Paper 1		Remarks
1	**(a)** Ellipse **(b)** Given: $PF+PG=2a$ By cosine rule, $PG^2=FP^2+FG^2-2(FP)(FG)\cos\theta$ $(2a-r)^2=r^2+(2c)^2-2r(2c)\cos\theta$ $4r(c\cos\theta-a)=4(c^2-a^2)$ $r=\dfrac{a^2-c^2}{a-c\cos\theta}$ $=\dfrac{a-\frac{c^2}{a}}{1-\frac{c}{a}\cos\theta}=\dfrac{l}{1-k\cos\theta}$, where $l=a-\dfrac{c^2}{a}$ and $k=\dfrac{c}{a}$ **(c)** k represents the eccentricity of the conic.	
2	$\dfrac{dy}{dx}+2xy=x^3$ Integrating factor $=e^{\int 2x\,dx}=e^{x^2}$ $\dfrac{d}{dx}\left(e^{x^2}y\right)=x^3e^{x^2}$ $e^{x^2}y=\dfrac{1}{2}\int x^2\cdot 2xe^{x^2}dx$ $=\dfrac{1}{2}\left[x^2e^{x^2}-\int 2xe^{x^2}\,dx\right]$ $=\dfrac{1}{2}\left[x^2e^{x^2}-e^{x^2}\right]+c$ $y=\dfrac{1}{2}(x^2-1)+ce^{-x^2}$ When $x=1$, $y=1$: $1=ce\Rightarrow c=e^{-1}$. $\therefore\ y=\dfrac{1}{2}(x^2-1)+e^{1-x^2}$	
3	$y=\dfrac{1}{6}x^3+\dfrac{1}{2x}$ $\dfrac{dy}{dx}=\dfrac{1}{2}\left(x^2-\dfrac{1}{x^2}\right)$ $1+\left(\dfrac{dy}{dx}\right)^2=1+\dfrac{1}{4}\left(x^4-2+\dfrac{1}{x^4}\right)=\dfrac{1}{4}\left(x^2+\dfrac{1}{x^2}\right)^2$ Length of arc $=s$	

	$= \int_a^t \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ $= \frac{1}{2}\int_a^t \left(x^2 + \frac{1}{x^2}\right)\mathrm{d}x \quad$ [since $a,\ t > 0$] $= \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_a^t$ $= \frac{1}{2}\left[\frac{t^3}{3} - \frac{1}{t} - \left(\frac{a^3}{3} - \frac{1}{a}\right)\right]$ $= \left(\frac{1}{6}t^3 + \frac{1}{2t}\right) - \frac{1}{t} - \frac{1}{2}\left(\frac{a^3}{3} - \frac{1}{a}\right)$ $= u - \frac{1}{t} - \frac{1}{2}\left(\frac{a^3}{3} - \frac{1}{a}\right)$ Therefore, $s = u - \frac{1}{t}$ for all t when $\frac{a^3}{3} - \frac{1}{a} = 0 \Rightarrow a = \sqrt[4]{3}$ (since $a > 0$).	
4	$z_1 + z_2 + z_3 = z_1 z_2 z_3$ $\Rightarrow \quad z_3(z_1 z_2 - 1) = z_1 + z_2$ **Case 1**: $z_1 z_2 = 1$. Then $\arg(z_1 z_2) = \arg(1)$ $\Rightarrow \arg(z_1) + \arg(z_2) = 0$ $\Rightarrow \arg(z_1) = -\arg(z_2)$ Then it cannot be that P_1 and P_2 are either both above or both below the real axis. **Case 2**: $z_1 z_2 \neq 1$. Then $z_3 = \frac{z_1 + z_2}{z_1 z_2 - 1}$ $= \frac{(a+c) + \mathrm{i}(b+d)}{(a+\mathrm{i}b)(c+\mathrm{i}d) - 1}$ $= \frac{(a+c) + \mathrm{i}(b+d)}{(ac - bd - 1) + \mathrm{i}(ad + bc)}$ $= \frac{\left[(a+c) + \mathrm{i}(b+d)\right]\left[(ac - bd - 1) - \mathrm{i}(ad + bc)\right]}{(ac - bd - 1)^2 + (ad + bc)^2}$ $\mathrm{Im}(z_3) = \frac{(b+d)(ac - bd - 1) - (a+c)(ad + bc)}{(ac - bd - 1)^2 + (ad + bc)^2}$ The denominator is non-negative. Numerator $= (b+d)(ac - bd - 1) - (a^2 d + abc + acd + bc^2)$ $= (b+d)(ac - bd - 1) - (b+d)ac - (a^2 d + bc^2)$ $= -(b+d)(bd + 1) - (a^2 d + c^2 b)$	

