Chapter 13: Complex Numbers

Content Outline

Complex numbers expressed in cartesian form
Include:
- extension of the number system from real numbers to complex numbers
- complex roots of quadratic equations
- conjugate of a complex number
- four operations of complex numbers
- equality of complex numbers
- conjugate roots of a polynomial equation with real coefficients

Complex numbers expressed in polar form
Include:
- representation of complex numbers in the Argand diagram
- complex numbers expressed in the form , $r(\cos\theta + \mathrm{i}\sin\theta)$, or $r\mathrm{e}^{\mathrm{i}\theta}$ where $r > 0$ and $-\pi < \theta \le \pi$ and vice versa
- calculation of modulus (r) and argument (θ) of a complex number
- multiplication and division of two complex numbers expressed in polar form

Useful textbooks on applications of Complex Numbers
Reference Textbooks:
(1) Ho Soo Thong, Tay Yong Chiang & Koh Khee Meng, "College Mathematics Syllabus C Vol 1", Pan Pacific Publications, Call Number: HO510
(2) H2 Mathematics For 'A' Level, Federick Ho, David Khor, Yui-P'ng Lam, B.S. Ong Volume 1, Call No 510.76 HO

1. Introduction

Recall what we have covered at GCE O level:

For any quadratic equation $ax^2 + bx + c = 0$, there are 3 cases:

(1) $b^2 - 4ac > 0$, there are 2 real and distinct roots

(2) $b^2 - 4ac = 0$, there are 2 repeated real roots

(3) $b^2 - 4ac < 0$, there is no real root.

In this chapter, we are dealing with case 3 for all polynomial equations.
Consider the equation $x^2 + 1 = 0$.
We know that the above equation has **no real roots**, since $\sqrt{-1}$ is not a real number. How then do we solve the equation $x^2 + 1 = 0$?

<u>1.1 Definition of Complex Number</u>

The quadratic equation $x^2+1=0$, i.e. $x^2=-1$ has no real roots since there is no real number whose square is -1. To solve this equation, we need to find a "number" whose square is -1. Let's suppose such a "number" **i** exists and define it such that

$\boxed{\mathbf{i}^2=-\mathbf{1}}$, where $\mathbf{i}=\sqrt{-\mathbf{1}}$.

Thus, $x^2+1=0 \Rightarrow x^2=-1$

$$\Rightarrow x=\pm\sqrt{-1}$$

$$\Rightarrow x=\pm\,\mathrm{i}.$$

Beyond the real number system, **i** is called the **unit imaginary number** in the system of complex numbers.

<u>Remark:</u>

From the definition of i, we see that $\mathrm{i}^2=-1$, $\mathrm{i}^3=-\mathrm{i}$ and $\mathrm{i}^4=1$. Can you tell what is the value of i^{79}?

With the introduction of 'i', all equations have solutions (roots), be they real or complex roots.

<u>Example 1</u>

Solve $x^2-2x+5=0$.

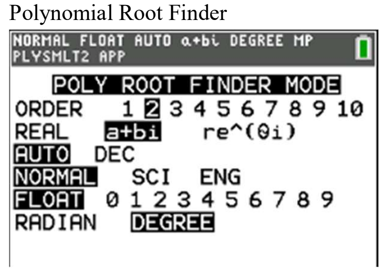
Solution:

$$x=\frac{2\pm\sqrt{4-4(1)(5)}}{2}$$

$$=\frac{2\pm\sqrt{-16}}{2}$$

<u>GC Corner</u>

Polynomial Root Finder



<u>Homework:</u> Verify your answer using the GC.

Any number denoted by z of the form $z=x+y\mathrm{i}$, where $x, y\in\mathbb{R}$ and $\mathrm{i}=\sqrt{-1}$, is called a **complex number**. x is called the real part of z, denoted by Re(z). y is called the imaginary part of z, denoted by Im(z).

The set of all complex numbers is denoted by $\mathbb{C}$. So $\mathbb{C}=\{x+y\mathrm{i}:\quad x,\ y\in\mathbb{R}\}$.

