Completing The Square

$x^2 - 3x + 7$

$x^2 - 3x + (-\frac{3}{2})^2 - (-\frac{3}{2})^2 + 7$

$(x - \frac{3}{2})^2 + \frac{19}{4}$

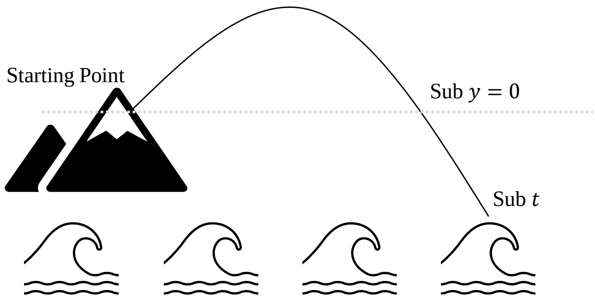
Coefficient of x^2 must be: Positive and One

Factorise if it is not e.g.
$-2x^2 + 4x + 8$
$= -2(x^2 - 2x - 4)$
Then you conduct C.T.S, on the expression in Bracket.

When x^2 *is not* 1

$$
\begin{aligned}
&-2x^2 + 4x + 8 \\
&= -2(x^2 - 2x - 4) \\
&= -2[(x^2 - 2x + (\frac{-2}{2})^2 - (\frac{-2}{2})^2 - 4] \\
&= -2[(x-1)^2 - 5] \\
&= -2(x-1)^2 + 10
\end{aligned}
$$

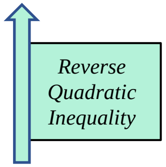
Application



Quadratic Inequalities

$(x+3)(x-4) > 0$
$x < -3$ *or* $x > 4$

$(x+3)(x-4) < 0$
$-3 < x < 4$



Step 1: Flush everything to the left and rearrange according to $ax^2 + bx + c$
Step 2: Simplify and rearrange according to $ax^2 + bx + c$
Step 3: Solve your quadratic inequalities

Note: Always ensure x^2 is Positive,
If it's negative, divide and FLIP Your INEQUALITY SIGN.

Reverse Inequalities

Step 1: Given that $x < -3$ or $x > 2$
Step 2: $(x+3)(x-2) > 0$ (Reverse and Form Back Original)
Step 3: $x^2 + x - 6 > 0$ (Expand)

Simultaneous Equations

Algebra

$x^2 + y^2 = 34 \ldots (1)$

$y + 3x = 14 \ldots (2)$

Using (2)

$$y = 14 - 3x$$

Substitute into (1)

$$x^2 + (14 - 3x)^2 = 34$$
$$x^2 + (196 - 84x + 9x^2) = 34$$
$$10x^2 - 84x + 162 = 0$$
$$(x - 3)(5x - 27) = 0$$
$$x = 3 \text{ or } x = \frac{27}{5}$$

Substitute into (2)

$y + 3(3) = 14$ $\quad$ $y + 3\left(\frac{27}{4}\right) = 14$

$y = 14 - 9 = 5$ $\quad$ $y = 14 - (3)\frac{27}{4} = -\frac{11}{5}$

Answer: $x = 3, y = 5$ $\quad$ $x = 5\frac{2}{5}, y = \ -2\frac{1}{5}$

Word Problems

The line $2x + 3y = 8$ meets the curve $2x^2 + 3y^2 = 110$ at the point A and B. Find the coordinates of A and B.

$$2x = 8 - 3y$$
$$x = \frac{8 - 3y}{2}$$

Substitute into (2)

$$2\left(\frac{8 - 3y}{2}\right)^2 + 3y^2 = 110$$

$$2\left(\frac{64 - 48y + 9y^2}{4}\right) + 3y^2 = 110$$

$$128 - 96y + 18y^2 + 12y^2 = 440$$

$$30y^2 - 96y - 312 = 0$$

$$10y^2 - 32 - 104 = 0$$

$$(y + 2)(5y - 26) = 0$$

$$y = -2, y = \frac{26}{5}$$

Substitute into (1)