	We prove by contradiction. Case 1: Assume that imaginary parts of z_1, z_2, z_3 are positive. Then $b+d>0$ and $bd>0$ $\Rightarrow \text{Im}(z_3)<0$ (contradiction) Case 2: Assume that imaginary parts of z_1, z_2, z_3 are negative. Then $b+d<0$ and $bd>0$ $\Rightarrow \text{Im}(z_3)>0$ (contradiction) Therefore, P_1, P_2, P_3 cannot be all above or all below the real axis of the Argand diagram.	
5	$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{2x} - \frac{1}{8}\mathrm{e}^{-2x}$ $1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1+4\mathrm{e}^{4x} - \frac{1}{2} + \frac{1}{64}\mathrm{e}^{-4x} = \left(2\mathrm{e}^{2x} + \frac{1}{8}\mathrm{e}^{-2x}\right)^2$ $A = 2\pi\int_0^{\ln 2} y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ $= 2\pi\int_0^{\ln 2}\left(\mathrm{e}^{2x} + \frac{1}{16}\mathrm{e}^{-2x}\right)\left(2\mathrm{e}^{2x} + \frac{1}{8}\mathrm{e}^{-2x}\right)\mathrm{d}x$ $= 2\pi\int_0^{\ln 2}\left(2\mathrm{e}^{4x} + \frac{1}{4} + \frac{1}{128}\mathrm{e}^{-4x}\right)\mathrm{d}x$ $= 2\pi\left[\frac{1}{2}\mathrm{e}^{4x} + \frac{1}{4}x - \frac{1}{512}\mathrm{e}^{-4x}\right]_0^{\ln 2}$ $= 2\pi\left[\frac{1}{2}\mathrm{e}^{4\ln 2} + \frac{1}{4}\ln 2 - \frac{1}{512}\mathrm{e}^{-4\ln 2} - \left(\frac{1}{2} - \frac{1}{512}\right)\right]$ $= 2\pi\left[\frac{1}{2}(16) + \frac{1}{4}\ln 2 - \frac{1}{512}\left(\frac{1}{16}\right) - \frac{255}{512}\right]$ $= \pi\left(\frac{61455}{4096} + \frac{1}{2}\ln 2\right)$	
6	**(a)** Let k be an eigenvalue of $\mathbf{M}$. $\det(\mathbf{M} - k\mathbf{I}) = 0$ $\Rightarrow \det\begin{pmatrix} a-k & b \\ b & c-k \end{pmatrix} = 0$ $(a-k)(c-k) - b^2 = 0$ $k^2 - (a+c)k + (ac-b^2) = 0$ $k = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2}$ $= \frac{(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}{2}$ Since $(a-c)^2 + 4b^2 \geq 0$ for all real values of *a*, *b*, *c*, there are real solutions for $k \Rightarrow$ the eigenvalues of $\mathbf{M}$ are real.	

	(b) For $\lambda=\mu$, we require $(a-c)^2+4b^2=0 \Rightarrow a=c$ and $b=0$. **(c)** Let $D=(a-c)^2+4b^2$. Then suppose $\lambda=\dfrac{a+c+\sqrt{D}}{2}$ and $\mu=\dfrac{a+c-\sqrt{D}}{2}$. To get the eigenvectors, • Solve $(\mathbf{M}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0} \Rightarrow \left[a-\left(\dfrac{a+c+\sqrt{D}}{2}\right)\right]x+by=0$ $\Rightarrow y=\dfrac{c-a+\sqrt{D}}{2b}x$ $\therefore \mathbf{x}=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\ \dfrac{c-a+\sqrt{D}}{2b}x\end{pmatrix}=\dfrac{1}{2b}\begin{pmatrix}2b\\c-a+\sqrt{D}\end{pmatrix}$ Eigenvectors corresponding to λ are $\alpha\begin{pmatrix}2b\\c-a+\sqrt{D}\end{pmatrix},\ \alpha\in\mathbb{R}\setminus\{0\}$. • Solve $(\mathbf{M}-\mu\mathbf{I})\mathbf{x}=\mathbf{0} \Rightarrow \left[a-\left(\dfrac{a+c-\sqrt{D}}{2}\right)\right]x+by=0$ $\Rightarrow y=\dfrac{c-a-\sqrt{D}}{2b}x$ Eigenvectors corresponding to μ are $\beta\begin{pmatrix}2b\\c-a-\sqrt{D}\end{pmatrix},\ \beta\in\mathbb{R}\setminus\{0\}$. Consider the dot product, $\alpha\begin{pmatrix}2b\\c-a+\sqrt{D}\end{pmatrix}\bullet\beta\begin{pmatrix}2b\\c-a-\sqrt{D}\end{pmatrix}=\alpha\beta\left(4b^2+(c-a)^2-D^2\right)$ $=\alpha\beta\left(4b^2+(c-a)^2-(a-c)^2-4b^2\right)$ $=0$ Therefore, every eigenvector corresponding to λ is perpendicular to every eigenvector corresponding to μ.	
7	**(a)(i)** $M_1=2M_0\left(1-b\sqrt{M_0}\right)=20000(1-100b)$ If the population remains constant, then $M_1=M_0=10000$ and we have $1=2(1-100b)\Rightarrow b=\dfrac{1}{200}$ **(ii)** $M_1<0\Rightarrow 1-100b<0$ $\Rightarrow b>\dfrac{1}{100}$	

