}^x}{\mathrm{e}^x-1}$. $\Rightarrow x=\frac{\mathrm{e}^y}{\mathrm{e}^y-1}$ $\Rightarrow x\mathrm{e}^y-x=\mathrm{e}^y$ $\Rightarrow (x-1)\mathrm{e}^y=x$ $\Rightarrow \mathrm{e}^y=\frac{x}{x-1}$ $\Rightarrow y=\ln\left(\frac{x}{x-1}\right)$ OR $\Rightarrow y\mathrm{e}^x-y=\mathrm{e}^x$ $\Rightarrow (y-1)\mathrm{e}^x=y$ $\Rightarrow \mathrm{e}^x=\frac{y}{y-1}$ $\Rightarrow x=\ln\left(\frac{y}{y-1}\right)$ $\Rightarrow f^{-1}(y)=\ln\left(\frac{y}{y-1}\right)$ $\frac{x}{x-1}>0\Rightarrow x(x-1)>0$ OR  Thus, $f(x)=\ln\left(\frac{x}{x-1}\right), x<0$ or $x>1$.	M1 – swap x and y OR let $x=f^{-1}(y)$ M1 – *correctly* making y OR x the subject A1 – correct rule for f^{-1} (M1) – valid attempt to solve >0 A1 – correct domain
2	Arithmetic Sequences and Series	**[Maximum mark: 6]**
(a)	**(i)** $u_1=\sum_{r=1}^{1}u_r=1^2+1-1=1$ **(ii)** $u_n=\sum_{r=1}^{n}u_r-\sum_{r=1}^{n-1}u_r=\left(n^2+n-1\right)-\left((n-1)^2+(n-1)-1\right)=2n,\ n\geq 2$	M1 – $n=1$ A1 M1 – use of "S_n-S_{n-1}" A1
(b)	The sequence is not arithmetic as the first three terms are 1, 4 and 6; and, $6-4\neq 4-1$. Note: This is a hence question.	R1 – no common difference with evidence A1 – A0 for invalid reasoning
3	Quadratic, Reciprocal and Modulus Functions	**[Maximum mark: 9]**
(a)	-2	A1
(b)	$2(x-1)^2-2=0\Rightarrow x-1=\pm1\Rightarrow x=0,2$	M1 – equate to 0 A1 – 0 **and** 2

Qn	Suggested Solutions	Marks
(c)	$(1,-0.5)$	A1A1
(d)		A1 – shape (2 symmetric branches w.r.t. $x=0$ above the x-axis) A1 – shape (2 symmetric branches w.r.t. $x=0$ below the x-axis) A1 – at least two correct vertical asymptotes A1 – horizontal asymptote
4	Trigonometric & 1-1 Functions and Derivatives	**[Maximum mark: 8]**
(a)	$g'(x)=1+\cos x\geq 0\Rightarrow x\in[-\pi,\pi]$ (entire domain)	A1 – g' A1 – A0 for $\mathbb{R}$
(b)	$g'(x)\geq 0$ and $g'(x)=0$ only when $x=\pm\pi$ imply that g is strictly increasing. (Award R1 as long as "strictly increasing" is deduced.) Thus, no horizontal line can intersect the graph of g more than once. Therefore, g is one-one.	R1 – strictly increasing R1 – HLT AG
(c)	Solving $g(x)=g^{-1}(x)$ is equivalent to solving $g(x)=x$. Thus, $x+\sin x=x\Rightarrow\sin x=0\Rightarrow x=0,\pm\pi$	(M1) – solving for $g(x)=x$ A1 – 0 A1A1 – $\pm\pi$
5	Counting Techniques	**[Maximum mark: 8]**
(a)	$(m+n)!$	A1
(b)	$n!(m+1)!$ Note: A0A0 for $n!+(m+1)!$	A1 – $n!$ A1 – $(m+1)!$
(c)	$2!(m+n-1)!$ Note: A0A0 if the factors were added instead	A1 – 2! A1 – $(m+n-1)!$
(d)	$n!\,{}_{n+1}C_n\times n!=n!\,{}_{n+1}P_n=n!(n+1)!$ Remark: Only the form $n!(n+1)!$ gets full marks. Note: A0A0A0 for $n!+(n+1)!$	A1 – $n!$ (M1) – ${}_{n+1}P_n$ A1 – $(n+1)!$

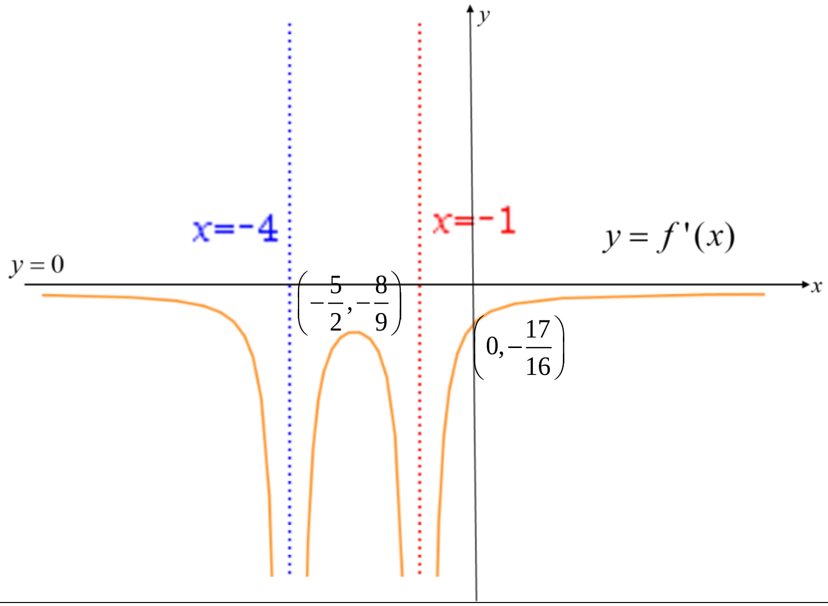
<table>
<tr><th>Qn</th><th>Suggested Solutions</th><th>Marks</th></tr>
<tr><td>6</td><td>Financial Application</td><td>[Maximum mark: 4]</td></tr>
<tr><td></td><td>Suppose the initial principal makes I as interest per year based on the interest rate r %, i.e., if we let P be the principal, then $I = P \times \frac{r}{100}$.

After 1 year, both bonds yield the same interest I .

On the 2nd year, bond A will still yield the same interest I , but Bond B will yield not only the interest from the original principal, which is still I , but also yield interest from the interest earned in the first year, i.e., $I \times \frac{r}{100}$. Thus, the total interest payments are:

<table>
<tr><th></th><th>Bond A</th><th>Bond B</th></tr>
<tr><td>Year 1</td><td>I</td><td>I</td></tr>
<tr><td>Year 2</td><td>I</td><td>$I + I \times \frac{r}{100}$</td></tr>
<tr><td>Total Interest</td><td>550</td><td>605</td></tr>
</table>
</td><td></td></tr>
<tr><td></td><td>From Bond A: $I + I = 550$</td><td>(A1) – Bond A</td></tr>
<tr><td></td><td>From Bond B: $I + \left(I + I \times \frac{r}{100} \right) = 605$</td><td>(A1) – Bond B</td></tr>
<tr><td></td><td>Thus, $I = 275$ and $275 \times \frac{r}{100} = 55$</td><td>M1 – valid attempt to find r</td></tr>
<tr><td></td><td>Therefore, the interest rate is $r = \frac{55}{275} \times 100\% = 20\%$</td><td>A1</td></tr>
<tr><td></td><td>OR</td><td>OR</td></tr>
<tr><td></td><td>Bond A: $550 = \frac{P \cdot r \cdot 2}{100} \Rightarrow P = \frac{27500}{r}$</td><td>(A1) – Bond A</td></tr>
<tr><td></td><td>Bond B: $P + 605 = P\left(1 + \frac{r}{100} \right)^2$</td><td>(A1) – Bond B</td></tr>
<tr><td></td><td>$\Rightarrow \frac{27500}{r} + 605 = \frac{27500}{r}\left(1 + \frac{2r}{100} + \frac{r^2}{10000} \right)$
$\Rightarrow 605 = 550 + \frac{275r}{100}$</td><td>M1 – valid attempt to find r</td></tr>
<tr><td></td><td>$\Rightarrow r = \frac{5500}{275} = 20$</td><td>A1</td></tr>
</table>

Qn	**Suggested Solutions**	**Marks**
7	Simultaneous eq, Rational function, Techniques of Differentiation, Gradient graph, decreasing function, concavity	**[Maximum mark: 18]**
(a)	$f(x)=\frac{ax+b}{x^2+cx+d}$ Since $x=-4,\ x=-1$ are asymptotes, $x^2+cx+d=(x+4)(x+1)=x^2+5x+4$ $\Rightarrow c=5,d=4$	M1 A1[$c=5,d=4$]
	$f(0)=\frac{5}{4}$ $\frac{b}{d}=\frac{5}{4}\Rightarrow b=5$ $f\left(-\frac{5}{2}\right)=0,$ $-\frac{5}{2}a+b=0\Rightarrow a=2$ $\therefore a=2,b=5,c=5,d=4.$	Results of Substitution of $f\left(-\frac{5}{2}\right)=0, f(0)=\frac{5}{4}$: A1A1 [$\frac{b}{d}=\frac{5}{4}$; $-\frac{5}{2}a+b=0$] AG [$a=2,b=5$]
(b)	$\frac{2x+5}{x^2+5x+4}=\frac{A}{x+4}+\frac{B}{x+1}$ $\Rightarrow 2x+5=A(x+1)+B(x+4)$ $\Rightarrow\begin{cases}A+B=2\\A+4B=5\end{cases}$ $\Rightarrow A=1,B=1$ [accept cover-up rule]	M1 A1A1
(c)	$f(x)=\frac{1}{x+4}+\frac{1}{x+1}$ $f'(x)=\frac{-1}{(x+4)^2}-\frac{1}{(x+1)^2}$ $f''(x)=\frac{2}{(x+4)^3}+\frac{2}{(x+1)^3}$ It is given that $f\left(-\frac{5}{2}\right)=0.$ $f''\left(-\frac{5}{2}\right)=2\left((-2.5+4)^{-3}+(-2.5+1)^{-3}\right)=2\left((1.5)^{-3}+(-1.5)^{-3}\right)=0$ Since $f''\left(\frac{-5}{2}\right)=0$ and $f''\left(\left(-\frac{5}{2}\right)^{-}\right)f''\left(\left(-\frac{5}{2}\right)^{+}\right)<0,$ hence $\left(-\frac{5}{2},0\right)$ is a point of inflexion.	M1 M1 R1 AG

Qn	Suggested Solutions	Marks
(d)	For $f'(x)<0$, $x\in\mathbb{R}\setminus\{-4,-1\}$ OR $x<-4$ or $-4<x<-1$ or $x>-1$.	A1A1
(e)	For $f''(x)>0, -4<x<-\frac{5}{2}$ or $x>-1$.	A1A1
(f)		A1 – shape (1st & 3rd piece) A1- shape(2nd piece) A1 – max point $\left(-\frac{5}{2},-\frac{8}{9}\right)$ & *y*-intercept $\left(0,-\frac{17}{16}\right)$ A1 – 3 asymptotes

8	Quadratic eq, complex no in cartesian form, square roots, modulus of complex no, double angle, solve trigo equation polynomial eq, FTA	**[Maximum mark:27]**
(a)(i)	$\left(2-2\mathrm{i}\right)^2 = 2^2+\left(2\mathrm{i}\right)^2-2\left(2\right)\left(2\mathrm{i}\right)$ $= 4-4-8\mathrm{i}$ $= -8\mathrm{i}$ (shown)	**M1** **AG**
(ii)	$\left(2+2\mathrm{i}\right)^2 = 8\mathrm{i}$	**A1**
(iii)	$2z^2-\left(2-2\mathrm{i}\right)z-5\mathrm{i}=0$ $z=\dfrac{\left(2-2\mathrm{i}\right)\pm\sqrt{\left(2-2\mathrm{i}\right)^2-4(2)(-5\mathrm{i})}}{4}$ $=\dfrac{\left(2-2\mathrm{i}\right)\pm\sqrt{-8\mathrm{i}+40\mathrm{i}}}{4}$ $=\dfrac{2\left(1-\mathrm{i}\right)\pm 4\sqrt{2\mathrm{i}}}{4}$ $=\dfrac{1}{2}\left(1-\mathrm{i}\right)\pm\sqrt{2\mathrm{i}}$ (shown)	**M1** **A1** **A1** **AG**
(iv)	From (ii), $\left(2+2\mathrm{i}\right)^2=8\mathrm{i}$, $\Rightarrow\left(1+\mathrm{i}\right)^2=2\mathrm{i}$ $\Rightarrow\sqrt{2\mathrm{i}}=1+\mathrm{i}$ Let $z_1=\dfrac{1}{2}\left(1-\mathrm{i}\right)+\sqrt{2\mathrm{i}}$ and $z_2=\dfrac{1}{2}\left(1-\mathrm{i}\right)-\sqrt{2\mathrm{i}}$ $z_1=\dfrac{1}{2}\left(1-\mathrm{i}\right)+\sqrt{2\mathrm{i}}=\dfrac{1}{2}\left(1-\mathrm{i}\right)+\left(1+\mathrm{i}\right)=\dfrac{3}{2}+\dfrac{1}{2}\mathrm{i}$ $z_2=\dfrac{1}{2}\left(1-\mathrm{i}\right)-\sqrt{2\mathrm{i}}=\dfrac{1}{2}\left(1-\mathrm{i}\right)-\left(1+\mathrm{i}\right)=-\dfrac{1}{2}-\dfrac{3}{2}\mathrm{i}$	**M1** **A1** **A1** **A1**
(b)(i)	$\lvert z\rvert=\sqrt{\cos^2\theta+\sin^2\theta}=1$	**M1A1**
(ii)	$z=\cos\theta+\mathrm{i}\sin\theta$ Since $\lvert z\rvert=1\Rightarrow z^*z=1$ $\Rightarrow\dfrac{1}{z}=z^*$ (deduced)	**R1**
(iii)	Since $\dfrac{1}{z}=z^*=\cos\theta-\mathrm{i}\sin\theta$, $z+\dfrac{1}{z}=\cos\theta+\mathrm{i}\sin\theta+\cos\theta-\mathrm{i}\sin\theta$ $=2\cos\theta$	**M1** **A1**