Example 2
Write down real and imaginary parts of z given in the table below.

z	Re(z)	Im(z)
$-3 + 2\text{i}$	-3	2
$2 - \text{i}$		
$-4 - 5\text{i}$		
2i		
-5		

IMPORTANT NOTE:
(1) A complex number can be expressed in various forms. The form $x + y\text{i}$ is called the **cartesian, rectangular or algebraic form**.
(2) Let $z = x + y\text{i}$. In general,
  (i) If ____, the complex number is purely **REAL.**
  (ii) If ____, the complex number is purely **IMAGINARY.**
  (iii) $x + y\text{i} = 0$ if and only if $x =$ ____ and $y =$_______.
  (iv) If w is another complex number defined by $w = a + b\text{i}$,
  then $z = w$ if and only if $x =$__and $y =$____.
  i.e. $x + y\text{i} = a + b\text{i} \Leftrightarrow x = a$ and $y = b$

1.2. The Complex Conjugate
The **complex conjugate** of z is denoted by z^* or $\overline{z}$. If $z = x + y\text{i}$ is a complex number, then the complex conjugate of z is $z^* = x - y\text{i}$. (note the change in sign before i).

Example 3
Write down complex conjugate z^* of z, given in the table below.

Z	z^*
$-3 + 2\text{i}$	$-3 - 2\text{i}$
$2 - \text{i}$	
-5i	
7	

If $z = x + y\text{i}$, it can be proven that z and z^* satisfy the following properties:

(1) $z + z^* = 2x$
(2) $z - z^* = 2y\text{i}$
(3) $zz^* = x^2 + y^2$, which is a real number
[Proof] Let $z = x + \text{i}y$, then $z^* = x - \text{i}y$.
$zz^* = (x + \text{i}y)(x - \text{i}y) = x^2 + y^2$, which is a real number

2. The Four Operations of Complex Numbers

Let $z_1 = 3-2\text{i}$ and $z_2 = -1+5\text{i}$. Then,

i) $z_1 + z_2 =$

ii) $z_1 - z_2 =$

iii) $z_1 z_2 =$

iv) $\frac{z_1}{z_2} =$

Note: The simplification of the division of two complex numbers is similar to the method of *rationalising the denominator*:

For real numbers: Rationalising Denominator

$$\frac{1}{1-\sqrt{2}} = \frac{1}{1-\sqrt{2}} \times \frac{1+\sqrt{2}}{1+\sqrt{2}}$$

$$= \frac{1+\sqrt{2}}{1-\left(\sqrt{2}\right)^2} = \frac{1+\sqrt{2}}{-1} = -1-\sqrt{2}$$

For complex numbers: Realising Denominator

$$\frac{\text{i}}{1-\sqrt{2}\text{i}} = \frac{\text{i}}{1-\sqrt{2}\text{i}} \times \frac{1+\sqrt{2}\text{i}}{1+\sqrt{2}\text{i}}$$

$$= \frac{\text{i}-\sqrt{2}}{1-\left(\sqrt{2}\text{i}\right)^2} = \frac{-\sqrt{2}+\text{i}}{1+2} = -\frac{\sqrt{2}}{3} + \frac{1}{3}\text{i}$$

Important Note: Whenever possible, you should use the GC to do your calculations unless exact values are required or No GC is allowed as specified in the question.

Homework: Verify your answers above with the use of the GC.

Example 4 (Do not use a calculator in answering this question)

(a) Simplify the following:

$(-5+2\text{i})(1-3\text{i}) =$

(b) It is given that $(1-3\text{i})(a+b\text{i}) = 2+\text{i}$, find the values of *a* and *b*.

$$(1-3\text{i})(a+b\text{i}) = 2+\text{i}$$

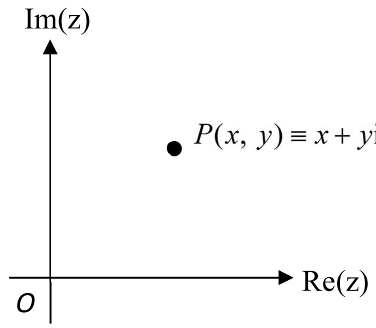
$$a+b\text{i} = \frac{2+\text{i}}{1-3\text{i}}$$

$$=$$

Think! Is there an alternative method for Example 4(b)? Hint: Look at Section 1.1

3. The Argand Diagram

A complex number $\boldsymbol{z = x + yi}$ can be represented by an ordered pair $\boldsymbol{P(x, y)}$ in a coordinate plane. Such a coordinate plane is called a **complex number plane** and the resulting diagram is called an **Argand diagram**.



Assuming $x > 0, y > 0$

Note:

(1) The x-axis is called the **real axis** and the y-axis the **imaginary axis**.

(2) All points $(x, 0)$ on the x-axis represent real numbers while all points $(0, y)$ on the y-axis represent purely imaginary numbers.

