# Check your Understanding (Exponential & Logarithmic Functions)

## Section 1: Exponential Functions

1. By letting $y = e^x$, find the values of $x$ for which $e^x - 2e^{-x} = 1$. [4]

$e^x - 2e^{-x} = 1$

$\Rightarrow y - \dfrac{2}{y} = 1$

$\Rightarrow y^2 - y - 2 = 0$

$\Rightarrow (y-2)(y+1) = 0$

$\Rightarrow y = 2 \text{ or } -1$

$\Rightarrow e^x = 2 \text{ or } -1$ (reject $-1$ since $e^x > 0$)

$\Rightarrow x = \ln 2$

2. By means of the substitution $u = 2^{2x+1}$, or otherwise, solve the equation:

$$\frac{1}{2^{2x+1}} - 2^{2x+1} = \frac{3}{2}$$ [3]

$\dfrac{1}{u} - u = \dfrac{3}{2}$

$2 - 2u^2 = 3u$

$2u^2 + 3u - 2 = 0$

$(2u-1)(u+2) = 0$

$u = \dfrac{1}{2} \quad or \quad u = -2$

$2^{2x+1} = \dfrac{1}{2} = 2^{-1}$

$2x + 1 = -1$

$x = -1$

3. By using the substitution $y=3^x$, or otherwise, solve the equation

$$2\left(3^{1-x}\right)-9^{1+\frac{x}{2}}=53. \qquad [5]$$

$$2\left(\frac{3}{3^x}\right)-9\left(3^x\right)=53$$

Let $y=3^x$.

$$2\left(\frac{3}{y}\right)-9y=53$$

$$9y^2+53y-6=0$$

$$\left(9y-1\right)\left(y+6\right)=0$$

$\therefore y=\frac{1}{9}$ or $y=-6$

So, $3^x=\frac{1}{9}$ or $3^x=-6$ (reject)

$x=-2$

4. Find the exact answer(s) for the equation $3e^x = 2e^{-x} - 1$. [4]

Given $3e^x=2e^{-x}-1$,

Let $y=e^x$, $\quad 3y=\frac{2}{y}-1$

$$3y^2+y-2=0$$

$$(3y-2)(y+1)=0$$

$y=\frac{2}{3}$ or $-1$.

So $\quad e^x=\frac{2}{3}$ or $e^x=-1$ [N.A.]

$\therefore x=\ln\frac{2}{3}$.

5. Show that $3^{3x-1} = 27\left(\frac{1}{9}\right)^{y}$ can be expressed as $3x + 2y = 4$. [2]

$3^{3x-1} = 27\left(\frac{1}{9}\right)^{y}$

$3^{3x-1} = 3^{3}(3^{-2})^{y}$

$3^{3x-1} = 3^{3}(3^{-2y})$

$3^{3x-1} = 3^{3-2y}$

$3x - 1 = 3 - 2y$

$3x + 2y = 4$ (Shown)

6. Find the exact value of $x$ for which $14(2^{-x}) + 33 = 5(2^{x})$. [5]

$14(2^{-x}) + 33 = 5(2^{x})$

Let $u = 2^{x}$

$\frac{14}{u} + 33 = 5u$

$5u^{2} - 33u - 14 = 0$

$(5u + 2)(u - 7) = 0$

$u = -\frac{2}{5}(NA);\ u = 7$

$2^{x} = 7 \Rightarrow x = \frac{\ln 7}{\ln 2}$

7. Find the exact value of $x$ such that $e^{2x} = 6 + e^{x}$. [4]

$e^{2x} = 6 + e^{x}$

Let $y = e^x$,

$y^2 - y - 6 = 0$

$(y + 2)(y - 3) = 0$

$y = -2$ or $y = 3$

$e^x = -2$ or $e^x = 3$

(Rejected since $e^x > 0$) $x = \ln 3$

8. Use an algebraic method to solve the simultaneous equations

$$\left(\frac{1}{4}\right)^{2x-3} = 32^{y+1},$$
$$3^{x+y^2} = 1.$$

[5]

$$\left(\frac{1}{4}\right)^{2x-3} = 32^{y+1}$$
$$\Rightarrow 2^{-2(2x-3)} = 2^{5(y+1)}$$
$$\Rightarrow -4x + 6 = 5y + 5$$
$$\Rightarrow 5y + 4x = 1 \quad .......(1)$$

$$3^{x+y^2} = 1$$
$$\Rightarrow x + y^2 = 0$$
$$\Rightarrow x = -y^2 \quad ......(2)$$