(iii)	$\frac{1}{z^2} = \left(\cos\theta - \mathrm{i}\sin\theta\right)^2 = \cos^2\theta - \sin^2\theta - 2\mathrm{i}\sin\theta\cos\theta$ ----(1) $z^2 = \left(\cos\theta + \mathrm{i}\sin\theta\right)^2 = \cos^2\theta - \sin^2\theta + 2\mathrm{i}\sin\theta\cos\theta$ ----(2) $(1)+(2) \Rightarrow z^2 + \frac{1}{z^2} = 2\left(\cos^2\theta - \sin^2\theta\right) = 2\cos 2\theta$ (shown) **Alternatively,** $z^2 + \frac{1}{z^2} = \left(z + \frac{1}{z}\right)^2 - 2$ $= \left(2\cos\theta\right)^2 - 2$ $= 4\cos^2\theta - 2$ $= 2\cos 2\theta$	**A1** **A1** **M1AG** **M1[** $z^2 + \frac{1}{z^2} = \left(z + \frac{1}{z}\right)^2 - 2$] **A1[** $z + \frac{1}{z} = 2\cos\theta$] **M1**[double angle] **AG**
(c)(i)	$P(z) = 5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$ Divide throughout by z^2, $5z^2 - 11z + 16 - 11\left(\frac{1}{z}\right) + 5\left(\frac{1}{z^2}\right) = 0$ $\Rightarrow 5\left(z^2 + \frac{1}{z^2}\right) - 11\left(z + \frac{1}{z}\right) + 16 = 0$ $\Rightarrow 5\left(2\cos 2\theta\right) - 11\left(2\cos\theta\right) + 16 = 0$ $\Rightarrow 5\left(2\cos^2\theta - 1\right) - 11\cos\theta + 8 = 0$ $\Rightarrow 10\cos^2\theta - 11\cos\theta + 3 = 0$ (shown)	**M1** **A1** **M1**[double angle] **AG**
(ii)	$\left(5\cos\theta - 3\right)\left(2\cos\theta - 1\right) = 0$ $\Rightarrow \cos\theta = \frac{3}{5}$ *or* $\cos\theta = \frac{1}{2}$ $\Rightarrow \sin\theta = \pm\frac{4}{5}$ *or* $\sin\theta = \pm\frac{\sqrt{3}}{2}$ $z = \cos\theta + \mathrm{i}\sin\theta = \frac{3}{5} \pm \mathrm{i}\frac{4}{5},\ \frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}$ Remark: If students were to forget ± *i.e.* $\sin\theta = \pm\frac{4}{5}$ *or* $\sin\theta = \pm\frac{\sqrt{3}}{2}$ By FTA, since all the coefficients of $P(z)$ are real, complex roots occur in conjugate pairs. Hence, the roots are $\frac{3}{5} \pm \mathrm{i}\frac{4}{5},\ \frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}$.	**M1** **A1A1** **A1A1A1A1**

STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

ST. JOSEPH'S INSTITUTION
YEAR 5 END OF YEAR EXAMINATION 2021

MATHEMATICS: ANALYSIS AND APPROACHES
HIGHER LEVEL
PAPER 2
Wednesday

13 October 2021
1.5 hours
0800 – 0930 hrs

INSTRUCTIONS TO CANDIDATES

- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A:** Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B:** Answer all questions using the foolscap paper provided.
- The use of a scientific or examination graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted apps and be ram cleared.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures.
- The maximum mark for this examination paper is *[**85** marks]*.
- This question paper consists of **9** printed pages including the Cover Sheet**.**
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	TOTAL
									/85

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to *show all working.*

SECTION A (40 marks)

Answer **all** questions in the **spaces** provided.

1 [Maximum mark: 4]

By making y the subject, solve the system of equations, for $0.6 < x < 1.3$ and $0.6 < y < 1.3$,

$$\sin x + \sin y = 1.5$$
$$\cos x + \cos y = 1.2\ .$$

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TURN OVER

2 **[Maximum mark: 6]**

(a) Differentiate $y = \ln\left[\frac{(x-2)(x+1)}{x^2}\right]$ with respect to x. [4]

(b) Find the maximum value of y. [2]

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3 **[Maximum mark: 7]**

α, β and γ are angles in a triangle.

(a) Show that $\cos\gamma = -\cos(\alpha + \beta)$. [2]

(b) Given that $\sin\alpha = \frac{5}{13}$ and $\cos\beta = -\frac{3}{5}$, without solving for α, β and γ, find the value of $\cos\gamma$. [5]

TURN OVER

4 **[Maximum mark: 7]**

It is given that $h(x) = \frac{1}{\sqrt[3]{8-3x}}$.

(a) **(i)** Find the binomial expansion for $h(x)$, in ascending powers of x, up to and including the term in x^2. Give the coefficients as **exact** fractions in their simplest form.

(ii) State the range of values of x for which the expansion is valid. [4]

(b) By putting $x = \frac{1}{16}$ into the expansion in **(a)(i)**, find an approximate value of $\sqrt[3]{16}$, as a fraction in its lowest terms, in the form $\frac{a}{b}$, where a and b are integers. [3]

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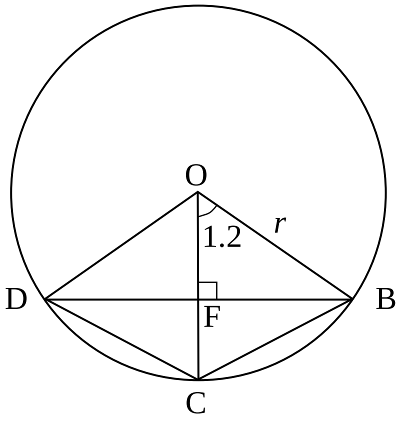
5 **[Maximum mark: 6]**

The diagram below shows a circle with centre O and radius *r* cm. The points B, C and D lie on the circumference of the circle. F is the point of intersection of lines BD and OC and BD is perpendicular to OC.



It is given that $\angle \mathrm{FOB} = 1.2$ radians.

(a) Find BF in terms of *r*. [1]

(b) Given that the area bounded by CF, BF and the minor arc BC is $20\ \mathrm{cm}^2$, find the value of *r*. [5]

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6 **[Maximum mark: 10]**

(a) Using the result

$$\sum_{r=1}^{n}\left(\frac{r}{2^r}\right) = 2 - \frac{n+2}{2^n},$$

show that

$$\sum_{r=1}^{n}(r-n)(2^{-r}+1) = 2\left(1-\frac{1}{2^n}\right) - \frac{1}{2}n(n+1)\,.$$

[7]

(b) Find the positive values of *n* such that

$$\sum_{r=1}^{n}(r-n)(2^{-r}+1) > -\ln 12\,.$$

[3]

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Do NOT write solutions on this page

SECTION B (45 marks)

Answer all questions on the foolscap paper provided. ***Please start each question on a new page.***

7 **[Maximum Mark: 15]**

Let $f(x) = \frac{\pi}{2}x - x \arctan x$.

(a) Find the value of $f''(1)$. [2]

(b) Using L'Hopital's Rule, show that as $x \to \infty$, $y \to 1$. [4]

(c) Explain why the inverse function f^{-1} exists. [1]

(d) Find the composite function $(f \circ f^{-1})(x)$, stating clearly its domain. [2]

(e) Solve the equation $f(x) = f^{-1}(x)$. [3]

The graph of $y = f(x)$ is mapped onto the graph of $y = g(x)$ through a sequence of transformations as follows:

I: a translation of magnitude 1 unit in the direction of the x-axis;

II: a stretch parallel to the x-axis by a factor of 2;

III: a stretch parallel to the y-axis by a factor of 4.

(f) Point $\mathrm{P}\left(1, \frac{\pi}{4}\right)$ lies on the graph of $y = f(x)$. Find the image of P on the graph of $y = g(x)$. [3]

8 **[Maximum Mark: 12]**

Let $\mathrm{e}^{xy} = -xy$, where $x > 0$ and $y < 0$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}$. [3]

(b) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2}{x}\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

(c) Find the value of k such that $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -\frac{k}{x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. [3]

(d) Hence, deduce and simplify an expression for $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}$ in terms of n, x and y. [4]

TURN OVER

9 **[Maximum Mark: 18]**

The function f is defined by $f(x) = \frac{12+16x-x^2}{x-6}$, $x \neq 6$.

(a) Express f in the form $f(x) = A + Bx + \frac{C}{x-6}$, where A, B and C are constants to be determined. [4]

(b) Sketch the graph of $y = f(x)$, indicating clearly the coordinates of any turning points, axes intercepts and asymptotes. [5]

(c) **(i)** Show that $f'(x) < 0$ for all $x \in \mathbb{R}, x \neq 6$.

(ii) Explain whether $f(x)$ is a decreasing function. [4]

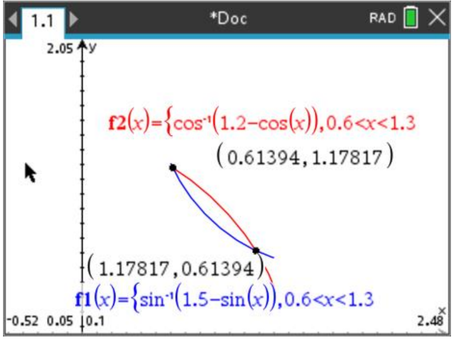
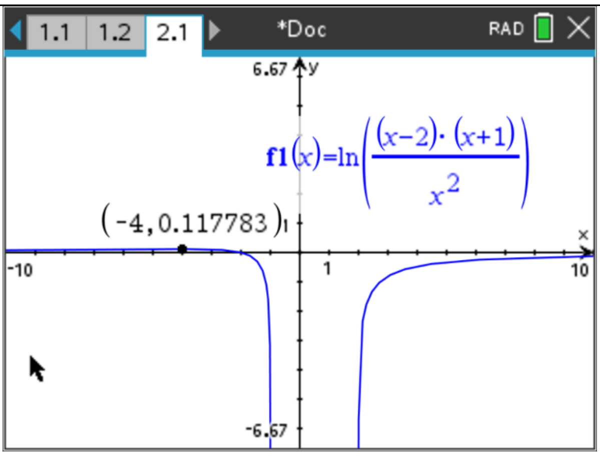
Let $g(x) = kx$, where $k \in \mathbb{R}$.

(d) Find the range of values of k for which the graphs of $y = f(x)$ and $y = g(x)$ do not intersect. [5]

End of Paper

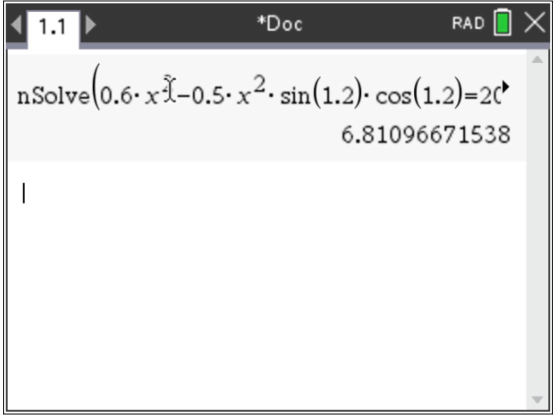
Year 5 HL MAA End of Year Examination 2021 Paper 2 (Markscheme)

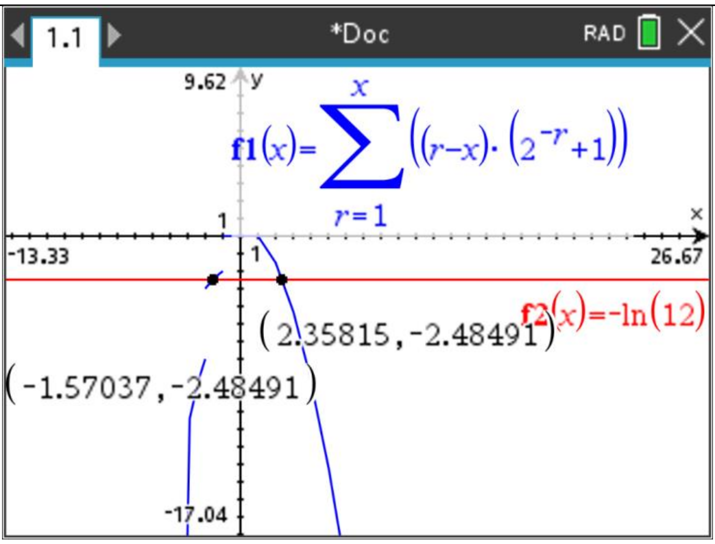
Section A

Qn	Suggested solution	Markscheme
1	***Trigonometry (Solving Equations)***	**[Marks: 4]**
	Rearrange equations in terms of y: $$y = \sin^{-1}(1.5 - \sin x)$$ $$y = \cos^{-1}(1.2 - \cos x)$$	**M1**
	Using GDC,  	**M1**
	$x = 0.614, y = 1.18,$	**A1**
	$x = 1.18, y = 0.614$	**A1**
2	***Differentiation***	**[Marks: 6]**
(a)	$y = \ln\frac{(x-2)(x+1)}{x^2} = \ln(x-2) + \ln(x+1) - \ln x^2$	**M1**
	$\frac{dy}{dx} = \frac{1}{x-2} + \frac{1}{x+1} - \frac{2}{x}$	**A1 A1 A1 o.e.**
(b)	 	**M1**
	Maximum $y = 0.118$	**A1**
3	***Trigonometry (Special angles, Formulae)***	**[Marks: 7]**
(a)	$$\cos\gamma = \cos[\pi - (\alpha + \beta)]$$	**M1**
	$$= \cos\pi\cos(\alpha+\beta) + \sin\pi\sin(\alpha+\beta)$$	**A1**