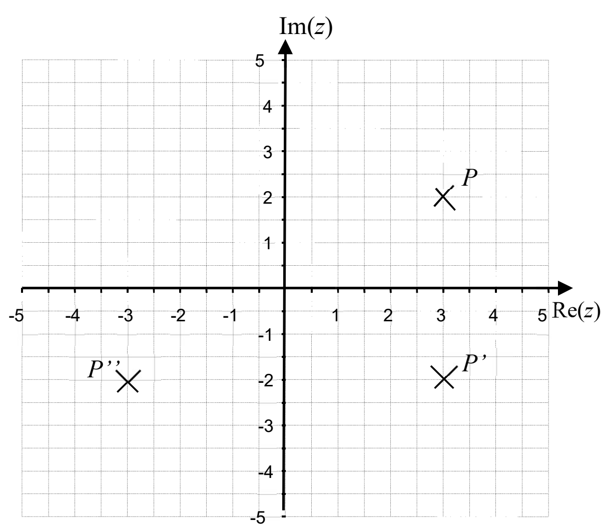
3.1. Representation of complex numbers in the Argand Diagram.

Example 5

Let the point P with coordinates (3, 2) represent the complex number $z = 3 + 2\mathrm{i}$.
On an Argand diagram, point P is plotted as shown below:

Plot on the same diagram, the points P' and P'' representing
z^* and $-z$ respectively.

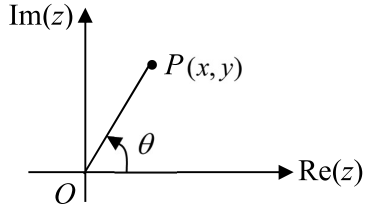
$z^* =$
$-z =$



Note: The point P' is a reflection of point P about the Re(z) axis.

3.2. Modulus and Argument

Let point P be represented by the complex number $z = x + y\text{i},\ x, y \in \mathbb{R}$, on an Argand diagram.

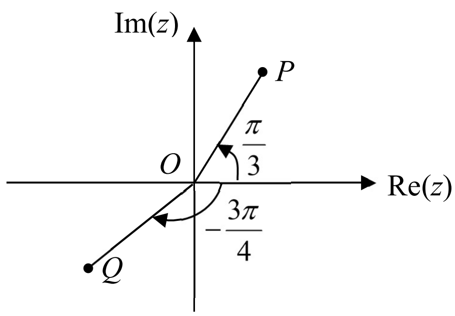


The **modulus** of the complex number z, denoted by $|z|$, is the length of OP.
By Pythagoras theorem, $|z| = r = \sqrt{x^2 + y^2}$.

The angle θ made by the positive x-axis and the line OP gives the **argument** of the complex number z, denoted by arg(z), where $-\pi < \arg(z) \le \pi$ (***principal argument*** range and expressed in radian mode)

Note: arg(z) is positive if it is measured in anti-clockwise direction and negative if it is measured in clockwise direction.

For example, if z is represented by the point P and w is represented by the point Q, as shown in the diagram below,



then $\arg(z) =$
and
$\arg(w) =$

(not $\frac{5\pi}{4}$, which is $> \pi$).

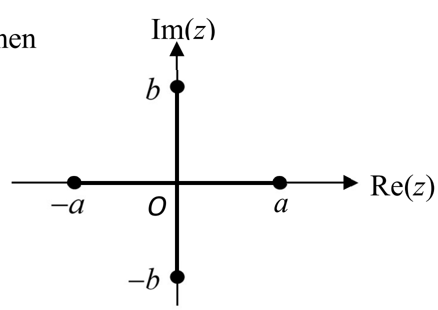
Note:
If a and b are positive real numbers, then

(i) $\arg(a) =$

(ii) $\arg(-a) =$

(iii) $\arg(b\text{i}) =$

(iv) $\arg(-b\text{i}) =$

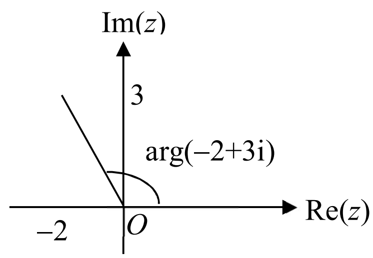


<u>Example 6</u> (Use of GC)

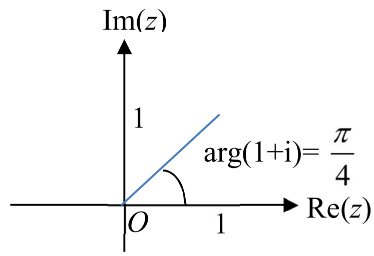
Find the modulus and argument of (a) $-2+3\text{i}$, (b) $1+\text{i}$, (c) $-1-\sqrt{3}\,\text{i}$, (d) $\sqrt{2}-\sqrt{2}\,\text{i}$

