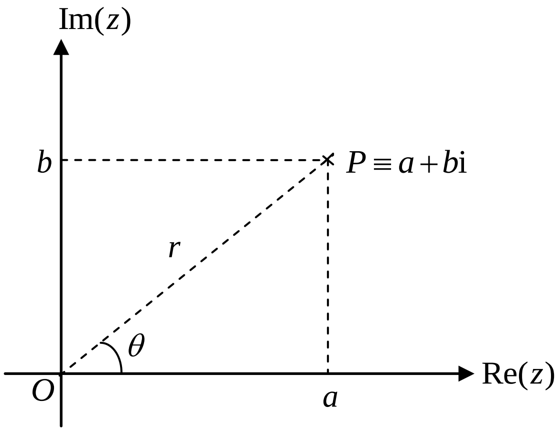
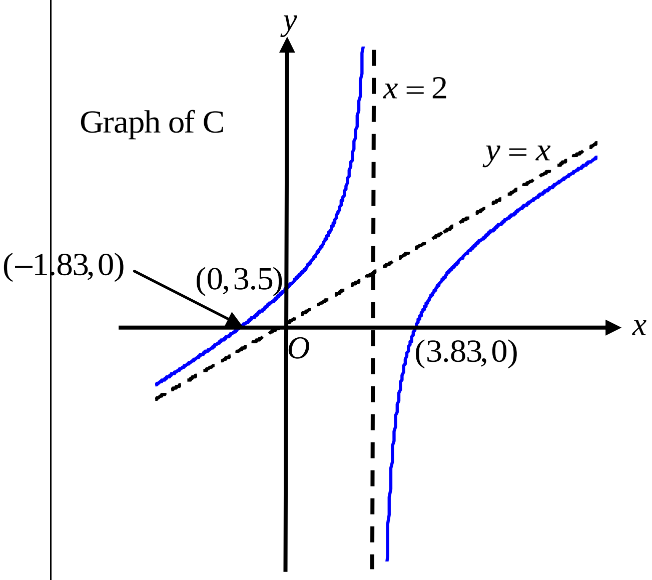
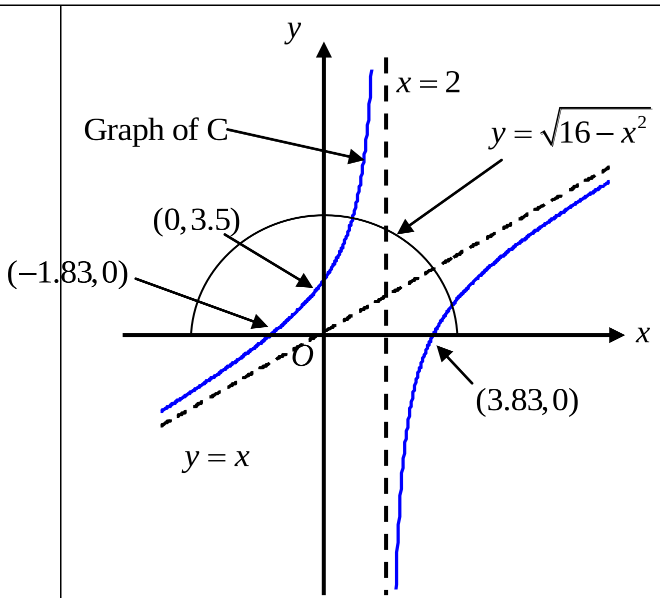
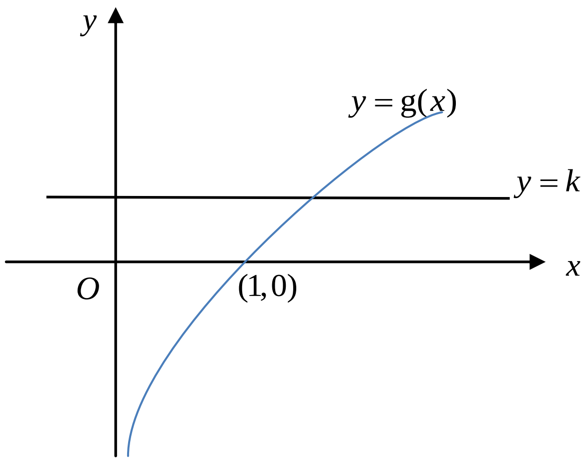
YISHUN JUNIOR COLLEGE
2015 JC2 PRELIMINARY EXAM PAPER 2
H2 MATHEMATICS
SOLUTION