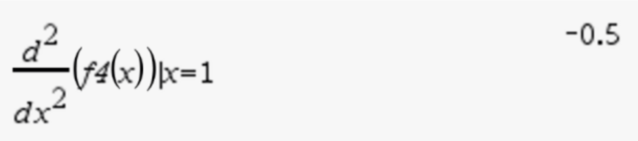
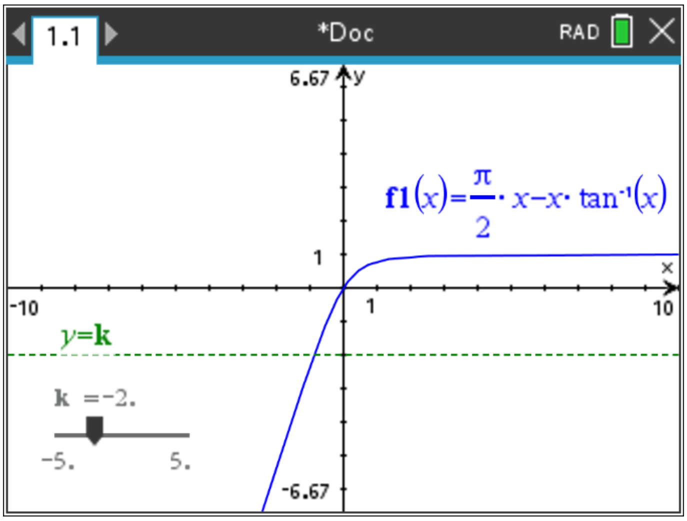
Qn	Suggested solution	Markscheme
	$= -\cos(\alpha+\beta)$	**AG**
(b)	$\cos\beta = -\frac{3}{5}$ implies β is in the 2nd quadrant, and α is in the 1st quadrant. β is obtuse and α is acute $\cos\gamma$ $= -\cos(\alpha+\beta)$ $= -\cos\alpha\cos\beta + \sin\alpha\sin\beta$ $= -\frac{12}{13}\left(-\frac{3}{5}\right) + \frac{5}{13}\left(\frac{4}{5}\right)$ $= \frac{56}{65}$	**(R1)** **M1** **A1** - $\frac{12}{13}$, **A1** - $\frac{4}{5}$ **A1**

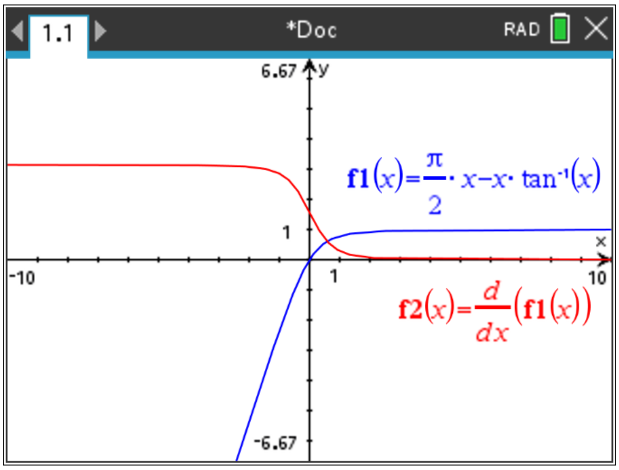
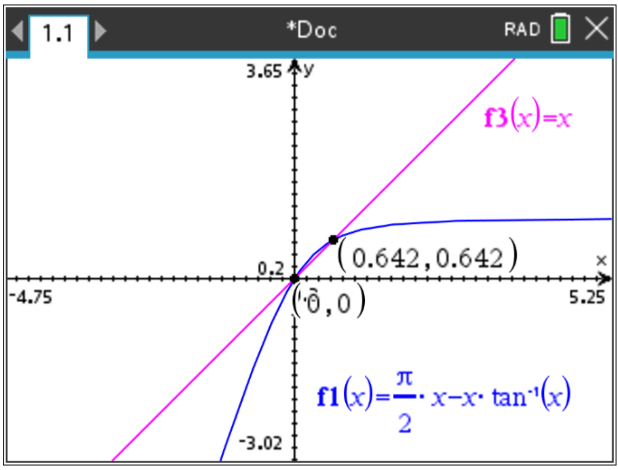
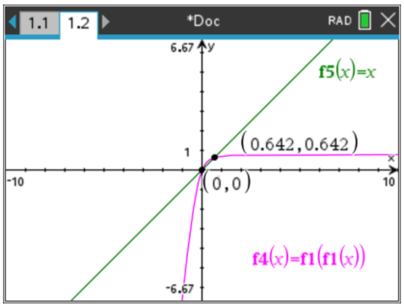
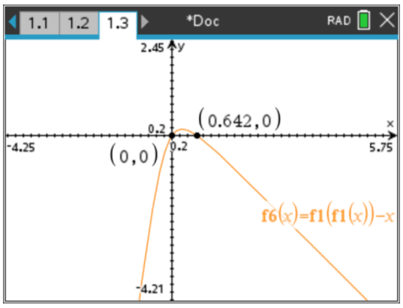
Qn	**Suggested solution**	**Markscheme**
4	***Binomial Expansion***	**[Marks: 7]**
(ai)	$\frac{1}{\sqrt[3]{8-3x}}$ $=(8-3x)^{-\frac{1}{3}}$	
	$=\frac{1}{2}\left(1-\frac{3}{8}x\right)^{-\frac{1}{3}}$	**M1 o.e.**
	$=\frac{1}{2}\left[1+\left(-\frac{1}{3}\right)\left(-\frac{3}{8}x\right)+\binom{-\frac{1}{3}}{2}\left(-\frac{3}{8}x\right)^2+\cdots\right]$	**A1**
	$=\frac{1}{2}\left[1+\frac{1}{8}x+\frac{1}{32}x^2+\cdots\right]$	
	$=\frac{1}{2}+\frac{1}{16}x+\frac{1}{64}x^2+\cdots$	**A1**
(aii)	$\lvert x\rvert<\frac{8}{3}$ or $-\frac{8}{3}<x<\frac{8}{3}$	**A1**
(b)	$\frac{1}{\sqrt[3]{8-3\left(\frac{1}{16}\right)}}=\frac{1}{2}+\frac{1}{16}\left(\frac{1}{16}\right)+\frac{1}{64}\left(\frac{1}{16}\right)^2+\cdots$	
	$\frac{1}{\sqrt[3]{\frac{125}{16}}}=\frac{1}{2}+\frac{1}{16}\left(\frac{1}{16}\right)+\frac{1}{64}\left(\frac{1}{16}\right)^2+\cdots$	**M1**
	$\frac{\sqrt[3]{16}}{5}=\frac{8257}{16384}$	**A1 ft**
	$\sqrt[3]{16}=\frac{41285}{16384}$	**A1 ft**
5	***Trigonometry***	**[Marks: 6]**
(a)	Using trigonometric ratio, $\sin 1.2=\frac{BF}{r}$ $BF=r\sin 1.2$	**A1 (accept 0.932r)**

Qn	Suggested solution	Markscheme
(b)	$OF = r\cos 1.2$	**A1**
	Area of sector OBC $= \frac{1}{2}r^2(1.2) = 0.6r^2$	**A1**
	Area of triangle OBF $= \frac{1}{2}(r\cos 1.2)(r\sin 1.2)$	**A1**
	Hence, $0.6r^2 - \frac{1}{2}(r\cos 1.2)(r\sin 1.2) = 20$ $0.6r^2 - \frac{1}{2}r^2(\cos 1.2)(\sin 1.2) = 20$ Using GDC,	**M1**
	 nSolve(0.6·x²−0.5·x²·sin(1.2)·cos(1.2)=2 6.81096671538 $r = 6.811 \approx 6.81\text{cm}$ (to 3 sf)	**A1**

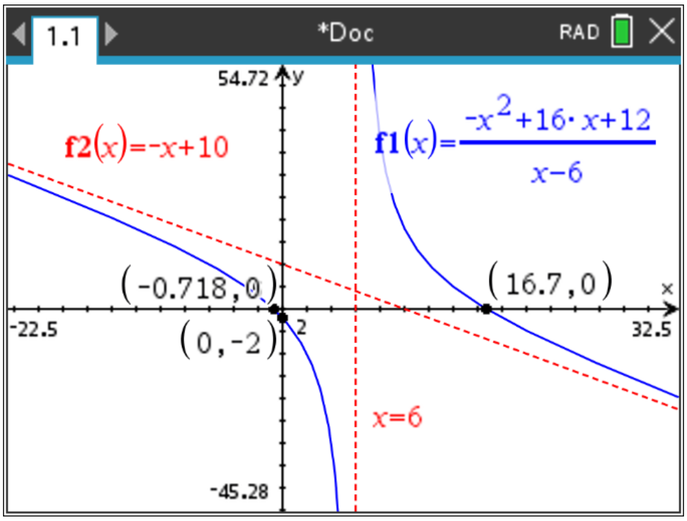
Qn	**Suggested solution**	**Markscheme**
6	***Sigma Notation***	**[Marks: 10]**
(a)	$\sum_{r=1}^{n}(r-n)(2^{-r}+1)$	
	$=\sum_{r=1}^{n}\left(\frac{r}{2^r}\right)-\sum_{r=1}^{n}\left(\frac{n}{2^r}\right)+\sum_{r=1}^{n}(r)-n(n)$	**M1 – arranging**
	$=2-\frac{n+2}{2^n}-n\left[\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}\right]+\frac{n}{2}(1+n)-n(n)$	**A1** $n(n)$ **M1A1 -2nd term GP S_n** **A1 -3rd term AP S_n**
	$=2-\frac{2}{2^n}-\frac{n}{2^n}-n\left(1-\frac{1}{2^n}\right)+\frac{1}{2}n(1+n)-n^2$	**M1**
	$=2\left(1-\frac{1}{2^n}\right)-\frac{n}{2^n}+\frac{n}{2^n}-n+\frac{1}{2}n(1+n)-n^2$	
	$=2\left(1-\frac{1}{2^n}\right)-n(n+1)+\frac{1}{2}n(1+n)$	**A1**
	$=2\left(1-\frac{1}{2^n}\right)-\frac{1}{2}n(1+n)$	**AG**
(b)		**(M1)** **(M1) – both intersections**
	$n=1,2$	**A1**

Section B

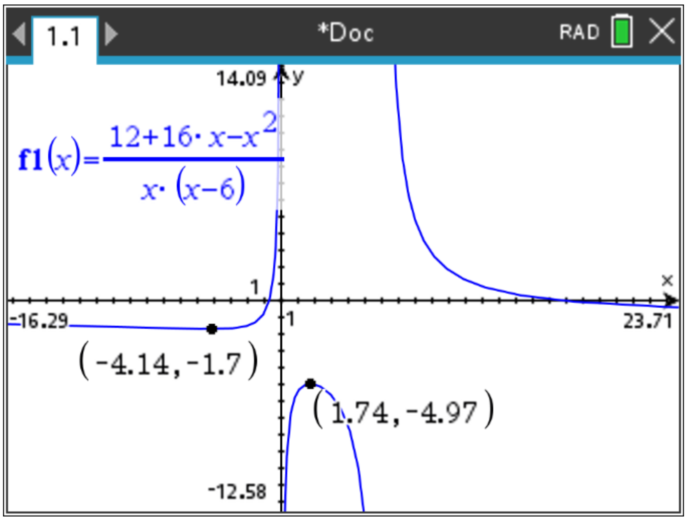
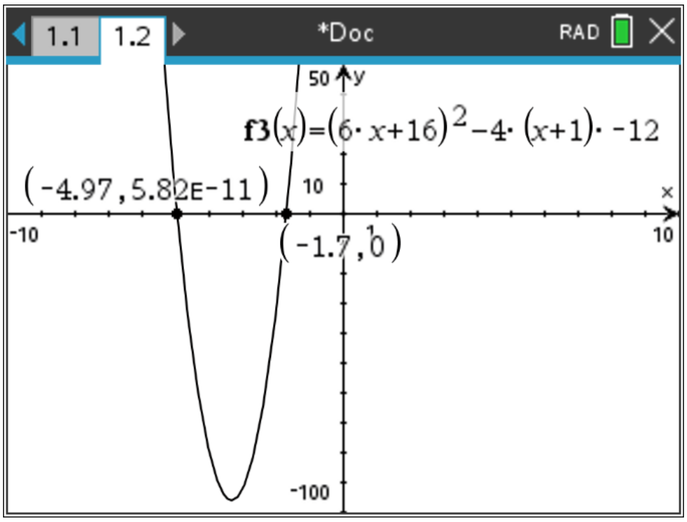
Qn	Suggested Solutions	Marks
7	***Functions and Calculus***	**[Marks: 15]**
(a)	Using GDC graph, plot the graph of $y=\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x)$, and substitute $x=1$.  $\frac{d^2}{dx^2}(f4(x))\vert x=1$ -0.5	**M1 A1**
(b)	Let L be the limit. $L=\lim_{x\to\infty}\left(\frac{\pi}{2}x-x\arctan x\right)$ Observe that the terms are in the form $\infty-\infty$, which can be set up to L'Hopital's rule. $L=\lim_{x\to\infty}\left(x\left(\frac{\pi}{2}-\arctan x\right)\right)$ $L=\lim_{x\to\infty}\frac{\left(\frac{\pi}{2}-\arctan x\right)}{\frac{1}{x}}$ $\left(\frac{0}{0}\text{, ready for L'Hopital's Rule}\right)$ $=\lim_{x\to\infty}\frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}}$ $=\lim_{x\to\infty}\frac{x^2}{1+x^2}\ \left(=\lim_{x\to\infty}\frac{1}{\frac{1}{x^2}+1}\right)$ $=1$	**M1** **M1 A1** **A1** **AG**
(c)	f^{-1} exists since f is 1-1 **EITHER** f passes horizontal line test 	**R1**