	(iii) As $n \to \infty, M_{n+1} \to L, M_n \to L$ $L = 2L\left(1 - b\sqrt{L}\right) \Rightarrow 1 - b\sqrt{L} = \frac{1}{2} \Rightarrow b\sqrt{L} = \frac{1}{2}$ $0 < L < 10000 \Rightarrow \; 0 < \sqrt{L} < 100 \Rightarrow 0 < b\sqrt{L} < 100b$ given $b > 0$ Therefore $100b > \frac{1}{2} \Rightarrow b > \frac{1}{200}$ From (ii), we gather that $b > \frac{1}{100}$ if L is to be positive. Hence, for $0 < L < 10000$, $\frac{1}{200} < b < \frac{1}{100}$ $L > 10000 \Rightarrow b\sqrt{L} > 100b \Rightarrow \frac{1}{2} > 100b$ given $b\sqrt{L} = \frac{1}{2}$ Therefore $0 < b < \frac{1}{200}$ **(b)** $M_2 = aM_1\left(1 - b\sqrt{M_1}\right) \Rightarrow 9680 = 2420a\left(1 - b\sqrt{2420}\right)$ $\Rightarrow 4 = a\left(1 - b\sqrt{2420}\right) \quad \cdots\cdots(1)$ $M_3 = aM_2\left(1 - b\sqrt{M_2}\right) \Rightarrow 2420 = 9680a\left(1 - b\sqrt{9680}\right)$ $\Rightarrow \frac{1}{4} = a\left(1 - 2b\sqrt{2420}\right) \quad \cdots\cdots(2)$ $\frac{(1)}{(2)}$: $16 = \left(\frac{1 - b\sqrt{2420}}{1 - 2b\sqrt{2420}}\right) \Rightarrow 32 - 32b\sqrt{2420} = 1 - b\sqrt{2420}$ $\Rightarrow 31b\sqrt{2420} = 15$ $b = \frac{15}{31\sqrt{2420}}$ $b = \frac{15\sqrt{2420}}{31(2420)} = \frac{15\left(22\sqrt{5}\right)}{31(2420)} = \frac{3\sqrt{5}}{682}$ From (1), $a = \frac{4}{1 - b\sqrt{2420}} = \frac{4}{1 - \left(\frac{3\sqrt{5}}{682}\right)\left(22\sqrt{5}\right)}$ $a = \frac{4}{1 - \frac{15}{31}} = \frac{4}{\frac{16}{31}} = \frac{31}{4}$ **(ii)** When $M_0 = 9679$ and $M_0 = 9681$, the model predicts that the population oscillates between values approximately 2420 and 9680 for the first few values of n but it eventually becomes negative in value when $n = 9$ or 10. Hence the proposed model is not appropriate.	

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(a) The linear system can be written as

$$\mathbf{Ax}=\mathbf{b}, \text{ where } \mathbf{A}=\begin{pmatrix}1&1&1\\2&-1&1\\4&1&3\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix},\ \mathbf{b}=\begin{pmatrix}\sin\theta\\\cos\theta\\1\end{pmatrix}.$$

$\det(\mathbf{A})=1(-4)-1(2)+1(6)=0$. This implies that the linear system has either infinitely many or no solutions.