Solution:

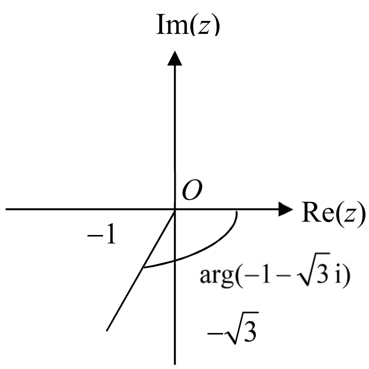
(a)



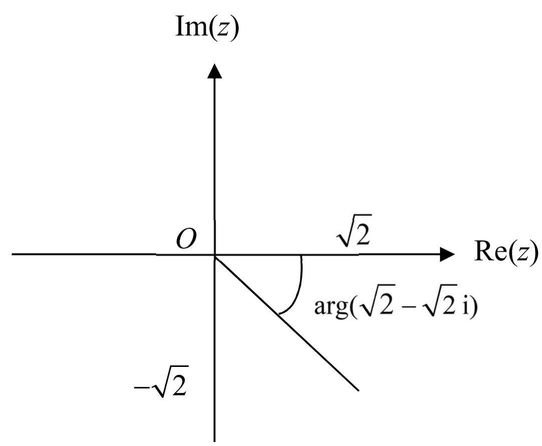
(b)



(c)



(d)

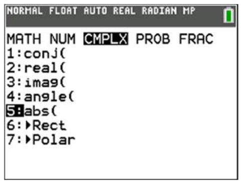


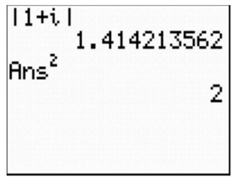
GC Corner

For a given complex number $z = x + \mathrm{i}y$, whenever $\left|\dfrac{y}{x}\right| = 1$ or $\dfrac{1}{\sqrt{3}}$ or $\sqrt{3}$, we should use GC to find the **exact** values of the modulus and argument of the complex number.

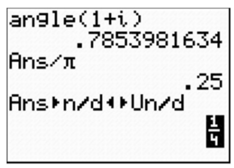
[Note: check to confirm that the mode is in **radian** mode.]

(i) To find $|z|$:





(ii) To find $\arg(z)$:



From GC, arg(z) = 0.7853981634

If 0.7853981634 = $k\pi$, then

$$k = \frac{0.7853981634}{\pi} = 0.25$$

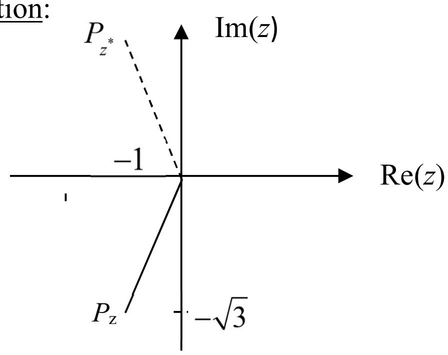
$$\therefore \quad \arg(z) = \frac{\pi}{4}$$

3.3. Properties of |z| and arg(z)

Example 7

Represent the complex number $z = -1 - \sqrt{3}\,\mathrm{i}$ and its conjugate on the Argand diagram below. Then find the modulus and argument of z and z^*.

Solution:



Property 1: The modulus of z and z^* is the same, i.e. $|z| = |z^*|$

Property 2: $\arg(z^*) = -\arg(z)$

z^* can be seen from Argand diagram as the reflection of complex number z about Re(z)-axis.

What do you observe if we find zz^*?

<u>Property 3:</u> $|z|^2 = zz^*$

[Proof] Let $z = x + \mathrm{i}y$, then $z^* = x - \mathrm{i}y$.

So $|z| = \sqrt{x^2 + y^2} = |z^*|$

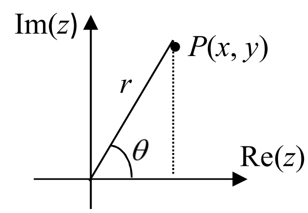
$zz^* = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2 = |z|^2$

4. Different Representations of Complex Numbers

4.1. Cartesian Form

$z = x + y\mathrm{i}$, where x is the real part and y is the imaginary part of z.

4.2. Polar - Trigonometric Form



From the Argand diagram on the right, we see that

$\cos\theta = \dfrac{x}{r} \Rightarrow x =$ $\qquad$ $\sin\theta = \dfrac{y}{r} \Rightarrow y =$

Therefore, $z = x + \mathrm{i}y$

$= (r\cos\theta) + \mathrm{i}(r\sin\theta)$

$= r(\cos\theta + \mathrm{i}\sin\theta)$

where r is the modulus of z and θ is the argument of z.
Note that $r > 0$ and $-\pi < \theta \leq \pi$.