Qn	Suggested Solutions	Marks
	OR $f'(x) > 0$ ⇒ Strictly increasing function  	**R1**
(d)	$(f \circ f^{-1})(x) = x,\ \ x \in \mathbb{R}, x < 1$	**A1 A1**
(e)	$f(x) = f^{-1}(x)$ Solve $f(x) = x$ using GDC graph  $x = 0$ or $x = 0.642$ **Alternatively,** $f(x) = f^{-1}(x)$ Solve $f \circ f(x) = x$ Or Solve $f \circ f(x) - x = 0$   	**M1** **A1 A1**

Qn	Suggested Solutions	Marks
(f)	Tracing the sequence of transformations, $\left(1,\frac{\pi}{4}\right) \to \left(2,\frac{\pi}{4}\right) \to (4,\pi)$ Accept non-exact answers: $(1,0.785) \to (2,0.785) \to (4,3.14)$	**M1** any valid method **A1 A1**
8	***Differentiation – Implicit and Higher Order Derivatives***	**[Marks: 12]**
(a)	$\mathrm{e}^{xy} = -xy$ Differentiating implicitly w.r.t. x, $\mathrm{e}^{xy}\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) = -\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)$ $x(\mathrm{e}^{xy}+1)\frac{\mathrm{d}y}{\mathrm{d}x} = -y(\mathrm{e}^{xy}+1)$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y(\mathrm{e}^{xy}+1)}{x(\mathrm{e}^{xy}+1)}, x \neq 0$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}$ **Alternatively**, Taking ln, we have $xy = \ln(-xy)$ $\Rightarrow xy = \ln x + \ln(-y)$ since $x > 0, y < 0$ Differentiating implicitly w.r.t. x, $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = \frac{1}{x} + \frac{1}{-y}(-1)\frac{\mathrm{d}y}{\mathrm{d}x}$ $\Rightarrow \left(x - \frac{1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x} - y$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{1}{x}-y}{x-\frac{1}{y}}$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-xy^2}{x^2y-x}$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(1-xy)}{x(xy-1)} = -\frac{y}{x}$	**M1** Implicit **M1** Product in chain **A1** (collecting $\frac{\mathrm{d}y}{\mathrm{d}x}$) **AG** **M1** Implicit **A1** (collecting $\frac{\mathrm{d}y}{\mathrm{d}x}$) **A1** **AG**
(b)	From (a), we have $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}$. $x\frac{\mathrm{d}y}{\mathrm{d}x} = -y$ Differentiating w.r.t. x, $x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\mathrm{d}y}{\mathrm{d}x}$ $\Rightarrow x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2}{x}\frac{\mathrm{d}y}{\mathrm{d}x}$ **Alternatively**, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right)$ $\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{x\frac{\mathrm{d}y}{\mathrm{d}x}-y}{x^2}$ $\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{x}\left(\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x}\right)$ $\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{x}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x}\right) \qquad \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2}{x}\frac{\mathrm{d}y}{\mathrm{d}x}$	**M1** Implicit **A1** Product **AG** **M1** Implicit **A1** Quotient **AG**

Qn	Suggested Solutions	Marks
(c)	$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2\frac{\mathrm{d}y}{\mathrm{d}x}$ Differentiating again w.r.t. x, $\Rightarrow x\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ $\Rightarrow x\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-3\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ $\Rightarrow \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-\frac{3}{x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ Hence, $k=3$	**M1** Implicit **M1** Product **A1**
(d)	Generalising from $\frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\ldots$ $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}=-\frac{n}{x}\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}$ $=\left(-\frac{n}{x}\right)\left(-\frac{n-1}{x}\frac{\mathrm{d}^{n-2}y}{\mathrm{d}x^{n-2}}\right)$ $=\left(-\frac{n}{x}\right)\left(-\frac{n-1}{x}\right)\ldots\left(-\frac{2}{x}\right)\left(-\frac{1}{x}\right)y$ $=\frac{(-1)^n\,n!}{x^n}y$	**M1** **M1** **A1** **A1**
9	***Functions, Sketching, Discriminant***	**[Marks: 17]**
(a)	By long division, $f(x)=\frac{12+16x-x^2}{x-6}$ $=-x+10+\frac{72}{x-6}$ $\begin{array}{r\|rrr} & -x & +10 & \\ \hline x-6 & -x^2 & +16x & +12 \\ & -x^2 & +6x & \\ \hline & & 10x & +12 \\ & & 10x & -60 \\ \hline & & & 72 \end{array}$	**(M1)** by any valid method, e.g. long division **A1 A1 A1** for each of A, B, C
(b)		**A1** Shape **A1** Vertical asymptote **A1** Oblique asymptote **A1** Both zeros **A1** y-intercept

Qn	Suggested Solutions	Marks
(c,i)	$f'(x) = -1 - \frac{72}{(x-6)^2}\ \ \forall x \in \mathbb{R}\backslash\{6\}$ Since $-\frac{72}{(x-6)^2} < 0$, therefore $f'(x) = -1 - \frac{72}{(x-6)^2}$ < -1 $< 0\ \ \forall x \in \mathbb{R}\backslash\{6\}$	**M1 A1** **R1** **AG**
(c,ii)	No, not decreasing. Counter-example: $f(1) = -\frac{27}{5}$, $f(14) = 5$. That is, although $1 < 14$, $f(1) \ngtr f(14)$ [For decreasing function, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ Although $f'(x) < 0$ where defined, the discontinuity at $x = 6$ is a source of counter-examples where $x_1 < x_2 \nRightarrow f(x_1) < f(x_2)$.]	**A1**

(d)	Equate $f(x) = g(x)$, $\frac{12+16x-x^2}{x-6} = kx$	
	Method 1: Transforming to comparing with $y = k$. $\frac{12+16x-x^2}{x(x-6)} = k$	**M1 A1**
	Sketch $= \frac{12+16x-x^2}{x(x-6)}$. Then $y = k$ is a horizontal line. 	**M1** Sketch Realise that as $x \to \infty, y \to -1$, which doesn't affect the answer.
	$-4.97 < k < -1.70$	**A1 A1**
	Method 2: Using Discriminant	
	$12 + 16x - x^2 = kx^2 - 6kx$ $(k+1)x^2 - (6k+16)x - 12 = 0$	**M1**
	For no intersection, we require $\Delta < 0$ $\Delta = (6k+16)^2 - 4(k+1)(-12) < 0$ Using GDC graph,	**M1**
		**M1**
	$-4.97 < k < -1.70$	**A1 A1**

STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

ST. JOSEPH'S INSTITUTION
YEAR 5 END OF YEAR EXAMINATION 2022

MATHEMATICS: ANALYSIS AND APPROACHES

HIGHER LEVEL

PAPER 1

Thursday

6 October 2022

1 hour 30 minutes

0800 – 0930 hrs

INSTRUCTIONS TO CANDIDATES

- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the foolscap paper provided.
- The use of calculators is **not** permitted in this paper.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is **[85 marks]**.
- This question paper consists of **9** printed pages including the Cover Sheet.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	TOTAL
								/ 85

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A (40 marks)

Answer **all** questions in the **spaces** provided.

1 [Maximum mark: 5]

In the expansion of $\left(k-\frac{1}{x}\right)^9$, where k is a non-zero real constant, the coefficient of the term in x^{-7} is –144. Find the possible values of k.

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2 **[Maximum mark: 6]**

Solve the equation $2\cos 2x - 4\cos x = 1$, for $-\pi \le x \le \pi$.

TURN OVER

3 **[Maximum mark: 7]**

Consider the cubic function $f\left(x\right)=-3x^3+8x^2-9x+10$, for $x\in\mathbb{R}$.

(a) Show that x = 2 is a root of $f(x)=0$. [1]

(b) Express $f\left(x\right)$ as a product of two factors. [3]

(c) Hence, show that there are exactly two real solutions for the equation

$$-3y^6+8y^4-9y^2+10=0.$$ [3]

4 **[Maximum mark: 7]**

The curve C has equation $x^2 \tan y = 9$, where $0 < y < \frac{\pi}{2}$ and $x \neq 0$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, simplifying your answer in terms of x. [4]

(b) Given that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{54(x^4 - 27)}{(x^4 + 81)^2}$, explain why the curve has a point of inflexion at $x = \sqrt[4]{27}$. [3]

TURN OVER

5 **[Maximum mark: 9]**

The complex numbers *u* and *w* are defined by $u = -1 + 7\mathrm{i}$ and $w = 3 + 4\mathrm{i}$ where $\mathrm{i}^2 = -1$.

(a) Find $u - 2w$ and $\frac{u}{w}$ in the form $x + \mathrm{i}y$, where *x* and *y* are real. [3]

Points A and B represent the complex numbers *u* and *w* respectively on an Argand diagram with O as the origin.

(b) Given that angle AOB = $\arg\left(\frac{u}{w}\right)$, find angle AOB. [2]

(c) Hence show that $\arctan\left(\frac{4}{3}\right) + \arctan 7 = \frac{3\pi}{4}$. [4]

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6 **[Maximum mark: 6]**

The table below shows the values of two functions *p* and *q* and their first derivatives when $x=\sqrt{3}$, $x=\frac{1}{\sqrt{3}}$ and $x=1$.

x	$p(x)$	$p'(x)$	$q(x)$	$q'(x)$
$\sqrt{3}$	2	$\sqrt{2}$	1	$\frac{1}{2}$
$\frac{1}{\sqrt{3}}$	1	5	$-\pi$	$-\sqrt{2}$
1	π	-2	-1	3

Find the derivative of the composite function $(p\circ q)(\cot 2x)$ when $x=\frac{\pi}{12}$.

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*Do **NOT** write solutions on this page.*

SECTION B (45 marks)

Answer **all** questions on the foolscap paper provided. **Please start each question on a new page.**

7 [Maximum mark: 20]

Consider an arithmetic series

$$\ln x + \ln\left(x^a\right) + \ln\left(\sqrt{x}\right) + \ldots \text{ where } x \in \mathbb{R},\ x > 1 \text{ and } a \in \mathbb{R},\ a \neq 0.$$

(a) Find the value of a. [3]

(b) Find the common difference of the series in the form $d \ln x$ where $d \in \mathbb{Q}$. [2]

(c) The sum of the first n terms of the series is equal to $\ln\left(x^{-4.5}\right)$.

Find the value of n. [5]

Consider the geometric series

$$\lg x + \lg\left(x^b\right) + \lg\left(\sqrt{x}\right) + \ldots \text{ where } x \in \mathbb{R},\ x > 1 \text{ and } b \in \mathbb{R},\ b \neq 0.$$

(d) Find the possible values of b. [5]

(e) The sum to infinity of the above series is equal to $2+\sqrt{2}$ when $b > 0$.

Find the value of x. [5]

8 [Maximum mark: 25]

The function f is defined by $f(x) = \dfrac{1}{x^2 - 2x - 8}$, where $x \in \mathbb{R}$, $x \neq p$, $x \neq q$ and $p < q$.

(a) Find the value of p and of q. [3]

(b) Sketch the graph of $y = f(x)$, clearly indicating any asymptotes with their equations. State clearly the coordinates of any local maximum or minimum points and any points of intersection with the coordinate axes. [5]

The function g is defined by $g(x) = \dfrac{1}{x^2 - 2x - 8}$, where $x \in \mathbb{R}$, $x > 4$.

(c) Find the inverse function of g and state its domain. [7]

(d) Sketch the graph of $y = g^{-1}(x)$ on the same sketch as the graph of $y = f(x)$. [3]

[Question 8 continues on the next page]

The function h is defined by $h(x)=\arctan\frac{x}{7}$, where $x\in\mathbb{R}$.

(e) Find the value of m such that $(h\circ g)(m)=\frac{\pi}{4}$.