$x = \frac{8-3(-2)}{2} = 7$ $\quad$ $x = \frac{8-3(\frac{26}{5})}{2} = -\frac{19}{5}$

Answer: $(7, -2)$ and $(-3\frac{4}{5}, 5\frac{1}{5})$

Important Concepts

Concept:

- There are 2 methods to solve for Simultaneous, either Substitution Method or Elimination Method.
- I highly recommend to use Substitution Method as I find that it is faster and easier.

Validation

For Simultaneous Equations,

- Validate by Substituting Your Final Answer back into the Original Question
- If the question is related to coordinates, ensure that you leave your answers in (x, y)

Completing The Square

Easy

Simplify

$$x^2 + 4x - 12$$

$$= x^2 + 4x + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2 - 12$$

$$= (x+2)^2 - 16$$

Hence, Solve

$$x^2 + 4x - 12 = 0$$

$$(x+2)^2 - 16 = 0$$

$$(x+2)^2 = 16$$

$$x + 2 = 4 \quad \text{or} \quad x + 2 = -4$$

Simplify

$$x^2 - 6x + 8$$

$$= x^2 - 6x + \left(-\frac{6}{2}\right)^2 - \left(-\frac{6}{2}\right)^2 + 8$$

$$= (x-3)^2 - 1$$

Hence, Solve

$$(x-3)^2 - 1 = 0$$

$$(x-3)^2 = 1$$

$$x - 3 = 1 \quad \text{or } x - 3 = -1$$

$$x = 4 \quad \text{or} \quad x = 2$$

Advance

Simplify

$$-x^2 - x + 2$$

$$= -(x^2 + x - 2)$$

$$= -[(x^2 + x + \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right) - 2]$$

$$= -[\left(x + \frac{1}{2}\right)^2 - \frac{5}{2}]$$

$$= -\left(x - \frac{1}{2}\right)^2 + \frac{5}{2}$$

Simplify

$$2x^2 - 5x + 9$$

$$= 2(x^2 - \frac{5}{2}x + \frac{9}{2})$$

$$= 2[(x^2 - \frac{5}{2}x + \left(-\frac{5}{4}\right)^2 - \left(-\frac{5}{4}\right)^2 + \frac{9}{2}]$$

$$= 2[\left(x - \frac{5}{4}\right)^2 + \frac{47}{16}]$$

$$= 2\left(x - \frac{5}{4}\right)^2 + \frac{47}{8}$$

Important Concepts

Concept:

1. The condition for Completing The Square is that the coefficient of x^2 MUST BE +1. If it is not +1, we need to FACTORISE the value to make it +1.
2. Be careful of the values you substitute in the bracket. Always include the **SIGN.**
3. When solving and completing the square, always solve by Square Rooting the values.

   NEVER expand back and solve by factorisation. That defeats the purpose of Completing The Square.

Validation

- After you get your final answer, re-expand back to make sure it gives you back the original answer.
- If you are solving, you can check your solution by using the Quadratic Equation Function in your calculator... they should be the same.

Graphical Methods

Method 1: Fully Factorised

Sketch $y = x^2 + 3x + 2$

1) Find the Roots

$$Sub\ y = 0$$
$$x^2 + 3x + 2 = 0$$
$$(x+2)(x+1) = 0$$
$$x = -1\ or\ x = -2$$

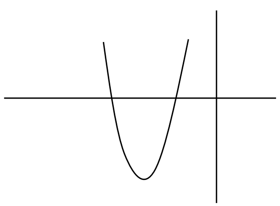
2) Find the y-intercept

$$Sub\ x = 0$$
$$y = 2$$

3) Find the turning point

$$\frac{Sum\ of\ Roots}{2} = \frac{-1 + -2}{2} = -1.5$$
$$Line\ of\ Symmetry\ x = -1.5$$
$$Sub\ x = -1.5$$
$$y = (-1.5)^2 + 3(-1.5) + 2 = -0.25$$