4.3. Polar – Exponential Form (denoted as "polar form" in GC)

Euler's Formula: $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$, where θ is in radians.

Proof:

Using Maclaurin's series,
$$\mathrm{e}^{\mathrm{i}\theta} = 1 + \mathrm{i}\theta + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^4}{4!} + \ldots$$
$$= 1 + \mathrm{i}\theta - \frac{\theta^2}{2!} - \mathrm{i}\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots$$
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots\right) + \mathrm{i}\left(\theta - \frac{\theta^3}{3!} + \ldots\right)$$
$$= \cos\theta + \mathrm{i}\sin\theta$$

Thus $z = x + y\mathrm{i} = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta}$, where $r = |z| = \sqrt{x^2 + y^2}$ and θ is the argument of z.

Example 8

Given that $z = 2\mathrm{e}^{-\mathrm{i}\frac{2}{3}\pi}$, represent the complex number z on an Argand diagram.

Solution:

Remarks

Note the different forms of a complex number z and its conjugate z^*:

Form	z	z^*
Cartesian/algebraic/rectangular	$x+\mathrm{i}y$	$x-\mathrm{i}y$
Modulus-argument/trigonometric/Polar	$r(\cos\theta+\mathrm{i}\sin\theta)$	$r\left[\cos\left(-\theta\right)+\mathrm{i}\sin\left(-\theta\right)\right]$ $=r(\cos\theta-\mathrm{i}\sin\theta)$
Modulus-argument/Euler's/exponential/Polar	$r\mathrm{e}^{\mathrm{i}\theta}$	$r\mathrm{e}^{-\mathrm{i}\theta}$

Example 9

Express $z = 1+\sqrt{3}\mathrm{i}$ in trigonometric and exponential forms.

Solution:

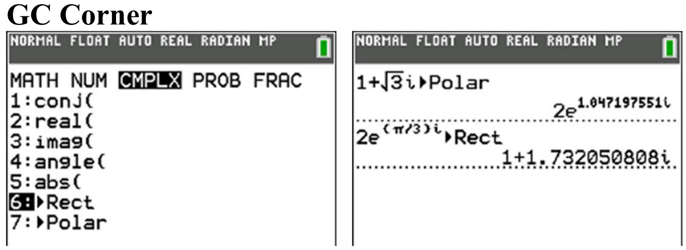
In this case, the modulus of z is $|z| = 2$ and the argument of z is $\arg(z) = \frac{\pi}{3}$.

Then $z =$ (polar – trigonometric form)

$=$ (polar – exponential form)

Note: You can use the GC to convert between Cartesian and polar forms. However, note that the expression given by the GC may not be expressed in exact form.

GC Corner



Example 10

(a) Convert the following complex numbers into trigonometric form and exponential form. (i) i (ii) -2 (iii) -4i

(b) Convert $\sqrt{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)$ into exponential and Cartesian form.

(c) Convert $2\mathrm{e}^{\mathrm{i}\frac{5}{6}\pi}$ into trigonometric and Cartesian form.

Solution:

(a) (i) $|\mathrm{i}| =$ $\arg(\mathrm{i}) =$

$\mathrm{i} =$

(ii) $|-2| =$ $\arg(-2) =$

$-2 =$

(iii) $|-4\mathrm{i}| =$ $\arg(-4\mathrm{i}) =$

$-4\mathrm{i} =$

(b) $\sqrt{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)=\sqrt{2}\cos\frac{\pi}{4}+\mathrm{i}\sqrt{2}\sin\frac{\pi}{4}=$

(c) $2\mathrm{e}^{\mathrm{i}\frac{5}{6}\pi} =$

5. Multiplication and Division of Complex Numbers in Polar Form

Earlier in Section 2, we saw how two complex numbers in the Cartesian form can be easily added up and subtracted from each another. However, evaluating $z_1 \times z_2$ and $\frac{z_1}{z_2}$ can be quite tedious in certain instances, e.g. when we want to find $\left(\sqrt{3}-\mathrm{i}\right)^3\left(-1+\sqrt{3}\mathrm{i}\right)$ or $\frac{\left(\sqrt{3}-\mathrm{i}\right)^3}{\left(-1+\sqrt{3}\mathrm{i}\right)}$.