Qn	Solution
1a	 $a = r\cos\theta$ and $b = r\sin\theta$ Therefore $a+b\mathrm{i} = r\cos\theta + (r\sin\theta)\mathrm{i} = r(\cos\theta + \mathrm{i}\sin\theta)$ Let P be the point on the Argand diagram representing the complex number $a+b\mathrm{i}$. r is the distance of the point P to the origin. θ is the directed angle made by the line segment joining origin to point P, with the positive real axis. Cartesian equation of locus: $x^2 + y^2 = 9$
1b	Given $z = \cos\alpha + \mathrm{i}\sin\alpha$ By De Moivre's Theorem $\frac{1}{z} = z^{-1} = (\cos\alpha + \mathrm{i}\sin\alpha)^{-1} = \cos\alpha - \mathrm{i}\sin\alpha$ OR $\frac{1}{z} = z^{-1} = \mathrm{e}^{-\mathrm{i}\alpha} = \cos\alpha - \mathrm{i}\sin\alpha$ OR $\frac{1}{z} = \frac{1}{\cos\alpha + \mathrm{i}\sin\alpha} \times \frac{\cos\alpha - \mathrm{i}\sin\alpha}{\cos\alpha - \mathrm{i}\sin\alpha}$ $= \frac{\cos\alpha - \mathrm{i}\sin\alpha}{\cos^2\alpha - \mathrm{i}^2\sin^2\alpha}$ $= \frac{\cos\alpha - \mathrm{i}\sin\alpha}{\cos^2\alpha + \sin^2\alpha}$ $= \cos\alpha - \mathrm{i}\sin\alpha$

	OR $zz^* = \lvert z\rvert^2 = 1 \Rightarrow \frac{1}{z} = z^* = \cos\alpha - \mathrm{i}\sin\alpha$ $z + \frac{1}{z} = \cos\alpha + \mathrm{i}\sin\alpha + \cos\alpha - \mathrm{i}\sin\alpha$ $= 2\cos\alpha$ (Shown) $z - \frac{1}{z} = \cos\alpha + \mathrm{i}\sin\alpha - (\cos\alpha - \mathrm{i}\sin\alpha)$ $= 2\mathrm{i}\sin\alpha$ (Shown) $\left(z + \frac{1}{z}\right) \div \left(z - \frac{1}{z}\right) = \frac{z^2+1}{z} \times \frac{z}{z^2-1}$ Hence $\frac{z^2+1}{z^2-1} = \frac{2\cos\alpha}{2\mathrm{i}\sin\alpha} = -\frac{\mathrm{i}\cos\alpha}{\sin\alpha}$ or $-\mathrm{i}\cot\alpha$
2i	$4y = x^2$ and $8(y-k) = x^2$ $4y = 8(y-k) \Rightarrow y = 2(y-k)$ $y = 2k$ $x^2 = 4(2k) = 8k$ $x = \pm 2\sqrt{2k}$ Diameter of the rim of the bowl $= 2\sqrt{2k} - \left(-2\sqrt{2k}\right)$ or $2\sqrt{2k} \times 2$ $= 4\sqrt{2k}$ (Shown)
2ii	Capacity of the bowl $= \pi\int_k^{2k} 8(y-k)\ \mathrm{d}y$ $= 8\pi\left[\frac{1}{2}y^2 - ky\right]_k^{2k}$ $= 8\pi\left[\frac{4}{2}k^2 - 2k^2 - \frac{1}{2}k^2 + k^2\right]$ $= 4\pi k^2$ Volume of material used for the bowl $= \pi\int_0^{2k} 4y\ \mathrm{d}y - 4\pi k^2$ $= \pi\left[2y^2\right]_0^{2k} - 4\pi k^2$ $= \pi\left[8k^2 - 0\right] - 4\pi k^2$ $= 4\pi k^2$ $=$ capacity of the bowl
2iii	Given $k = 4$ Required area $= 2\int_0^{2\sqrt{2k}} \frac{1}{8}x^2 + 4 - \frac{1}{4}x^2\ \mathrm{d}x$