Sub (2) into (1):
$$5y - 4y^2 = 1$$
$$4y^2 - 5y + 1 = 0$$
$$(4y - 1)(y - 1) = 0$$
$$\begin{cases} y = \frac{1}{4} \\ x = -\frac{1}{16} \end{cases} \quad \text{or} \quad \begin{cases} y = 1 \\ x = -1 \end{cases}$$

9. Solve the simultaneous equations $2^x - 5^y = 3$ and $2^{x-3} = 21 - 5^{y-2}$. [4]

Let $A = 2^x$, $B = 5^y$

$A - B = 3$ … (1)

$\frac{A}{8} = 21 - \frac{B}{25} \Rightarrow 25A + 8B = 4200$ …. (2)

Solving (1) & (2), $A = 128$, $B = 125$

Hence $2^x = 128 = 2^7 \Rightarrow x = 7$

and $5^y = 125 = 5^3 \Rightarrow y = 3$

## Section 2: Logarithmic Functions

1. (a) It is given that $w = \log_9 x$. Find $\frac{1}{x}$, in terms of $w$, [2]

(b) Given that $\ln x + 3\ln y - \ln 2 = \ln x^4 - \ln 54$, find the value of $\frac{x}{y}$. [4]

(a) $w = \log_9 x \Rightarrow x = 9^w \Rightarrow \frac{1}{x} = 9^{-w}$

(b) $\ln x + 3\ln y - \ln 2 = \ln x^4 - \ln 54$

$$\Rightarrow \ln\left(\frac{xy^3}{2}\right) = \ln\left(\frac{x^4}{54}\right)$$

$$\Rightarrow \frac{xy^3}{2} = \frac{x^4}{54}$$

$$\Rightarrow \left(\frac{x}{y}\right)^3 = 3^3$$

$$\Rightarrow \frac{x}{y} = 3$$

2. Solve the equation: $3+\log_2 x=\log_2(x+3)+\log_2(x-3)$ [3]

$$3+\log_2 x=\log_2(x+3)+\log_2(x-3)$$
$$\log_2(8x)=\log_2\left((x+3)(x-3)\right)$$
$$8x=(x+3)(x-3)$$
$$x^2-8x-9=0$$
$$(x-9)(x+1)=0$$
$$x=9 \quad or \quad x=-1$$
$$\log_2\left(\frac{x}{2}\right), x>0 \text{ so x}=-1 \text{ is not valid}$$
$$x=9$$

3. (i) Show that $\log_7(7x)-1=\log_7 x$. [2]

(ii) Hence solve the simultaneous equations $3^{3x-1}=27\left(\frac{1}{9}\right)^y$ and

$\log_7(7x)-1=\log_7(3y+5)$. [2]

(i)

$$\log_7(7x)-1$$
$$=\log_7 7x-\log_7 7$$
$$=\log_7\frac{7x}{7}$$
$$=\log_7 x \ \text{ (Shown)}$$

(ii)

$$3^{3x-1}=27\left(\frac{1}{9}\right)^y$$
$$3^{3x-1}=3^3\left(3^{-2}\right)^y$$
$$3^{3x-1}=3^{3-2y}$$
$$3x+2y=4 \text{ --------- (1)}$$

$$\log_7(7x) - 1 = \log_7(3y+5)$$
$$\log_7 x = \log_7(3y+5)$$
$$x = 3y+5 \quad \text{---------- (2)}$$

Solving (1) and (2)

$x = 2$ and $y = -1$

4. Solve the equations, leaving your answers in 3 significant figures where necessary.

(a) $2\lg(x-1) = \lg 5$, [3]

(b) $5^{y+1} = 7^y$. [2]

(a) $2\lg(x-1) = \lg 5$

$\lg(x-1)^2 = \lg 5$

$(x-1)^2 = 5$

$x^2 - 2x + 1 = 5$

$x^2 - 2x - 4 = 0$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2}$$

$$x = \frac{2 \pm \sqrt{4 \times 5}}{2}$$

$x = 3.24$ or $x = -1.24$ (reject) because lg (-ve) is undefined

(b)