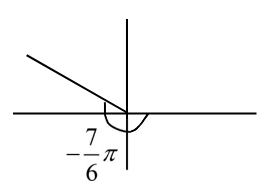
Now, let us express $z_1=\left(\sqrt{3}-\mathrm{i}\right)^3$ and $z_2=-1+\sqrt{3}\mathrm{i}$ in exponential form:

$$z_1=\left(2\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}\right)^3=8\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}} \quad \text{and} \quad z_2=2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}.$$

Then, using laws of indices, we have

$$z_1z_2=\left(8\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}\right)\left(2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\right)=16\mathrm{e}^{\left(-\mathrm{i}\frac{\pi}{2}+\mathrm{i}\frac{2\pi}{3}\right)}=16\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$$
$$=16\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)$$
$$=8\sqrt{3}+8\mathrm{i}$$

$$\frac{z_1}{z_2}=\frac{8\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}}{2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}}=4\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}-\mathrm{i}\frac{2\pi}{3}}=4\mathrm{e}^{-\mathrm{i}\frac{7}{6}\pi}=4\mathrm{e}^{\mathrm{i}\frac{5}{6}\pi}$$
$$=4\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)$$
$$=-2\sqrt{3}+2\mathrm{i}$$



Hence, if $z_1 = r_1 \mathrm{e}^{\mathrm{i}\theta_1} = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2 \mathrm{e}^{\mathrm{i}\theta_2} = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$, then

$$\boxed{\begin{aligned} z_1 z_2 &= r_1 r_2 \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)} = r_1 r_2[\cos(\theta_1+\theta_2) + \mathrm{i}\sin(\theta_1+\theta_2)], \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2}\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + \mathrm{i}\sin(\theta_1-\theta_2)]. \end{aligned}}$$

We see that it is *easier to multiply and divide complex numbers in the polar form*.

<u>**Note**</u>:
We should always check that the final argument found lies within the principal range $(-\pi, \pi]$. If the angle lies outside the range, we must **add or subtract a constant multiple of 2π**.

<u>**Example 11**</u>

Given that $z_1 = 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}$ and $z_2 = 5\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$, find

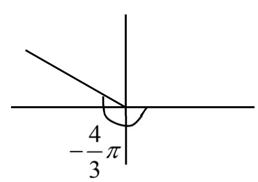
(a) $\dfrac{z_1}{z_2}$ (b) $z_1 z_2$ (c) $(z_1)^8$ in exponential form.

Solution:

a) $\dfrac{z_1}{z_2} =$

b) $z_1 z_2 =$

c) $(z_1)^8 =$



<u>**5.1. More properties of $|z|$ and $\arg(z)$**</u>

Continued from Section 3.3, we have more properties of $|z|$ and $\arg(z)$:

<u>Property 4:</u> $|z_1 z_2| = |z_1| \cdot |z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

<u>Property 5:</u> $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ and $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

<u>Property 6</u>: (as an extension <u>to property 4</u>) $|z^n| = |z|^n$ and $\arg(z^n) = n\arg(z)$.

Proof of 4 & 5:

Let $z_1 = r_1 \mathrm{e}^{\mathrm{i}\theta_1}$, where $r_1 = |z_1|$, $\theta_1 = \arg(z_1)$, and $z_2 = r_2 \mathrm{e}^{\mathrm{i}\theta_2}$, where $r_2 = |z_2|$, $\theta_2 = \arg(z_2)$.
Then,

$$z_1 z_2 = \left(r_1 \mathrm{e}^{\mathrm{i}\theta_1}\right)\left(r_2 \mathrm{e}^{\mathrm{i}\theta_2}\right) = r_1 r_2 \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}.$$

$$\therefore\ |z_1 z_2| = r_1 r_2 = |z_1||z_2| \text{ and } \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2).$$

Now, $$\frac{z_1}{z_2} = \frac{\left(r_1 \mathrm{e}^{\mathrm{i}\theta_1}\right)}{\left(r_2 \mathrm{e}^{\mathrm{i}\theta_2}\right)} = \frac{r_1}{r_2}\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)}$$

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \text{ and } \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

Example 12

The complex number w has modulus $\sqrt{2}$ and argument $-\frac{3}{4}\pi$, and the complex number z has modulus 2 and argument $-\frac{1}{3}\pi$. Find the modulus and the argument of (i) wz (ii) $\frac{w}{z^2}$.