(b) $$\left(\begin{array}{ccc|c}1&1&1&\sin\theta\\2&-1&1&\cos\theta\\4&1&3&1\end{array}\right)\xrightarrow[R_4\to R_3-4R_1]{R_2\to R_2-2R_1}\left(\begin{array}{ccc|c}1&1&1&\sin\theta\\0&-3&-1&\cos\theta-2\sin\theta\\0&-3&-1&1-4\sin\theta\end{array}\right)$$

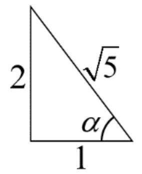
$$\xrightarrow{R_3\to R_3-R_2}\left(\begin{array}{ccc|c}1&1&1&\sin\theta\\0&-3&-1&\cos\theta-2\sin\theta\\0&0&0&1-\cos\theta-2\sin\theta\end{array}\right)$$

Solutions exist if and only if

$1-\cos\theta-2\sin\theta=0$

$\Rightarrow \cos\theta+2\sin\theta=1$

$\sqrt{5}\cos(\theta-\alpha)=1$, where $\tan\alpha=2$



$$\theta-\alpha=\cos^{-1}\frac{1}{\sqrt{5}}\quad [\text{since } -\alpha<\theta-\alpha<2\pi-\alpha]$$

$$=\alpha$$

$$\therefore \theta=2\alpha=2\tan^{-1}2.$$

(c) From GC, $S_0=\left\{\frac{1}{3}-\frac{2}{3}z,\ -\frac{1}{3}-\frac{1}{3}z,\ z\in\mathbb{R}\right\}$.

The two solution sets represent distinct parallel lines with direction vector $\begin{pmatrix}-2\\-1\\3\end{pmatrix}$.

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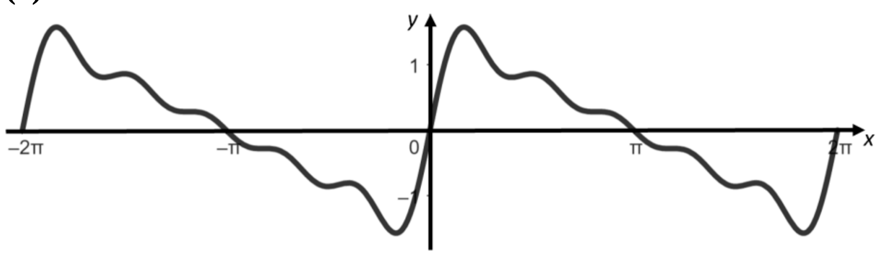
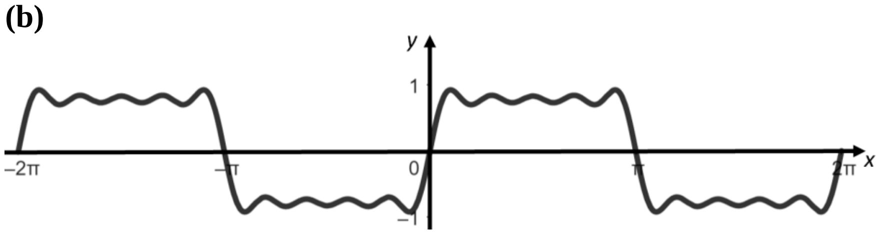
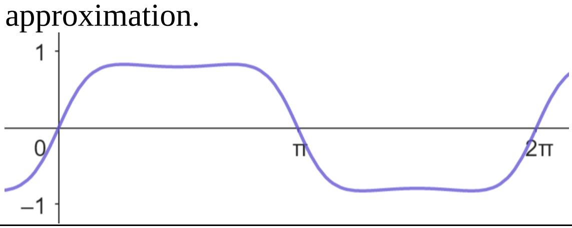
(a)



(b)



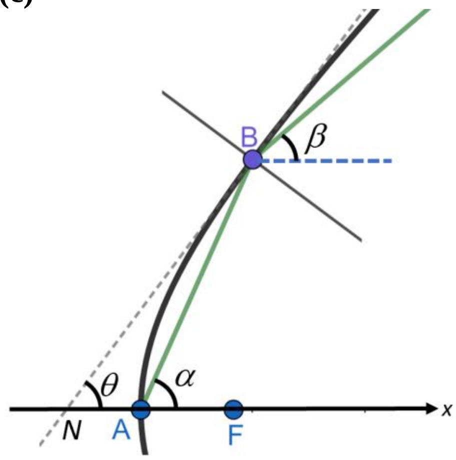
(c)(i) $y_3=\sum_{r=1}^{\infty}\left(\frac{1}{2^{2r-2}}\sin(2r-1)x\right)$