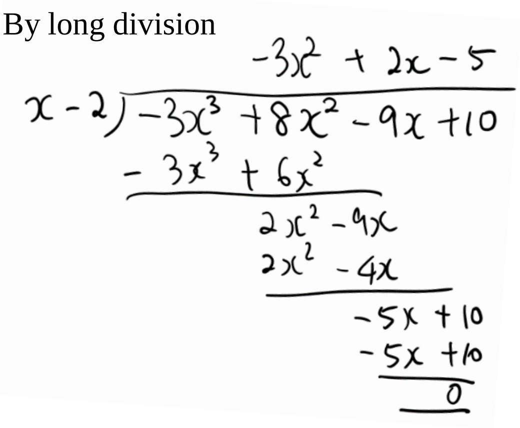
Give your answer in the form $a+\frac{b}{7}\sqrt{c}$, where a, b, $c\in\mathbb{Z}^{+}$. [7]

End of Paper

Year 5 Mathematics: Analysis and Approaches HL End of Year Examination 2022 Paper 1 (Mark Scheme)

Qn	**Suggested Solutions for Section A**	**Marks**
1	**SL1.9 Binomial theorem**	**[Maximum mark: 5]**
	General term	
	$= \binom{9}{r}\left(-\frac{1}{x}\right)^{r} k^{9-r}$	**(M1)** **A1**
	For $x^{-7}, r = 7$	
	Identifying the correct term as	
	$\binom{9}{7} k^{2}\left(-\frac{1}{x}\right)^{7}$ or $\binom{9}{2}\left(-\frac{1}{x}\right)^{7} k^{2}$	**M1**
	Thus $\binom{9}{7} k^{2}(-1)^{7} = -144$	
	Since $\binom{9}{2} = 36$ or $\binom{9}{7} = 36$	**A1**
	$k^{2} = 4$	
	$k = \pm 2$	**A1**
2	**SL3.6, SL3.8 Double angle, Trigo equation**	**[Maximum mark: 6]**
	$2\cos 2x - 4\cos x = 1$	
	$4\cos^{2}x - 2 - 4\cos x - 1 = 0$	**M1**
	$4\cos^{2}x - 4\cos x - 3 = 0$	
	Using $4\cos^{2}x - 4\cos x - 3 = 0$,	
	$(2\cos x + 1)(2\cos x - 3) = 0$	**M1**
	$\cos x = -\frac{1}{2}$ or	**A1**
	$\cos x = \frac{3}{2}$ (rejected because $-1 \leq \cos x \leq 1$)	**A1 (with rejection)**
	Reference angle = $\frac{\pi}{3}$	
	$x = -\frac{2\pi}{3}$ or $\frac{2\pi}{3}$	**A1A1**

Year 5 Mathematics: Analysis and Approaches HL End of Year Examination 2022 Paper 1 (Mark Scheme)

Qn	Suggested Solutions for Section A	Marks
3	**AHL2.12 Polynomial functions**	**[Maximum mark: 7]**
(a)	$f(2) = -3(8) + 8(4) - 9(2) + 10$ $f(2) = 0$ Therefore by Factor Theorem, $x = 2$ is a root of $f(x) = 0$.	**A1** **AG**
(b)	Since $f(2) = 0$, **METHOD ONE** $-3x^3 + 8x^2 - 9x + 10$ $= (x-2)(-3x^2 + bx - 5)$, where b is a constant. Compare coeff of x: $-9 = -2b - 5$ $b = 2$ $f(x) = (x-2)(-3x^2 + 2x - 5)$ **METHOD TWO** By long division  Therefore $f(x) = (x-2)(-3x^2 + 2x - 5)$	**(M1)** **by inspection** **A1** **A1** **(M1)** **A1** **A1**
(c)	$-3y^6 + 8y^4 - 9y^2 + 10 = 0$ $\Rightarrow (y^2 - 2)(-3y^4 + 2y^2 - 5) = 0$ $\Rightarrow y^2 = 2$ or $3y^4 - 2y^2 + 5 = 0$ Since $y = \pm\sqrt{2}$ and $3y^4 - 2y^2 + 5 = 0$ has no real solution as its discriminant = $-56 < 0$, therefore $-3y^6 + 8y^4 - 9y^2 + 10 = 0$ has exactly two real solutions which are $y = \pm\sqrt{2}$.	**M1** **(for replacement of *x*)** **A1** **R1** **(discriminant value must be evaluated and shown explicitly and must be correct)** **AG**

Qn	**Suggested Solutions for Section A**	**Marks**
4	**SL5.8, AHL5.14 Inflexion point, Implicit differentiation**	**[Maximum mark: 7]**
(a)	**Method One:** $x^2 \tan y = 9$ Differentiating wrt x,	
	$$2x\tan y + x^2\sec^2 y\,\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$	**A1A1** **(for each term)**
	Using $\sec^2 y = 1 + \tan^2 y$,	**M1** (use $\sec^2 y$ identity)
	$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x \times \frac{9}{x^2}}{x^2\left(1 + \frac{81}{x^4}\right)}$$	
	$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-18x}{x^4 + 81}$$	**A1**
	Method Two: Make $\tan y$ the subject. $$\tan y = \frac{9}{x^2}$$	
	$$\sec^2 y\,\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{18}{x^3}$$	**A1A1** **(LHS and RHS)**
	$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{18}{x^3} \times \cos^2 y$$	**M1** **(use toa-cah-soh)**
	$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{18}{x^3} \times \frac{x^4}{x^4 + 81}$$	
	$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-18x}{x^4 + 81}$$	**A1**
	Method Three: $$y = \tan^{-1}\left(\frac{9}{x^2}\right)$$	**M1** **(make y the subject)**
	$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1 + \left(\frac{9}{x^2}\right)^2} \times \left(\frac{(-2)(9)}{x^3}\right)$$	**A1A1** **(arctan deriv and chain rule)**
	$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-18x}{x^4 + 81}$$	**A1**

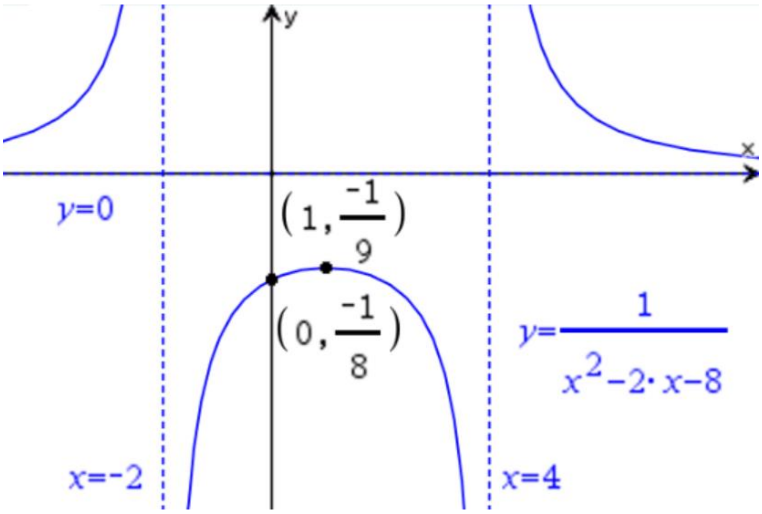
Qn	**Suggested Solutions for Section A**	**Marks**
(b)	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{54(x^4-27)}{(x^4+81)^2}$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ $x^4 - 27 = 0$ $x = \sqrt[4]{27}$	**M1** (find x)
	As x approaches $\left(\sqrt[4]{27}\right)^-$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$; As x approaches $\left(\sqrt[4]{27}\right)^+$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}x > 0.$	**A1** (correct sign of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$)
	Since $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ changes sign as x approaches $\left(\sqrt[4]{27}\right)^-$ to $\left(\sqrt[4]{27}\right)^+$	**R1**
	Therefore at $= \sqrt[4]{27}$, C has a point of inflexion.	**AG**
5	**AHL1.12 Complex numbers (modulus and argument)**	**[Maximum mark: 9]**
(a)	$u - 2w$ $= -1 + 7\mathrm{i} - (6 + 8\mathrm{i})$ $= -7 - \mathrm{i}$	**A1**
	$\frac{u}{w} = \frac{(-1+7\mathrm{i})(3-4\mathrm{i})}{(3+4\mathrm{i})(3-4\mathrm{i})}$	**M1** (multiply with the correct conjugate)
	$\frac{-3+21\mathrm{i}+4\mathrm{i}+28}{9+16}$ $= 1 + \mathrm{i}$	**A1**
(b)	Angle AOB $= \arg\left(\frac{u}{w}\right)$ $= \arctan(1)$	**M1**
	$= \frac{\pi}{4}$ (next year you will learn that $\arg(u) - \arg(w) = \arg\left(\frac{u}{w}\right)$)	**A1**

Year 5 Mathematics: Analysis and Approaches HL End of Year Examination 2022 Paper 1 (Mark Scheme)

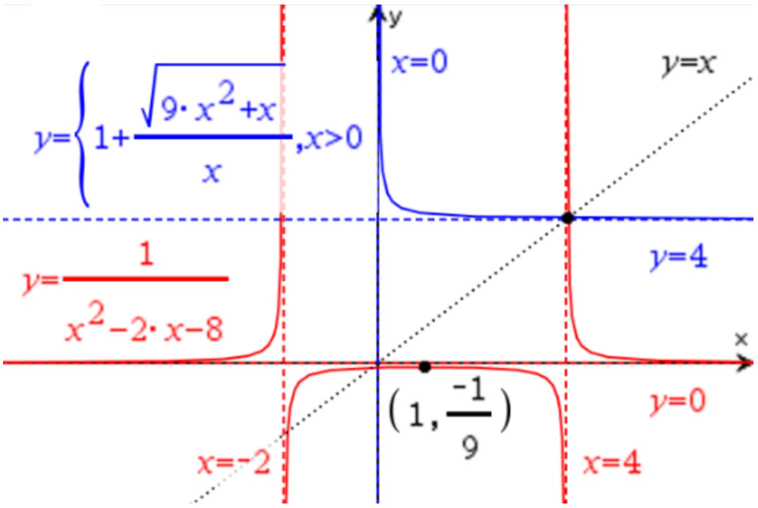
Qn	Suggested Solutions for Section A	Marks
(c)	Since $\arg(w) = \text{arc tan}\left(\frac{4}{3}\right)$ and $\arg(u) = \pi - \text{arc tan}\left\lvert\frac{7}{-1}\right\rvert$ (because $\arg(u)$ lies in the 2nd quadrant) $\Rightarrow \text{arc tan}\ 7 = \pi - \arg(u)$ $\text{arc tan}\ 7 = \pi - [\ \arg(w) + \left(\frac{\pi}{4}\right)\ ]$ $\text{arc tan}\left(\frac{4}{3}\right) + \text{arc tan}\ 7 = \pi - \left(\frac{\pi}{4}\right)$ $\therefore \text{arc tan}\left(\frac{4}{3}\right) + \text{arc tan}\ 7 = \frac{3\pi}{4}$	**A1** **(value of arg(w))** **M1** **(find arg (*u*) in 2nd quad)** **A1** **(follows the M1 above; arg (*u*) in terms of arg (*w*) and angle AOB)** **A1 (AG)**
6	**AHL5.15 Differentiation techniques**	**[Maximum mark: 6]**
	$\frac{\mathrm{d}}{\mathrm{d}x}(p\circ q)(\cot 2x)$ $= \underbrace{(p'(\ q(\cot 2x)\ )\times q'(\cot 2x)}_{*}\times(-2\operatorname{cosec}^2 2x)$ $\left.\frac{\mathrm{d}}{\mathrm{d}x}(p\circ q)(\cot 2x)\right\rvert_{x=\frac{\pi}{12}}$ $= (p'(\ q(\cot\frac{\pi}{6})\ )\times q'(\cot\frac{\pi}{6})\times(-2\operatorname{cosec}^2\frac{\pi}{6})$ $= (p'(\ q(\sqrt{3})\times q'(\sqrt{3})\times(-2\times\frac{1}{0.25})$ $= (\ p'(1)\ )\times\frac{1}{2}\times(-8)$ $= 8$	**M1A1** (chain rule) * **A1** (for $-2\operatorname{cosec}^2 2x$) **(M1)** **sub** **A1** (for $\cot\frac{\pi}{6}=\sqrt{3}$) **A1**

Year 5 Mathematics: Analysis and Approaches HL End of Year Examination 2022 Paper 1 (Mark Scheme)

Qn	Suggested Solutions for Section B	Marks
7	**SL1.2, SL1.3, SL1.5, SL1.7, SL1.8 – Arithmetic Series, Geometric Series, Exponents and Logarithms**	**[Maximum mark: 20]**
(a)	$\ln(x^a)-\ln x=\ln(\sqrt{x})-\ln(x^a)$ $a-1=\frac{1}{2}-a$ $a=\frac{3}{4}$	**M1 for equating common difference** **A1** **A1**
(b)	Common difference = $\ln(x^a)-\ln x$ or $\ln(\sqrt{x})-\ln(x^a)$ Common difference = $\ln(x^{a-1})$ or $\ln\left(x^{\frac{1}{2}-a}\right)$ Common difference = $-\frac{1}{4}\ln x$	**M1** **A1 FT**
(c)	$S_n=\frac{n}{2}\left[2\ln x+(n-1)\left(\frac{-1}{4}\ln x\right)\right]$ $S_n=\left(n+\frac{n-n^2}{8}\right)\ln x$ or $\left(\frac{9n-n^2}{8}\right)\ln x$ or $\left(\frac{n(9-n)}{8}\right)\ln x$ $n+\frac{n-n^2}{8}=-4.5$ or $\frac{9n-n^2}{8}=-4.5$ or $\frac{n(9-n)}{8}=-4.5$ $n^2-9n-36=0$ $(n+3)(n-12)=0$ Since $n>0$, $n=12$	**M1** **A1 FT** **M1 for equating sum** **A1 FT for correct factorisation** **A1 FT**
(d)	$\frac{\lg(x^b)}{\lg x}=\frac{\lg(\sqrt{x})}{\lg(x^b)}$ $\frac{b}{1}=\frac{\frac{1}{2}}{b}$ $b^2=\frac{1}{2}$ $b=\pm\frac{1}{\sqrt{2}}$ or $\pm\frac{\sqrt{2}}{2}$ **Do NOT accept** $b=\pm\sqrt{\frac{1}{2}}$ **.**	**M1 for equating common ratio** **M1 for power law** **A1** **A1 A1**

Qn	Suggested Solutions for Section B	Marks
(e)	$S_\infty = \frac{\lg x}{1-b} = \frac{\lg x}{1-\frac{\sqrt{2}}{2}}$ $S_\infty = \frac{2\lg x}{2-\sqrt{2}}$ $\frac{2\lg x}{2-\sqrt{2}} = 2+\sqrt{2}$ $\lg x = 1$ $x = 10$	**M1** **A1 FT for using $b > 0$** **M1 for equating sum** **A1 FT for correct simplifying using $b > 0$** **A1 FT for using $b > 0$**
8	**SL2.2 , SL2.3, SL2.5, SL2.6, SL2.7, SL2.8, SL2.10, AHL2.13, AHL2.14, AHL2.16, AHL3.9 – Function and Graph, Rational Function, Reciprocal Function, Inverse Function, Composite Function and Inverse Trigonometric Function**	**[Maximum mark: 25]**
(a)	$x^2-2x-8=(x+2)(x-4)$ $p=-2$, $q=4$	**M1** **A1 A1**
(b)	 Shape of the graph of $y = f(x)$ with 3 parts or sections Vertical asymptotes: $x = -2$, $x = 4$ Horizontal asymptote: $y = 0$ (x-axis) Coordinates of maximum point: $\left(1, -\frac{1}{9}\right)$ Coordinates of y-intercept: $\left(0, -\frac{1}{8}\right)$	**A1** **A1** **A1** **A1** **A1**