Solution:

Example 13

Express $z=\dfrac{1+\mathrm{i}}{\sqrt{3}-\mathrm{i}}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. Hence show that $\cos\dfrac{5\pi}{12}=\dfrac{\sqrt{3}-1}{2\sqrt{2}}$ and $\sin\dfrac{5\pi}{12}=\dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

Solution:

From (1) and (2), we have $\dfrac{\sqrt{2}}{2}\left(\cos\dfrac{5\pi}{12}+\mathrm{i}\sin\dfrac{5\pi}{12}\right)=\left(\dfrac{\sqrt{3}-1}{4}\right)+\mathrm{i}\left(\dfrac{\sqrt{3}+1}{4}\right)$.

Equating real parts: $\dfrac{\sqrt{2}}{2}\cos\dfrac{5\pi}{12}=\left(\dfrac{\sqrt{3}-1}{4}\right)=\dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

$$\cos\frac{5\pi}{12}=\frac{2}{\sqrt{2}}\left(\frac{\sqrt{3}-1}{4}\right)=\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Equating imaginary parts: $\dfrac{\sqrt{2}}{2}\sin\dfrac{5\pi}{12}=\dfrac{\sqrt{3}+1}{4}$

$$\sin\frac{5\pi}{12}=\frac{2}{\sqrt{2}}\left(\frac{\sqrt{3}+1}{4}\right)=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

<u>Example 14</u> **PJC Prelim 9758/2017/02/Q4**

Do not use a graphing calculator in answering this question.

The complex number z is given by $z=-1+\mathrm{i}c$, where c is a non-zero real number. Given that $\dfrac{z^n}{z^*}$ is purely real, find

(i) the possible values of c when $n=2$, [4]

(ii) the three smallest positive integer values of n when $c=\sqrt{3}$. [5]

Solution:

(i)

$$\frac{z^2}{z^*}=\frac{(-1+\mathrm{i}c)^2}{(-1-\mathrm{i}c)}$$

$$=\frac{1-\mathrm{i}2c-c^2}{(-1-\mathrm{i}c)}$$

$$=\frac{1-\mathrm{i}2c-c^2}{(-1-\mathrm{i}c)}\times\frac{(-1+\mathrm{i}c)}{(-1+\mathrm{i}c)}$$

$$=\frac{-1+\mathrm{i}c+\mathrm{i}2c+2c^2+c^2-\mathrm{i}c^3}{1+c^2}$$

Since $\dfrac{z^2}{z^*}$ is purely real,

(ii)

$z = -1 + \mathrm{i}\sqrt{3}$

$|z| = 2,$ $\arg(z) = \dfrac{2\pi}{3}$ $|z^*| =$ $\arg(z^*) =$

$$\frac{z^n}{z^*} = \frac{\left(2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\right)^n}{2\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}} = 2^{n-1}\mathrm{e}^{\mathrm{i}\left(\frac{2n\pi}{3} + \frac{2\pi}{3}\right)}$$

6. Solving Simple Equations involving Complex Numbers

There are 3 main approaches to solving equations involving complex numbers.

- using complex conjugates
- comparing real and imaginary parts
- solving simultaneous equations by elimination/substitution

6.1. Solution of equation using conjugate pair of Roots

Example 15

Solve the following equations: (a) $z^2 - 2z + 10 = 0$, (b) $z^2 - 2\mathrm{i}z + 10 = 0$.

Solution:

(a) $z = \dfrac{2 \pm \sqrt{4 - 4(1)(10)}}{2}$ (b) $z = \dfrac{2\mathrm{i} \pm \sqrt{(-2\mathrm{i})^2 - 4(1)(10)}}{2}$

What do you notice about the roots of equation (a) and (b)?

In equation (a), the roots ______________________________.

In equation (b), the roots ______________________________.

Key Result:

The ***Complex Conjugate Root Theorem*** states that the complex roots of a **polynomial equation with real coefficients** occur in conjugate pairs.
If $P(z)$ is a polynomial with **real coefficients**, and if a complex number $a + b\mathrm{i}$ is a root of the equation $P(z) = 0$, then its complex conjugate $a - b\mathrm{i}$ will also be a root of the equation $P(z) = 0$.

Example 16

Given that one of the roots is complex in each of the following equations, which equations will have roots occuring in conjugate pairs?

(a) $2z^4 - 5z^3 + 2z^2 - \pi z + 3 = 0$

(b) $z^3 - \mathrm{i}z^2 + 3 = 0$

(c) $z^3 - 3\ln z + 2 = 0$

Solution:

Example 17

Given that $z^3+z^2+7z+7=0$, verify that one of the roots of the equation is $\sqrt{7}$ i and hence find the other 2 roots without using a graphing calculator.