	$=2\int_0^{4\sqrt{2}} 4-\frac{1}{8}x^2 \mathrm{d}x$ $=30.16988933$ $=30.170$ (3 d.p.)
3i	C: $y=\frac{x^2+qx-7}{x-2}$, $q<0$ $y=x+q+2+\frac{2q-3}{x-2}$ or $y=x+q+2+\frac{\text{const}}{x-2}$ Asymptotes: $x=2$ and $y=x+q+2$
3ii	If $y=x$ is an asymptote of C, then $x+q+2=x$ Thus $q+2=0 \Rightarrow q=-2$ (Shown) 
3iii	$\frac{x^2-2x-7}{(x-2)\sqrt{16-x^2}} \le 1$ **---- (*)** $\frac{x^2-2x-7}{x-2} \le \sqrt{16-x^2}$

	 x-coordinates of points of intersection: 0.170, 3.98 Solving (*), $-4<x\leq 0.170$ or $2<x\leq 3.98$
4i	$\mathrm{g}(x)=\ln x$, $x>0$ $\mathrm{h}(x)=\dfrac{1}{x}$, $x>0$  Any line $y=k$ where $k\in\mathbb{R}$, cuts the graph of g at most once $\Rightarrow$ g is a $1-1$ function and therefore function g^{-1} exists. Let $y=\mathrm{g}(x)=\ln x$ $\mathrm{g}^{-1}(y)=x=\mathrm{e}^{y}$ $\therefore \mathrm{g}^{-1}(x)=\mathrm{e}^{x}$, $D_{\mathrm{g}^{-1}}=R_{\mathrm{g}}=\mathbb{R}$ and $R_{\mathrm{g}^{-1}}=D_{\mathrm{g}}=\mathbb{R}^{+}$
4ii	Since $R_{\mathrm{g}}=\mathbb{R}\not\subseteq D_{\mathrm{h}}=\mathbb{R}^{+}$, composite function hg does not exist.