$5^{y+1} = 7^y$

$(y+1)\lg 5 = y\lg 7$

$$\frac{y+1}{y} = \frac{\lg 7}{\lg 5}$$

$y + 1 = 1.209y$

$y = 4.78$

5. Given that $\log_2 x = p$ and $\log_2 y = q$, express $\log_2 y^x$ in terms of $p$ and $q$. [1]

$$\log_2 y^x = x\log_2 y$$
$$= (2^p)q$$

**Section 3: Word Problems**

1. Damian decides to deposit a fixed sum of money into his new bank account at the start of each year. He also decides that he will not draw any money out of the account, but just leave it, and any interest, to build up. At the end of $n$ years, when the interest for that year has been added, he will have a total of $\$5200\left(1.04^n - 1\right)$.

(i) How much will there be in Damian's account at the end of 9 years? Give your answer to the nearest dollar. [1]

(ii) After how many complete years will Damian have, for the first time, at least \$3500 in his account? [3]

(i)

$$5200\left(1.04^9 - 1\right) = \$2201.22 \approx \$2201$$

(ii)

$$5200\left(1.04^n - 1\right) > 3500$$

$$1.04^n > \frac{3500}{5200} + 1$$
$$n\lg 1.04 > \lg\left(\frac{3500}{5200} + 1\right)$$
$$n > \frac{\lg\left(\frac{3500}{5200} + 1\right)}{\lg 1.04}$$
$$n > 13.1$$

Ans: 14 years

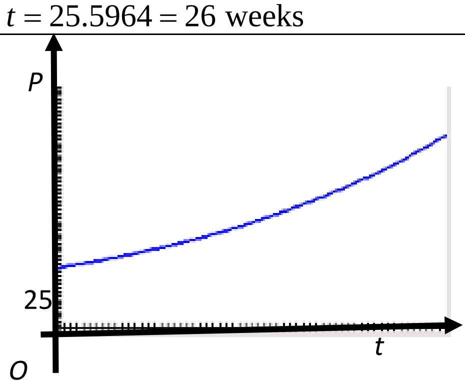
2. A biologist introduced a batch of 25 butterflies into a garden for breeding. The butterfly population, $P$, after $t$ weeks, is given by $P = 5 + Ae^{kt}$, where $A$ and $k$ are positive constants. After 12 weeks, there are 31 butterflies.

**(i)** Show that $A = 20$. [2]

**(ii)** Determine the value of $k$. [2]

**(iii)** Find the number of weeks it takes for the population to increase by 60% from the beginning. Leave your answer to the nearest integer. [3]

**(iv)** Sketch the graph of $P$ against $t$. [2]

| | |
|---|---|
| 2i | $P = 5 + Ae^{kt}$<br>$25 = 5 + Ae^{k(0)}$<br>$25 = 5 + A$<br>$A = 20$ |
| ii | $P = 5 + 20e^{kt}$<br>$31 = 5 + 20e^{k(12)}$<br>$26 = 20e^{k(12)}$<br>$e^{k(12)} = 1.3$<br>$12k = \ln(1.3)$<br>$k = 0.021863 = 0.0219$ (3sf) |
| iii | $1.6 \times 25 = 40$<br>$40 = 5 + 20e^{0.021863t}$<br>$e^{0.021863t} = 1.75$<br>$\ln e^{0.021863t} = \ln 1.75$<br>$0.021863t = 0.55961$<br>$t = 25.5964 = 26$ weeks |
| iv |  |

3. Desert locusts are a certain species of short-horned grasshoppers that are usually found in the deserts of Eutoria. When a drought begins, they start to breed quickly and destroy crops.

The population $P$ of the locusts (in thousands) after $t$ weeks of drought, is modelled by the equation

$$P = 10e^{0.5t} + k,$$

where $k$ is a positive real number.