	$= \text{Im}\left(\sum_{r=1}^{\infty}\left(\frac{1}{2^{2r-2}} e^{i(2r-1)x}\right)\right)$ $= \text{Im}\left(\frac{e^{ix}}{1-2^{-2}e^{i2x}}\right)$ (using sum of GP formula) $= \text{Im}\left(\frac{e^{ix}(1-2^{-2}e^{-i2x})}{(1-2^{-2}e^{i2x})(1-2^{-2}e^{-i2x})}\right)$ $= \text{Im}\left(\frac{e^{ix}-2^{-2}e^{-ix}}{1-2^{-2}(2\cos 2x)+2^{-4}}\right)$ $= \frac{\sin x - 2^{-2}(-\sin x)}{\frac{17}{16}-\frac{1}{2}\cos 2x}$ $= \frac{20\sin x}{17-8\cos 2x}$ (shown) **(ii)** $\frac{dy_3}{dx} = \frac{(17-8\cos 2x)(20\cos x)-(20\sin x)(16\sin 2x)}{(17-8\cos 2x)^2}$ When $x=0$, $\frac{dy_3}{dx} = \frac{9(20)}{9^2} = \frac{20}{9} \approx 2.22$ We expect a good approximation for the square-wave curve to have a very steep gradient at the origin. However, the gradient for this approximate curve is only around 2.22, which does not reflect the sharp edges of the original curve. Therefore, y_3 is not a good approximation.  	
10	**(a)** Foci are $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$. **(b)** Equation of light beam: $y = \tan\alpha(x-1)$ $\Rightarrow x = 1+(\cot\alpha)y$ Since B is the point of intersection between the light beam and the hyperbola, $1+2(\cot\alpha)y+(\cot^2\alpha)y^2-y^2=1$ $(\cot^2\alpha-1)y^2+2(\cot\alpha)y=0$ Since $y \neq 0$, $(\cot^2\alpha-1)y=-2(\cot\alpha)$ $y=\frac{-2\cot\alpha}{\cot^2\alpha-1}=\frac{-2\tan\alpha}{1-\tan^2\alpha}$ $=-\tan 2\alpha$ $\Rightarrow x^2 = 1+\tan^2 2\alpha = \sec^2 2\alpha$ Note that $A(1, 0)$ lies on this curve, so the branch of the hyperbola is the one where $x \geq 1$.	

For x to be positive, we choose $x = -\sec 2\alpha$
(since $\sec 2\alpha < 0$ when $45° < \alpha < 90°$).

Hence, B has coordinates $(-\sec 2\alpha, -\tan 2\alpha)$.

(c)



Let N be the point of intersection of the tangent at B and the x-axis and θ be the angle BNA.

$x^2 - y^2 = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{x}{y}$

At B, $\dfrac{dy}{dx} = \dfrac{\sec 2\alpha}{\tan 2\alpha} = \dfrac{1}{\sin 2\alpha}$.

Therefore, $\tan\theta = \dfrac{1}{\sin 2\alpha} = \dfrac{1+t^2}{2t} = \dfrac{1}{2}\left(t + \dfrac{1}{t}\right)$

$$\Rightarrow \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{t + \frac{1}{t}}{1 - \frac{1}{4}\left(t + \frac{1}{t}\right)^2} = -\frac{4(t^3 + t)}{(t^2-1)^2}$$

$\angle ABN = \alpha - \theta$

Since angle of incidence is equal to angle of reflection,

$\beta = \theta - \angle ABN = 2\theta - \alpha$

	$\tan\beta = \tan(2\theta-\alpha)$ $=\dfrac{\tan 2\theta-\tan\alpha}{1+\tan 2\theta\tan\alpha}$ $=\dfrac{-\dfrac{4(t^3+t)}{(t^2-1)^2}-t}{1-\dfrac{4(t^4+t^2)}{(t^2-1)^2}}$ $=-\dfrac{4(t^3+t)+t(t^2-1)^2}{(t^2-1)^2-4(t^4+t^2)}$ $=-\dfrac{4t^3+4t+t^5-2t^3+t}{t^4-2t^2+1-4(t^4+t^2)}$ $=\dfrac{t^5+2t^3+5t}{3t^4+6t^2-1}$	