<table>
<tr><td>4iii</td><td>$\mathrm{gh}(x)=\mathrm{g}(\mathrm{h}(x))=\mathrm{g}\left(\frac{1}{x}\right)=\ln\frac{1}{x}=-\ln x$
$D_{\mathrm{gh}}=D_{\mathrm{h}}=\mathbb{R}^{+}$
$\mathrm{gh}: x \mapsto -\ln x,\ x>0$
$R_{\mathrm{gh}}=\mathbb{R}$</td></tr>
<tr><td>4iv</td><td>$\mathrm{h}^2(x)=\mathrm{h}\left(\frac{1}{x}\right)=1\div\frac{1}{x}=x$
$\mathrm{h}^9(x)=\mathrm{h}(\mathrm{h}^2\mathrm{h}^2\mathrm{h}^2\mathrm{h}^2(x))=\mathrm{h}(x)=\frac{1}{x}$</td></tr>
<tr><td>4v(a)</td><td>$\mathrm{g}(x)=\ln x$ and $\mathrm{gh}(x)=-\ln x$
Graphs of g and gh are reflections of each other in the $x-$axis.</td></tr>
<tr><td>4v(b)</td><td>Graphs of g and g^{-1} are reflections of each other in the line $y=x$.</td></tr>
<tr><td></td><td></td></tr>
<tr><td>5i</td><td><table><tr><td>Cohort</td><td>600 Males</td><td>400 Females</td></tr><tr><td>Sample size</td><td>$0.05\times600=30$</td><td>20</td></tr></table>**Randomly** select 30 males and 20 females from 600 males and 400 females respectively to be surveyed.</td></tr>
<tr><td>5ii</td><td>A stratified sample is preferable in this context as the population is **fairly represented**.</td></tr>
<tr><td></td><td></td></tr>
<tr><td>6</td><td>$\mu=\frac{22.2+26.2}{2}=24.2$
$\overline{X}\sim\mathrm{N}\left(\mu,\frac{\sigma^2}{4}\right)$
$\mathrm{P}\left(\overline{X}<22.2\right)=0.0128$
$\mathrm{P}\left(Z<\frac{22.2-24.2}{\sigma/2}\right)=0.0128$
$-\frac{4}{\sigma}=-2.2322$
$\sigma=1.79$ (3 sf)</td></tr>
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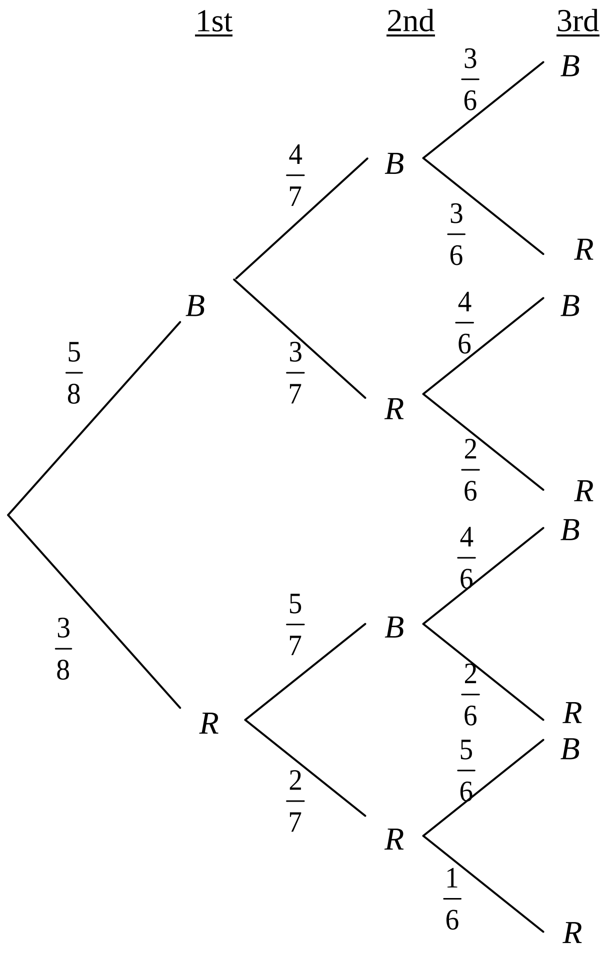
<tr>
<td>7</td>
<td>B – Black; R – Red

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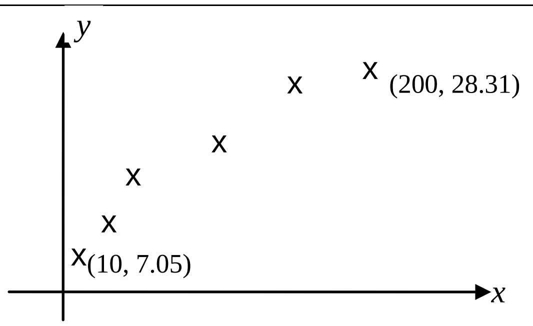
<tr>
<td>7i</td>
<td>Req. probability
$=\frac{5}{8}\times\frac{3}{7}\times\frac{2}{6}+\frac{3}{8}\times\frac{5}{7}\times\frac{2}{6}+\frac{3}{8}\times\frac{2}{7}$
$=\frac{2}{7}$