Qn	Suggested Solutions for Section B	Marks
(c)	Let $y = g(x)$ and replace/substitute y by x.	**M1**
	$y=\frac{1}{x^2-2x-8}, x>4$	
	Either	
	$y=\frac{1}{(x-1)^2-9}, x>4$	**M1 for completing square**
	$(x-1)^2-9=\frac{1}{y}, x>4$	
	$(x-1)^2=9+\frac{1}{y}, x>4$ or $(x-1)^2=\frac{9y+1}{y}, x>4$	**M1 for simplifying**
	Since $x>4$, $x-1=\sqrt{9+\frac{1}{y}}$ or $x-1=\sqrt{\frac{9y+1}{y}}$	**A1** **A1 for rejecting with reason, $x>4$**
	Or	
	$yx^2-2yx-8y=1, x>4$ $yx^2-2yx-8y-1=0, x>4$	**M1 for quadratic equation in x**
	$x=\frac{2y\pm\sqrt{4y^2+4y(8y+1)}}{2y}, x>4$	**M1 for using general solution formula**
	Since $x>4$, $x=1+\frac{\sqrt{36y^2+4y}}{2y}$	**A1** **A1 for rejecting with reason, $x>4$**
	$x=1+\sqrt{9+\frac{1}{y}}$ or $x=1+\sqrt{\frac{9y+1}{y}}$ or $x=1+\frac{\sqrt{9y^2+y}}{y}$	
	Replace or substitute x by y gives	
	$y=1+\sqrt{9+\frac{1}{x}}$ or $y=1+\sqrt{\frac{9x+1}{x}}$ or $y=1+\frac{\sqrt{9x^2+x}}{x}$	
	$g^{-1}(x)=1+\sqrt{9+\frac{1}{x}}, x\in\mathbb{R}, x>0$ or	**A1 for $g^{-1}(x)$ = rule** **A1 for domain**
	$g^{-1}(x)=1+\sqrt{\frac{9x+1}{x}}, x\in\mathbb{R}, x>0$ or	
	$g^{-1}(x)=1+\frac{\sqrt{9x^2+x}}{x}, x\in\mathbb{R}, x>0$	

Qn	Suggested Solutions for Section B	Marks
(d)		
	Shape of the graph of $y = g^{-1}(x)$ **intersecting** the graph of $y = f(x)$ **only once at $x > 4$**	**A1**
	Vertical asymptotes: $x = 0$ (y-axis)	**A1 FT**
	Horizontal asymptote: $y = 4$	**A1 FT**
	Do NOT penalise for omission of the line $y = x$.	
(e)	$h(x) = \arctan\frac{x}{7}$	
	$h(g(m)) = \arctan\left(\frac{g(m)}{7}\right)$ or $h^{-1}(x) = 7\tan x$	**M1 for composite function or inverse function of *h***
	$h(g(m)) = \frac{\pi}{4}$ gives $\arctan\left(\frac{g(m)}{7}\right) = \frac{\pi}{4}$ or $g(m) = h^{-1}\left(\frac{\pi}{4}\right)$	
	$g(m) = 7\tan\frac{\pi}{4} = 7$	**A1**
	Either	
	$m = g^{-1}(7)$	**A1**
	$m = 1 + \frac{\sqrt{9(7^2)+7}}{7}$	**M1 using $g^{-1}(x)$**
	$m = 1 + \frac{\sqrt{7(9\times7+1)}}{7} = 1 + \frac{\sqrt{7\times64}}{7} = 1 + \frac{8}{7}\sqrt{7}$	**A1 for a = 1** **A1 for b = 8** **A1 for c = 7**

Qn	Suggested Solutions for Section B	Marks
	Or $\frac{1}{m^2-2m-8}=7$, $m>4$ gives $7m^2-14m-57=0$, $m>4$ $m=\frac{14\pm\sqrt{(14)^2+4(7)(57)}}{2(7)}$ $m=\frac{14\pm\sqrt{4(7)(7+57)}}{2(7)}=\frac{14\pm\sqrt{4(7)(64)}}{2(7)}=1\pm\frac{\sqrt{7\times64}}{7}$ $m=1\pm\frac{8}{7}\sqrt{7}$ Since $m>4$, $m=1+\frac{8}{7}\sqrt{7}$	**A1 for quadratic equation** **M1 using general solution for quadratic equation** **A1 for *a* = 1** **A1 for *b* = 8** **A1 for *c* = 7**

STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

ST. JOSEPH'S INSTITUTION
YEAR 5 END OF YEAR EXAMINATION 2022

MATHEMATICS: ANALYSIS AND APPROACHES
HIGHER LEVEL
PAPER 2
Wednesday

12 October 2022
1 hour 30 minutes
0800 – 0930 hrs

INSTRUCTIONS TO CANDIDATES

- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A:** Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B:** Answer all questions using the foolscap paper provided.
- The use of a scientific or examination graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted apps and be ram cleared.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures.
- The maximum mark for this examination paper is *[**85** marks]*.
- This question paper consists of **9** printed pages including the Cover Sheet**.**
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	TOTAL
									/85

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to *show all working.*

SECTION A (40 marks)

Answer **all** questions in the **spaces** provided.

1 [Maximum mark: 4]

Find the complex numbers z and w which satisfy the simultaneous equations

$$\frac{z}{w} = 2\mathrm{i} - 1$$

$$(3 + 2\mathrm{i})z = w + 2\mathrm{i}, \quad \text{where } \mathrm{i}^2 = -1.$$

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2 **[Maximum mark: 9]**

(a) Obtain the series expansion of $\sqrt{3+4x}$, up to and including the term in x^2. [4]

(b) Explain why it is not suitable to use $x = 1$ to give an approximation for $\sqrt{7}$. [2]

(c) Use $x = -\frac{1}{6}$ to find an approximation for $\sqrt{7}$, giving your answer **correct to 3 decimal places**. [3]

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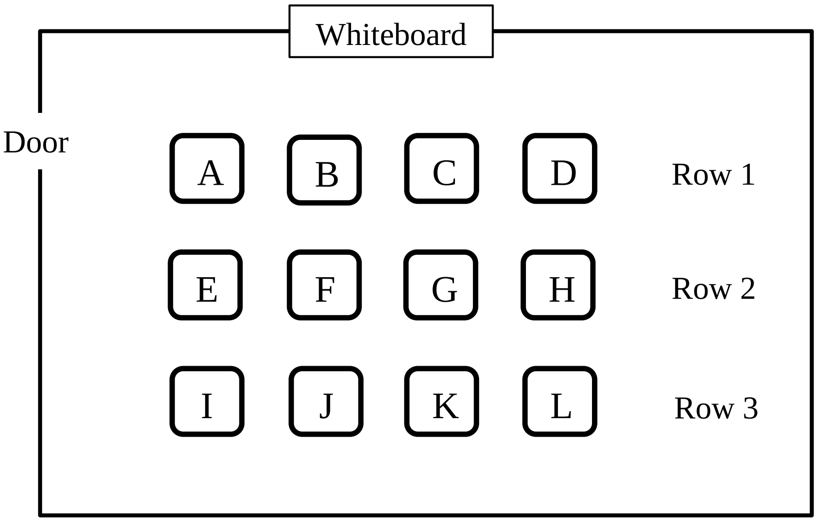
3 **[Maximum mark: 7]**

A classroom has 12 seats, as shown in the diagram below.



Find the number of seating arrangements for Johnston and his 11 classmates if

(a) there are no restrictions; [1]

(b) Johnston wants to sit in Row 1; [2]

(c) two of his classmates Samson and Jackson must not sit next to each other in the same row. [4]

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4 **[Maximum mark: 7]**

Mr. Jay places a 30% deposit for a car and takes out a loan of \$80 500 from a bank for the remaining amount. The bank charges an **annual** interest rate of 3.2% compounded monthly and Mr. Jay makes a payment of \$$x$ to the bank at the end of each month.

(a) Find the minimum value of x, to the nearest integer, if he wants to finish repaying the loan within 7 years. [3]

(b) Write down the purchase price of the car, to the nearest dollar. [1]

(c) If the value of the car depreciates to half its original value at the end of the fifth year, find the annual depreciation rate of the car.

State an assumption made in your calculation. [3]

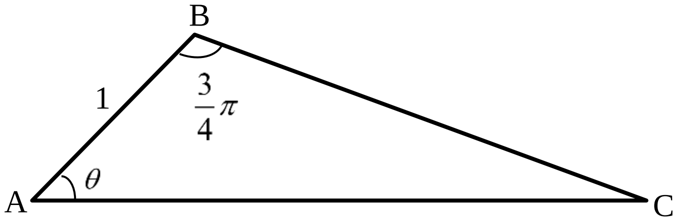
5 **[Maximum mark: 5]**

Using l’Hôpital’s rule, find the exact value of

$$\lim_{x\to 0}\frac{1}{\sqrt{3x}}\left(\frac{\pi}{2}-\arccos\left(\sqrt{x}\right)\right).$$

6 **[Maximum mark: 8]**

Consider a triangle ABC, where AB = 1 cm, $\text{B}\hat{\text{A}}\text{C} = \theta$ rad and $\text{A}\hat{\text{B}}\text{C} = \frac{3}{4}\pi$ rad.



(a) Show that $\text{AC} = \dfrac{1}{\cos\theta - \sin\theta}$. [4]

(b) Given that the area of triangle ABC is $\frac{7}{4}$ cm^2, find the value of θ and length of AC. [4]

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Do NOT write solutions on this page

SECTION B (45 marks)

Answer all questions on the foolscap paper provided. ***Please start each question on a new page.***

7 **[Maximum Mark: 16]**

The function h is defined by $h(x)=\sqrt{16-(x-3)^2}$, $x\in(a,\,b)$, where $a,\,b\in\mathbb{R}$.

(a) Find the value of a and of b such that the domain $D_h=(a,\,b)$ is the largest possible. [2]

(b) Find the equation of the line of symmetry on the graph $y=h(x)$. [2]

(c) The graph of h goes through a translation of $\begin{pmatrix} p \\ 0 \end{pmatrix}$ such that the resulting graph is an even function.

(i) State the value of p.

(ii) Show that $h(x-p)$ is an even function. [4]

(d) Given another function $g(x)=\dfrac{1}{x},\ x\in\mathbb{R},\ x\neq 0$, find

(i) the rule for the function $g\circ h$,

(ii) the range of the function $g\circ h$. [4]

Let $k(x)=-h(3x)$.

(e) The point $\mathrm{A}(m,\,2)$ on the graph of h is mapped to the point A' on the graph of k.

(i) Find the possible values of m.

(ii) Hence find the corresponding coordinates of A'. [4]

Do NOT write solutions on this page

8 [Maximum Mark: 13]

(a) (i) Find the roots of $w^2 = -2\text{i}$ in the form $p + q\text{i}$, where $p, q \in \mathbb{R}$.

(ii) Hence solve $(1 + \text{i}w)^2 = -2\text{i}$. [5]

(b) Suppose $z_1 = -2\text{i}$ and $z_2 = \text{i} - \sqrt{3}$.

(i) Verify that z_1 and z_2 are roots of the equation $z^3 = 8\text{i}$, where $z \in \mathbb{C}$.

(ii) Find the third root z_3 in the form $a + b\text{i}$, where$a, b \in \mathbb{R}$. [6]

(c) On an Argand diagram, z_1, z_2 and z_3 are represented by the points A, B and C respectively.

Find the area of triangle ABC. [2]

9 [Maximum Mark: 16]

Let $f(x) = b(x+1)\text{e}^{-2x}$, where $b > 0, x \in \mathbb{R}.$,

(a) (i) Find $f'(x)$ in terms of b.

(ii) Hence find the coordinates of the maximum point of $y = f(x)$ in terms of b. [4]

(b) (i) Given that $f(0) = 3$, show that $b = 3$.

(ii) Sketch the graph of $y = f(x)$, indicating clearly the coordinates of the turning points, axial intercepts and the equation of the asymptote.

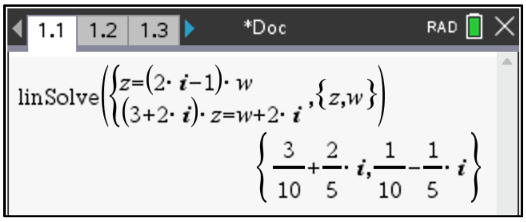
(iii) State $\lim\limits_{x \to \infty} f(x)$.