**(i)** Given that the population of the desert locusts is $77\,880$ after 2 weeks, show that $k = 50.7$, correct to 3 significant figures. [2]

**(ii)** Using a non-calculator method,

**(a)** find the initial population of the desert locusts, [2]

**(b)** determine the number of complete weeks for the population of desert locusts to first exceed 5 million. [3]

**(iii)** Sketch, in the context of this question, the graph of $P$ against $t$, stating the coordinates of any points of intersection with the axes. [1]

A pest control company has been employed by the government of Eutoria to tackle the problem caused by the desert locusts. The company claims that the population $Q$ of the locusts (in thousands) after $t$ days of administering a biological pesticide is modelled by the equation

$$Q = 100e^{2-0.3t} - 44.$$

**(iv)** Determine the number of complete days for the population of the desert locusts to first fall below 1000. [2]

**(v)** Comment on the suitability of the company's model, justifying your answer. [1]

| | |
|---|---|
| **(i)** | At $t = 2$<br>$10e^{0.5(2)} + k = 77.88$<br>$k = 50.697 \approx 50.7$ (shown) |
| **(ii) (a)** | At $t = 0$<br>$P = 10e^{0} + 50.7 = 60.7$<br>Initial population is 60,700 |
| **(ii) (b)** | $P = 10e^{0.5t} + 50.7 > 5000$<br>$e^{0.5t} > 494.93$<br>$0.5t > \ln 494.93$<br>$t > 12.409$<br>Number of weeks is 13 |

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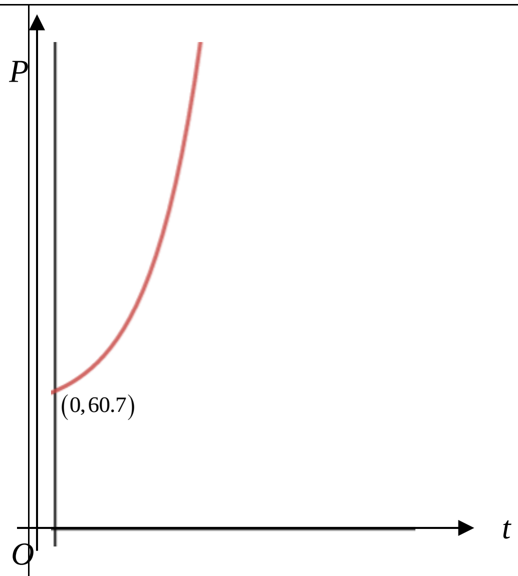
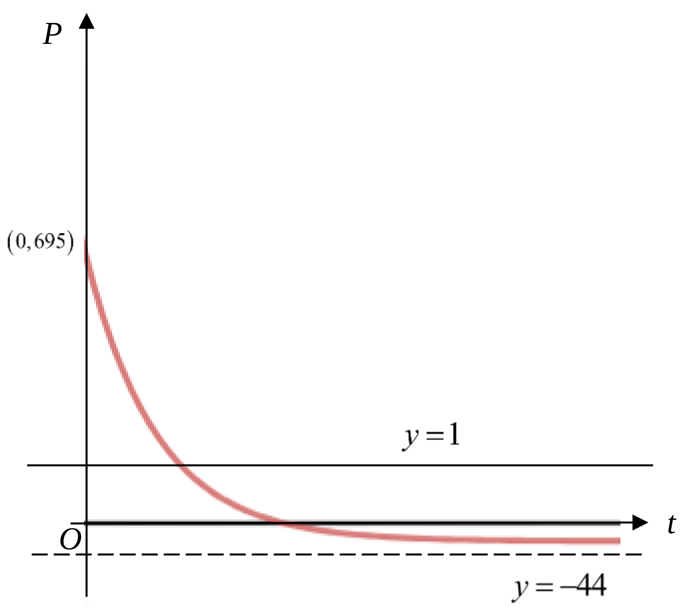
<tr><td>(iii)</td><td>
</td></tr>
<tr><td>(iv)</td><td><u>Method 1:</u><br>$Q = 100e^{2-0.3t} - 44 < 1$<br>$e^{2-0.3t} < 0.45$<br>$2 - 0.3t < \ln 0.45$<br>$t > 9.328$<br>Number of complete days is 10<br><u>Method 2:</u><br>
<br>From GC,<br>$t = 9.328$<br>Number of complete days is 10</td></tr>
<tr><td>(v)</td><td>Claim is not valid because if $t > 9.40$ days, the number of desert locusts is less than 0.</td></tr>
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