Alternatively:
Req. probability
$=\frac{5}{8}\times\frac{3}{7}\times\frac{2}{6}\times\frac{3!}{2!}+\frac{3}{8}\times\frac{2}{7}\times\frac{1}{6}$
$=\frac{2}{7}$</td>
</tr>
<tr>
<td>7ii</td>
<td>P(3rd ball red)
$=\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}+\frac{5}{8}\times\frac{3}{7}\times\frac{2}{6}+\frac{3}{8}\times\frac{5}{7}\times\frac{2}{6}+\frac{3}{8}\times\frac{2}{7}\times\frac{1}{6}$
$=\frac{3}{8}$</td>
</tr>
</table>

	$\mathrm{P}(\text{1st and 3rd ball red})$ $=\frac{3}{8}\times\frac{5}{7}\times\frac{2}{6}+\frac{3}{8}\times\frac{2}{7}\times\frac{1}{6}$ $=\frac{3}{28}$ $\mathrm{P}(\text{1st ball red} \mid \text{3rd ball red})$ $=\frac{\mathrm{P}(\text{1st and 3rd ball red})}{\mathrm{P}(\text{3rd ball red})}$ $=\frac{3/28}{3/8}=\frac{2}{7}$
7iii	$\mathrm{P}(\text{at least 2 red} \mid \text{1st ball red})$ $=\frac{5}{7}\times\frac{2}{6}+\frac{2}{7}=\frac{11}{21}$ $\neq\frac{2}{7}=\mathrm{P}(\text{at least 2 red})$ Hence events are not independent.
8i	$H_0 : \mu = 50$ $H_1 : \mu > 50$ Under H_0, the test statistic is $Z=\frac{\overline{X}-\mu}{s/\sqrt{n}} \sim \mathrm{N}(0,1)$ approximately (by CLT) where μ = 50, $\bar{x}=55$, n = 60, $s^2=\frac{60}{59}(19.3)^2$ p- value = 0.0233 Since the p-value < 0.05 (the significance level), we reject H_0 and conclude that at the 5% level, there is sufficient evidence that the mean time required by a person to complete the sum has increased.
8ii	Since the sample size (60) is large, CLT can be used to approximate the sample mean time required to complete the sum to a normal distribution. Hence any assumption is not necessary.
8iii	p-value = 2(0.0233) = 0.0466 < 0.05 Since the p-value < 0.05 (the significance level), we reject H_0 and conclude that at the 5% level, there is sufficient evidence that the mean time required by a person to complete the sum has changed.
9a	M A T H E I C S M A T No repeated letters : $^{8}P_{5}=6720$

	1 pair of repeated letters: ${}^{3}C_{1}\times{}^{7}C_{3}\times\frac{5!}{2!}=6300$ 2 pairs of repeated letters: ${}^{3}C_{2}\times{}^{6}C_{1}\times\frac{5!}{2!2!}=540$ Number of codewords = 6720 + 6300 + 540 = 13560
9bi	Required probability $=\dfrac{{}^{6}C_{1}\times\frac{5!}{2!2!}}{13560}=\dfrac{3}{226}$ or 0.0133 (3 sf)
9bii	The codeword contains 4 vowels, hence only 1 consonant is contained in the codeword. There are 5 choices of consonants. Required probability $=\dfrac{{}^{5}C_{1}\times3!\times{}^{4}C_{2}}{{}^{5}C_{1}\times\frac{5!}{2!}}$ $=\dfrac{3}{5}$
10i	
10ii	**(a)** y on x: $r=0.980$ (3 sf) **(b)** y on $\sqrt{x}$: $r=0.991$ (3 sf)
10iii	From (i), the scatter diagram shows that the rate of increase of y decreases as x increases. Hence the model $y=c+d\sqrt{x}$ should be a better model. This is confirmed by the r value in (ii), which shows that $y=c+d\sqrt{x}$ has $r=0.991$ which is closer to 1.
10iv	$y=0.12102+2.0112\sqrt{x}$ $=0.121+2.01\sqrt{x}$ (3 sf) $x=45\Rightarrow y=13.6$ Since $r\approx1$ and estimate is obtained via interpolation, hence estimate is reliable.
10v	From GC, $\left(\overline{\sqrt{x}}=8.6834,\bar{y}=17.585\right)$ Hence a possible point is: $(75.4, 17.6)$ (3 sf)