Solution:
Sub $z=\sqrt{7}\text{i}$ into the equation,

$$z^3+z^2+7z+7=\left(\sqrt{7}\text{i}\right)^3+(\sqrt{7}\text{i})^2+7\sqrt{7}\text{i}+7=-7\sqrt{7}\text{i}-7+7\sqrt{7}\text{i}+7=0.$$

Therefore, $z=\sqrt{7}\text{i}$ is a root of the equation.

Since the polynomial equation has real coefficients, ______________ is another root.

So $z^3+z^2+7z+7=$

$=$

(by long division or compare coefficient)

$z=-1,\sqrt{7}\text{i},-\sqrt{7}\text{i}$

Thus, the roots of the equation are -1, $\sqrt{7}\text{i}$ or $-\sqrt{7}\text{i}$

Example 18

If one of the roots of the equation $z^3-z^2+3z+5=0$ is $1-2\text{i}$, find the other two roots without using a graphing calculator.

Solution:
Since $1-2\text{i}$ is a root and the equation $z^3-z^2+3z+5=0$ has real coefficients, then ______ will be another root. The third root must be a ________________________

$z^3-z^2+3z+5=$

Comparing constant terms:

Therefore, the other two roots are

6.2. Solving equation by comparing real and imaginary parts

Refer to page 3: If $z = x + \mathrm{i}y$ and $w = a + \mathrm{i}b$, then $z = w$ implies $x = a$ and $y = b$.

Example 19

Solve the equation $z^2 = 4+2\sqrt{5}\mathrm{i}$, without using a graphing calculator, expressing them in the form $x+y\mathrm{i}$, where $x, y \in \mathbb{R}$.

Solution:

Let $z = x + y\mathrm{i}$ where $x, y \in \mathbb{R}$

$$(x+y\mathrm{i})^2 = 4+2\sqrt{5}\mathrm{i}$$

$$x^2 - y^2 + 2xy\mathrm{i} = 4+2\sqrt{5}\mathrm{i}$$

Comparing real and imaginary parts,

Substituting (2) into (1): $\left(\dfrac{\sqrt{5}}{y}\right)^2 - y^2 = 4$

$y^4 + 4y^2 - 5 = 0$

Always advantageous to check answer with GC

Substituting into (2): When $y = 1$, we have $x = \sqrt{5}$.

When $y = -1$, we have $x = -\sqrt{5}$.

Hence, the roots are

(Note that in Example 19, the roots obtained are NOT in conjugate pair as not all the coefficients of the polynomial equation are real).

6.3. Solving equations by elimination/ substitution

Example 20 **[N98/P2/12]**
By eliminating *w*, or otherwise, solve the simultaneous equations:

$z = w + 3\mathrm{i} + 2$ (1)

$z^2 - \mathrm{i}w + 5 - 2\mathrm{i} = 0$ (2)

Solution:
Method 1: (Substitution Method)
Take the simpler equation and make *w* the subject: From (1), $w = z - 3\mathrm{i} - 2$ (3)
Substitute into the other equation to get: $z^2 - \mathrm{i}(z - 3\mathrm{i} - 2) + 5 - 2\mathrm{i} = 0$

$$z^2 - \mathrm{i}z + 2 = 0$$

Solving:

Thus, when $z = 2\mathrm{i}$, $w = -2 - \mathrm{i}$
when $z = -\mathrm{i}$, $w = -2 - 4\mathrm{i}$

Method 2: (Elimination method).

Multiplying (1) by____, we have:____________________________ (4)

(2) – (4):

$$\begin{aligned} z &= \frac{\mathrm{i} \pm \sqrt{(-\mathrm{i})^2 - 4(2)}}{2} \\ &= \frac{\mathrm{i} \pm \sqrt{-9}}{2} \\ &= \frac{\mathrm{i} \pm 3\mathrm{i}}{2} \\ &= 2\mathrm{i} \text{ or } -\mathrm{i} \end{aligned}$$

Substituting into (1): When $z = 2\mathrm{i}$, $w = 2\mathrm{i} - 3\mathrm{i} - 2 = -2 - \mathrm{i}$
When $z = -\mathrm{i}$, $w = -\mathrm{i} - 3\mathrm{i} - 2 = -2 - 4\mathrm{i}$

Appendix on O level Trigonometry

Special Angles

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sinθ	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
cosθ	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$
tanθ	$\frac{\sqrt{0}}{\sqrt{4}}$	$\frac{\sqrt{1}}{\sqrt{3}}$	$\frac{\sqrt{2}}{\sqrt{2}}$	$\frac{\sqrt{3}}{\sqrt{1}}$	$\frac{\sqrt{4}}{\sqrt{0}}$

4 quadrants