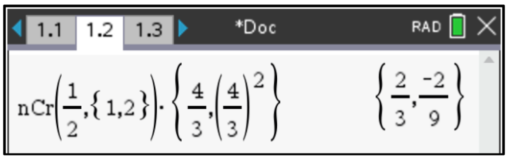
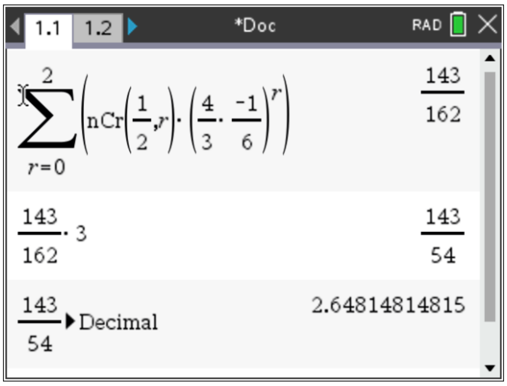
(iv) Find the coordinates of the point of inflexion and justify your answer. [9]

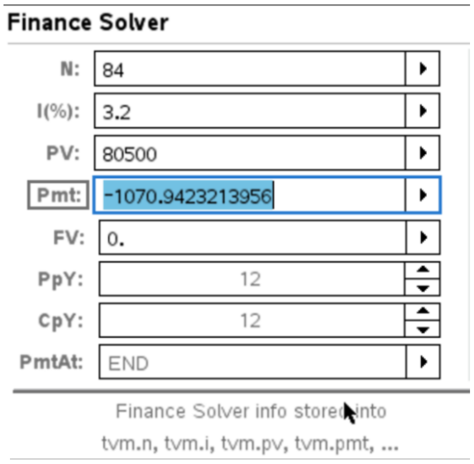
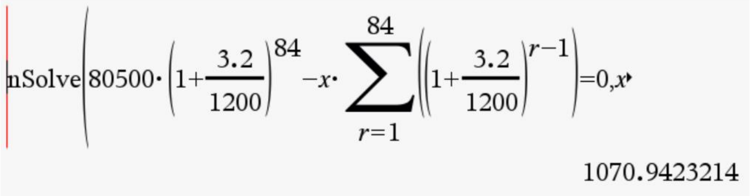
(c) Using $b = 3$, find the exact range of values of k for which $\left[f(x)\right]^2 = k$ has three distinct real roots. [3]

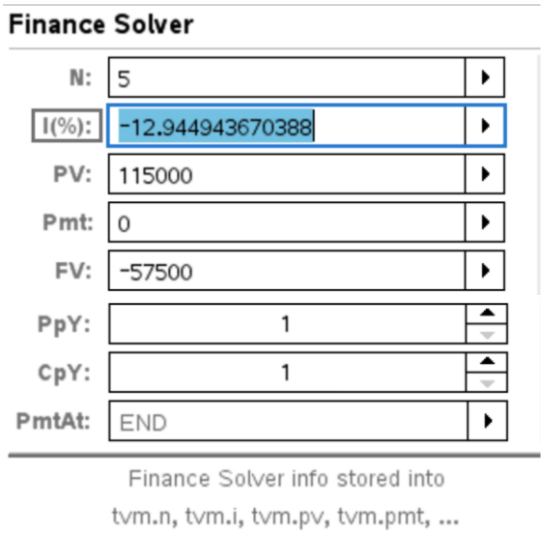
End of Paper

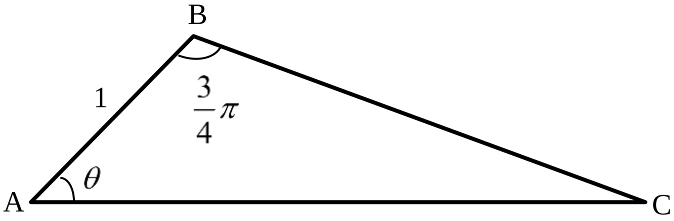
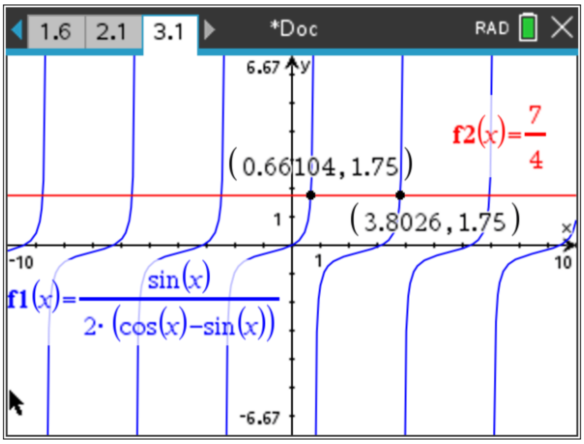
Year 5 HL MAA End of Year Examination 2022 Paper 2 (Mark Scheme)

Qn	Suggested Solutions	Marks
1	**Complex Numbers – Simultaneous Linear Equations**	**[Maximum mark: 4]**
	$\frac{z}{w}=2\mathrm{i}-1 \quad \Rightarrow \quad z=(2\mathrm{i}-1)w$ .... (1) $(3+2\mathrm{i})z=w+2\mathrm{i}$ .... (2)	
	Method 1: GDC linsolve	**Method 1**
	 $\mathrm{linSolve}\left(\begin{cases} z=(2\cdot i-1)\cdot w \\ (3+2\cdot i)\cdot z=w+2\cdot i\end{cases},\{z,w\}\right)$ $\left\{\frac{3}{10}+\frac{2}{5}\cdot i,\frac{1}{10}-\frac{1}{5}\cdot i\right\}$	M2 Use of linsolve
	$z=0.3+0.4\mathrm{i}$	A1
	$w=0.1-0.2\mathrm{i}$	A1
	Method 2: Analytical	**Method 2**
	Subst (1) into (2), $(3+2\mathrm{i})(2\mathrm{i}-1)w=w+2\mathrm{i}$ $(-7+4\mathrm{i})w-w=2\mathrm{i}$	M1 Any valid method for solving simultaneous linear equations
	$w=\frac{2\mathrm{i}}{-8+4\mathrm{i}}$	A1 Correct formulation
	$=\frac{\mathrm{i}}{-4+2\mathrm{i}}\cdot\frac{-4-2\mathrm{i}}{-4-2\mathrm{i}}$	
	$=\frac{2-4\mathrm{i}}{20}=\frac{1}{10}-\frac{1}{5}\mathrm{i} \quad (=0.1-0.2\mathrm{i})$	A1
	Subst into (1), we have $z=(2\mathrm{i}-1)\left(\frac{1}{10}-\frac{1}{5}\mathrm{i}\right)$	
	$=\frac{3}{10}+\frac{2}{5}\mathrm{i} \quad (=0.3+0.4\mathrm{i})$	A1

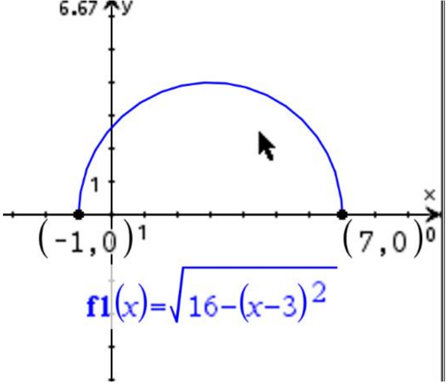
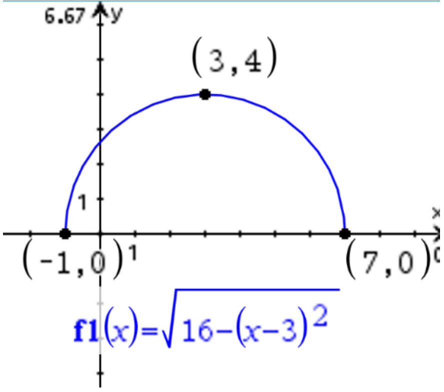
Qn	**Suggested Solutions**	**Marks**
2	**Binomial Series for Rational Powers**	**[Maximum mark: 9]**
(a)	$\sqrt{3+4x}=\sqrt{3}\sqrt{1+\frac{4x}{3}}$ $=\sqrt{3}\left[1+{}^{\frac{1}{2}}C_1\left(\frac{4x}{3}\right)+{}^{\frac{1}{2}}C_2\left(\frac{4x}{3}\right)^2+\dots\right]$ $=\sqrt{3}\left[1+\frac{\frac{1}{2}}{1!}\left(\frac{4x}{3}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{4x}{3}\right)^2+\dots\right]$ $=\sqrt{3}\left[1+\frac{2}{3}x-\frac{2}{9}x^2+\dots\right]$ $\left(=\sqrt{3}+\frac{2\sqrt{3}}{3}x-\frac{2\sqrt{3}}{9}x^2+\dots\right)$ Using GDC for ${}^{\frac{1}{2}}C_1$ and ${}^{\frac{1}{2}}C_2$  $\text{nCr}\left(\frac{1}{2},\{1,2\}\right)\cdot\left\{\frac{4}{3},\left(\frac{4}{3}\right)^2\right\}$ $\left\{\frac{2}{3},\frac{-2}{9}\right\}$	M1 Apply binomial expansion A1 A1 ${}^{\frac{1}{2}}C_1=\frac{1}{2}$ and ${}^{\frac{1}{2}}C_2=-\frac{1}{8}$ (Can be evaluated by GDC) A1 Either form
(b)	The series expansion is valid for $\left\|\frac{4x}{3}\right\|<1$, i.e. $\|x\|<\frac{3}{4}$. Since $x=1$ is outside the validity range, it is not suitable.	A1 Validity range R1
(c)	$\sqrt{3+4\left(-\frac{1}{6}\right)}\approx\sqrt{3}\left[1+\frac{2}{3}\left(-\frac{1}{6}\right)-\frac{2}{9}\left(-\frac{1}{6}\right)^2\right]$ $\sqrt{\frac{7}{3}}\approx\sqrt{3}\left[\frac{143}{162}\right]$ $\sqrt{7}\approx\frac{429}{162}=\frac{143}{54}=2.648\ \text{(3 dp)}$  $\sum_{r=0}^{2}\left(\text{nCr}\left(\frac{1}{2},r\right)\cdot\left(\frac{4}{3}\cdot\frac{-1}{6}\right)^r\right)$ $\frac{143}{162}$ $\frac{143}{162}\cdot 3$ $\frac{143}{54}$ $\frac{143}{54}$▸Decimal 2.64814814815 *Note*: An answer of 2.646 receives no credit, as it is a direct pressing of $\sqrt{7}$ on the GDC or calculator.	M1 Substitution A1 $\sqrt{\frac{7}{3}}$ A1 (No FT)

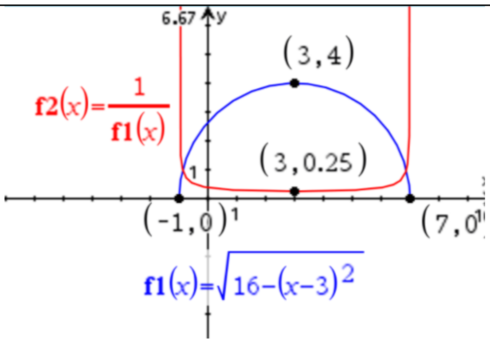
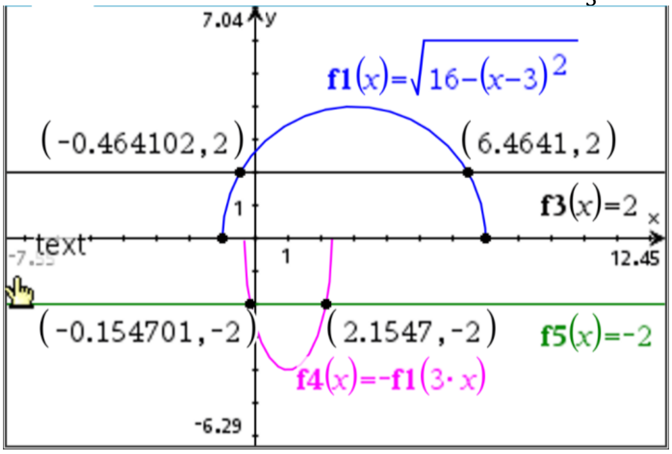
Qn	Suggested Solutions	Marks
3	**P&C – Permutations**	**[Maximum mark: 7]**
(a)	No. of ways without restrictions $=12!=479001600$	A1
(b)	No. of ways $={}^4C_1\times 11!=159667200$	M1 (${}^4C_1$) A1
(c)	**Method 1: Complement** No. of ways $=12!-{}^9C_1\times 2!\times 10!=413683200$ **Method 2: Cases** Case (1) In the same row, e.g. AC, AD, BD $3\times 3\times 2!\times 10!=65318400$ Case (2) Different rows: ${}^3C_2\times 2!\times 4\times 4\times 10!=348364800$ Total no. of ways $=348364800+65318400=413683200$	**Method 1** M1 ($12!-$) M1 (${}^9C_1\times 2!$) M1 (10!) A1 **Method 2** M1 ($3\times 3\times 2!$) M1 (${}^3C_2\times 2!$) M1 (10!) A1
4	**Financial Mathematics**	**[Maximum mark: 7]**
(a)	**Method 1: GDC Finance Solver**  Finance Solver N: 84 I(%): 3.2 PV: 80500 Pmt: -1070.9423213956 FV: 0. PpY: 12 CpY: 12 PmtAt: END Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ... Using GDC Finance Solver with the above values, Minimum repayment value = 1071 (to nearest integer) **Method 2: Analytical** Loan balance after *n* months $=80500\left(1+\frac{3.2}{1200}\right)^{n}-x\sum_{r=1}^{n}\left(1+\frac{3.2}{1200}\right)^{r-1}$  $\text{nSolve}\left(80500\cdot\left(1+\frac{3.2}{1200}\right)^{84}-x\cdot\sum_{r=1}^{84}\left(\left(1+\frac{3.2}{1200}\right)^{r-1}\right)=0,x\right.$ 1070.9423214 Minimum repayment value = 1071 (to nearest integer)	**Method 1** M1 Using finance solver to find "pmt"; at least 3 of 5 parameters correct A1 $\left\{\begin{array}{l}N=84\\ I=3.2\%\\ PV=80500\\ Ppy=12\\ FV=0\end{array}\right\}$ A1 **Method 2** M1 Correct formulation M1 (by GDC nSolve or Graph-Table, or GP) A1
(b)	Purchase price $=\frac{80500}{0.7}=\$115000$	A1