11i	Assume that the average rate at which people arrive at the bus stop is constant throughout, and that the arrivals at the bus stop are independent of other arrivals. The average rate may not be constant because for example, no one will take the bus during the wee hours where there are no bus services. The arrivals may not be independent because for example, family members will arrive at the bus stop at the same time for a family trip.
11ii	X: no. of people who arrive at the bus stop in a period of 2 minutes. $X \sim \text{Po}(4)$ $\text{P}(X=5)=0.1562934519$ $=0.156$ (3 sf)
11iii	Y: no. of people who arrive at the bus stop in a period of 1 minute. $Y \sim \text{Po}(2)$ $\text{P}(\text{at most 1 in 1st minute} \mid \text{5 in 2 minutes})$ $=\dfrac{\text{P}(Y_1=0)\text{P}(Y_2=5)+\text{P}(Y_1=1)\text{P}(Y_2=4)}{\text{P}(X=5)}$ $=\dfrac{0.0293050222}{0.1562934519}$ $=0.1875$ (4 sf) **Alternatively:** A person who arrived in the 2 minute period has an equal chance to arrive in the 1st or 2nd min. Req. probability $=\dfrac{\text{"0 in 1st min"}+\text{"1 in 1st min"}}{2^5}=\dfrac{1+5}{2^5}=\dfrac{3}{16}$ $=0.1875$
11iv	A: no. of people who arrive at the bus stop in a period of 10 min $A \sim \text{Po}(20)$ $\lambda=20>10$. Hence $A \sim \text{N}(20, 20)$ approx. $\text{P}(A \le 22) \xrightarrow{c.c.} \text{P}(A \le 22.5)$ $=0.712$ (3 sf)
12i	X: no. of successful two point shots out of n attempts. $X \sim \text{B}(n, 0.6)$ $\text{P}(X \ge 1)>0.99$ $\text{P}(X=0)<0.01$ $n=5, \quad \text{P}(X=0)=0.0102$ (3 sf) $n=6, \quad \text{P}(X=0)=0.00410$ (3 sf) Hence he needs to attempt at least 6 shots.
12ii	Y: no. of successful three point shots out of 60 attempts. $Y \sim \text{B}(60, 0.4)$ $n=60$ large, $np=24>5$, $nq=36>5$

	$Y \sim \mathrm{N}(24, 14.4)$ approx $\mathrm{P}(21 \le Y \le 28) \xrightarrow{c.c.} \mathrm{P}(20.5 \le Y \le 28.5)$ $= 0.704$ (3 sf)
12iii	*A*, *B*: no. of successful two point and three point shots out of 10 and 5 attempts respectively. *C*: points scored in a game. Then $C = 2A + 3B$ $\mathrm{E}(C) = 2\mathrm{E}(A) + 3\mathrm{E}(B)$ $\quad \mathrm{Var}(C) = 2^2\,\mathrm{Var}(A) + 3^2\,\mathrm{Var}(B)$ $= 2(10)(0.6) + 3(5)(0.4)$ $= 4(10)(0.6)(0.4) + 9(5)(0.4)(0.6)$ $= 18$ $\quad = 20.4$ $\overline{C} \sim \mathrm{N}\left(18, \dfrac{20.4}{50}\right)$ approx by CLT $\mathrm{P}(\overline{C} \ge 17) = 0.941$ (3 sf)