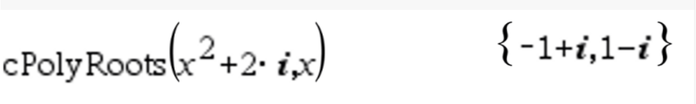
Qn	Suggested Solutions	Marks
(c)	<u>**Method 1: GDC Finance Solver**</u>	<u>**Method 1**</u>
	 Finance Solver N: 5 I(%): -12.944943670388 PV: 115000 Pmt: 0 FV: -57500 PpY: 1 CpY: 1 PmtAt: END Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ...	M1 $\left\{\begin{matrix} N=5 \\ FV=-\frac{1}{2}PV \end{matrix}\right\}$ A1 (note: as long as $FV=-\frac{1}{2}PV$, the answer will be 12.9)
	$\therefore$ Annual depreciation rate $=12.9\%$ (3 sf)	
	<u>**Method 2**</u>**: Analytical**	<u>**Method 2**</u>
	$PV\left(1-\frac{x}{100}\right)^5=\frac{1}{2}PV$	M1 Correct formulation
	$\left(1-\frac{x}{100}\right)^5=\frac{1}{2}$	
	$x=-100\left[\left(\frac{1}{2}\right)^{\frac{1}{5}}-1\right]$	
	$=12.944..$	
	$\therefore$ Annual depreciation rate $=12.9\%$ (3 sf)	A1 Condone "-12.9%"
	<u>Assumption</u> The depreciation rate is constant over the 5 years.	R1
5	**Calculus – Limits using l'Hôpital's rule**	**[Maximum mark: 5]**
	$\lim\limits_{x\to 0}\frac{1}{\sqrt{3x}}\left(\frac{\pi}{2}-\arccos\left(\sqrt{x}\right)\right)$ $\qquad(\infty)\cdot(0)$	
	$=\lim\limits_{x\to 0}\frac{\frac{\pi}{2}-\arccos\left(\sqrt{x}\right)}{\sqrt{3x}}$ $\qquad\left(\frac{0}{0}\right)$	(M1) Checking condition to use LH
	$=\lim\limits_{x\to 0}\frac{-\left(-\frac{1}{\sqrt{1-x}}\right)\cdot\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{3x}}\cdot 3}$ $\qquad$(by l'Hopital's Rule)	A1 o.e. Numerator A1 o.e. Denominator
	$=\lim\limits_{x\to 0}\frac{1}{\sqrt{3}\sqrt{1-x}}$	A1
	$=\frac{1}{\sqrt{3}}$	A1

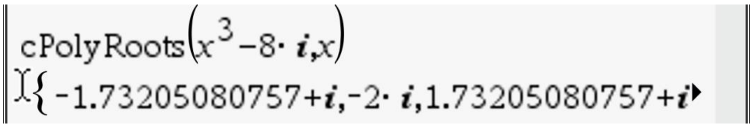
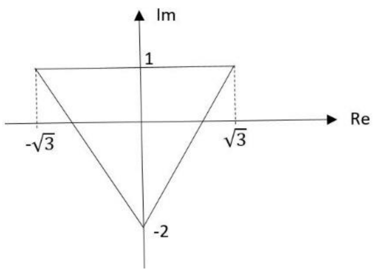
Qn	**Suggested Solutions**	**Marks**
6	**Trigonometry – Sine, Cosine Rules, Area of Triangle**	**[Maximum mark: 8]**
(a)	 Using Sine Rule,	
	$\frac{AC}{\sin\left(\frac{3}{4}\pi\right)}=\frac{1}{\sin\left(\pi-\frac{3}{4}\pi-\theta\right)}$	M1 Sine Rule with AC
	$AC=\frac{\sin\left(\frac{3}{4}\pi\right)}{\sin\left(\frac{\pi}{4}-\theta\right)}$	A1 $\sin\left(\frac{\pi}{4}-\theta\right)$
	$AC=\frac{\frac{1}{\sqrt{2}}}{\sin\left(\frac{1}{4}\pi\right)\cos\theta-\cos\left(\frac{1}{4}\pi\right)\sin\theta}$	M1 Use of Compound Angle formula
	$AC=\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta}$	A1 Evaluating Special angles
	$AC=\frac{1}{\cos\theta-\sin\theta}$ (Shown)	AG
(b)	$\text{Area of triangle ABC}=\frac{1}{2}(1)\left(\frac{1}{\cos\theta-\sin\theta}\right)\sin\theta$	M1
	$\frac{7}{4}=\frac{1}{2}\left(\frac{\sin\theta}{\cos\theta-\sin\theta}\right)$ (*)	
	<u>**Method 1: GDC Graph**</u>  	M1

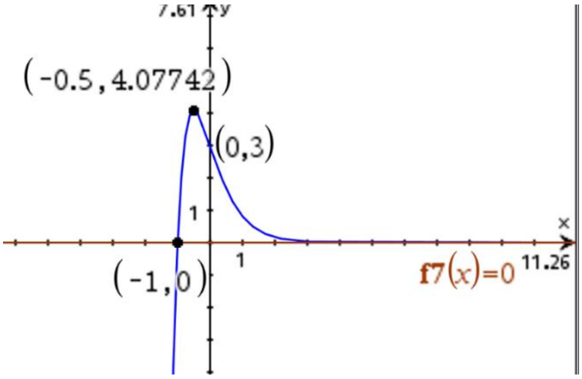
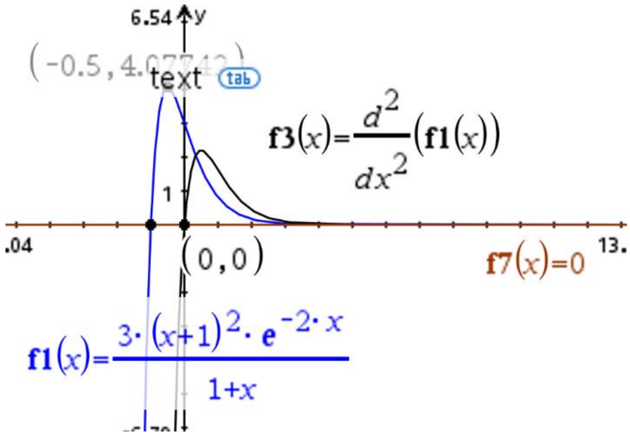
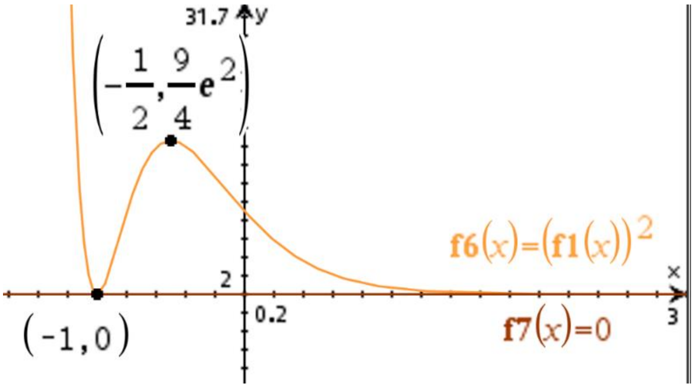
	Using GDC (Graph) $\theta = 0.66104$ $= 0.661 \text{ rad}$	A1
	Method 2: Analytical From $(*)$, $7(\cos\theta - \sin\theta) = 2\sin\theta$ $7\cos\theta = 9\sin\theta$ $\tan\theta = \frac{7}{9}$	M1
	Solving, $\theta = 0.66104$ $= 0.661 \text{ rad}$	A1
	Continuing from either method: $\text{AC} = \frac{1}{\cos\theta - \sin\theta}$ $= \frac{1}{\cos(0.66104) - \sin(0.66104)}$ $= 5.70 \text{ (to 3 s.f.)}$	A1

Qn	**Suggested Solutions**	**Marks**
7	**Functions, Graphs and Transformation**	**[Maximum mark: 16]**
(a)	Largest domain occurs when $16-(x-3)^2 \ge 0$ $(4-(x-3))(4+(x-3)) \ge 0$ $\therefore D_{\max h} = (-1, 7)$ **Alternatively:** Accept GDC graphing  $\therefore D_{\max h} = (-1, 7)$	**A1** **A1 - both** **M1** **A1 - both**
(b)	Using GDC to find maximum point of $h(x)$,  Maximum point at (3, 4) Line of symmetry: $x = 3$	**(M1) – attempt to use max. pt o.e. to determine shift** **A1**
(c)(i)	The curve has go through the translation $\begin{pmatrix}-3\\0\end{pmatrix}$ p = -3	**A1**
(c)ii)	$h(-x-(-3))$ $= \sqrt{16-(-x+3-3)^2}$ $= \sqrt{16-(x+3-3)^2}$ $= h(x-(-3))$ Hence, $h(-x-p)$ is an even function	**(M1) attempt to sub value of *p*** **A1 – replace –*x* with *x*** **R1** **AG**
(d)(i)	$gh(x) = \dfrac{1}{\sqrt{16-(x-3)^2}}$	**A1**

(d)(ii)	 $R_{gh} = [\frac{1}{4}, \infty)$	**(M1)** **A1A1 o.e** Accept $R_{gh} \geq \frac{1}{4}$
(e)(i)	Transformation results in $(m, 2)$ to become $(\frac{m}{3}, -2)$  $m = 3 \pm 2\sqrt{3}$ *or* $-0.464, 6.46$	**M1** **A1-both**
(ii)	Possible coordinates of A' are $(-0.155, -2)$ and $(2.15, -2)$. Or Possible coordinates of A' are $\left(1 - \frac{2\sqrt{3}}{3}, -2\right)$ and $\left(1 + \frac{2\sqrt{3}}{3}, -2\right)$.	**A1A1**
8	**Complex Numbers**	**[Maximum mark: 13]**
(a)(i)	Let $p + q\text{i}$ be one root of $w^2 = -2\text{i}$. $(p + q\text{i})^2 = -2\text{i}$ $p^2 - q^2 + 2pq\text{i} = -2\text{i}$ Comparing Real and Imaginary components, $2pq = -2; \ \ p^2 - q^2 = 0$ Solving p and q, $p = \pm 1; q = \mp 1$ $\therefore$ $1 - \text{i}$ and $-1 + \text{i}$ are roots of -2i.	**A1 A1**

	Alternatively, using GDC	
	 $\text{cPolyRoots}(x^2+2\cdot i,x)$ $\{-1+i,1-i\}$	
	$\therefore$ $1-\mathrm{i}$ and $-1+\mathrm{i}$ are roots of $-2\mathrm{i}$.	**A1A1**
(ii)	Replace w with $(1+\mathrm{i}w)$	
	$1+\mathrm{i}w = 1-\mathrm{i},\quad -1+\mathrm{i}$	**A1**
	$\mathrm{i}w = -\mathrm{i},\quad -2+\mathrm{i}$	
	$w=-1,\ \ 2\mathrm{i}+1$	**A1A1**
(b)(i)	$(-2\mathrm{i})^3 = -8\mathrm{i}^3 = 8\mathrm{i}$	**A1**
	$\left(\mathrm{i}-\sqrt{3}\right)^3 = \mathrm{i}^3 - 3\sqrt{3}\mathrm{i}^2 + 3\left(\sqrt{3}^2\mathrm{i}\right) - \sqrt{3}^3$	**A1**
	$= -\mathrm{i} + 3\sqrt{3} + 9\mathrm{i} - 3\sqrt{3}$	**M1**
	$= 8\mathrm{i}$	**A1**
(ii)	Let the third root be z_3.	
	Using sum of roots = 0	
	$-2\mathrm{i} + \mathrm{i} - \sqrt{3} + z_3 = 0$	**M1**
	$z_3 = \sqrt{3} + \mathrm{i}$	**A1**
	Alternatively, using Product of roots	
	$(-2\mathrm{i})\left(\mathrm{i}-\sqrt{3}\right)z_3 = 8\mathrm{i}$	**M1**
	$(2+2\sqrt{3}\mathrm{i})z_3 = 8\mathrm{i}$	
	$z_3 = \dfrac{8\mathrm{i}}{2+2\sqrt{3}\mathrm{i}} = \sqrt{3}+\mathrm{i}$	**A1**
	(Accept GDC $z_3 = 1.73 + \mathrm{i}$)	

(c)	**Alternatively, using GDC polyroots**  cPolyRoots$(x^3-8\cdot i,x)$ {-1.73205080757+i,-2·i,1.73205080757+i▸ $z_3=\sqrt{3}+\mathrm{i}$ Area of triangle  $=\frac{1}{2}(2\sqrt{3})(3)$ $=3\sqrt{3}\ or\ 5.20$	**M1** **A1** **M1** **A1**
9	**Differentiation**	**[Maximum mark: 16]**
(a)(i)	$f(x)=b(x+1)\mathrm{e}^{-2x}, x\neq-1$ $f'(x)=b[(x+1)(-2\mathrm{e}^{-2x})+\mathrm{e}^{-2x}]$ $=\ b\mathrm{e}^{-2x}(-2x-1)$	**M1 product rule** **A1**
(ii)	$f'(x)=0\ \Rightarrow x=-\frac{1}{2}$ Sub. $x=-\frac{1}{2}, f\left(-\frac{1}{2}\right)=\frac{b\mathrm{e}}{2}$ Maximum point at $\left(-\frac{1}{2},\ \frac{be}{2}\right)$	**A1** **A1** accept (-0.5,1.36b)
(b)(i)	$f(0)=\frac{b(1)^2\mathrm{e}^0}{1}=3$ $b=3$	**A1** **AG**

(ii)		**A1 maximum pt** **A1 horizontal asymp** **A1 x and y-intercepts** **A1 shape**
(iii)	$\lim_{x \to \infty} f(x) = 0$	**A1**
(iv)	 Since $f''(0^-) \cdot f''(0^+) < 0$, (change in sign of $f''(x)$), Point of inflexion at (0, 3)	**M1 (with graph)** **R1** **A1**
(c)	 $0 < k < \frac{9}{4}e^2$	**M1** **A1A1 (exact only)**