Turning point $(-1.5, -0.25)$



Method 2: Completing the Square

Sketch $y = (x+2)^2 - 9$

1) Find the turning point

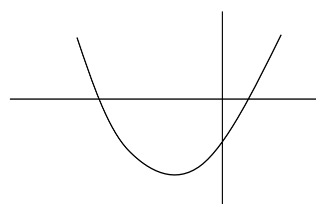
$$(-2, -9)$$

2) Find the y-intercept

$$Sub\ x = 0$$
$$y = (2)^2 - 9 = -5$$

3) Find the roots

$$Sub\ x = 0$$
$$(x+2)^2 - 9 = 0$$
$$(x+2)^2 = 9$$
$$x + 2 = 3\ or\ x + 2 = -3$$
$$x = 1\ or\ x = -5$$



Important Concepts

Concept:
There are two types of graph sketching.
1) Fully Factorised Equation
2) Completing The Square Equation
The steps are different due to the ease & convenience of finding the points.
When drawing a quadratic graph, I'm interested in knowing 3 things:
1) Roots 2) Y Intercept 3) Turning Point

Finding Roots and Y Intercepts are similar for both types of graphs.
However, the key difference is in finding the Turning Point.
Look at how I obtained the turning point for both methods!

Careless:
Coefficient of x^2 - Happy or Sad Face
Coordinates (x, y) vs Value – Number
Line of Symmetry - Equation

Validation

After you sketch your graph, make sure that the values make sense.
Curve, Turning Point, Roots, Y Intercept,

Applications

Word Problems

The path of a water jet can be modelled by the quadratic function $y = C(x-1.2)^2 + 2.25$, where x m is the horizontal distance it travels, y m is the height of the water above the ground and C is a constant. The initial height of the water jet is 1.05 m above the ground.

(i) Find the value of C.
(ii) Find the maximum height above the ground that the water jet reaches.
(iii) Find the value of x for which the water jet is 1.05m above the ground again.
(iv) Find the maximum horizontal distance travelled by the water jet

(i) $y = C(x-1.2)^2 + 2.25$
Sub $x = 0$, $y = 1.05$

$$1.05 = C(-1.2)^2 + 2.25$$
$$-1.2 = C(1.44)$$

$C = -\frac{5}{6}$ or -0.833

(ii) 2.25mm

(iii) $y = -\frac{5}{6}(x-1.2)^2 + 2.25$
Sub $y = 1.05$

$$1.05 = -\frac{5}{6}(x-1.2)^2 + 2.25$$
$$-1.2 = -\frac{5}{6}(x-1.2)^2$$
$$1.44 = (x-1.2)^2$$

1.2 or $-1.2 = x - 1.2$
$x = 2.4$ or $0(NA)$

(iii) $y = -\frac{5}{6}(x-1.2)^2 + 2.25$
Sub $y = 0$

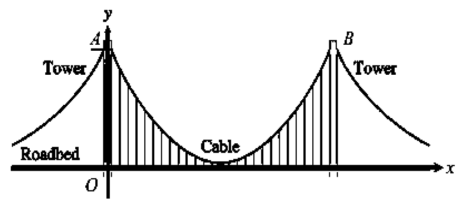
$$0 = -\frac{5}{6}(x-1.2)^2 + 2.25$$
$$-2.25 = -\frac{5}{6}(x-1.2)^2$$
$$2.7 = (x-1.2)^2$$

$\sqrt{2.7}$ or $-\sqrt{2.7} = x - 1.2$
$x = 2.84$ or $-0.443(NA)$
Max horizontal distance = 2.84m

Word Problems

A support cable for a bridge is parabolic in shape. The cable is supported by 25 m tall towers A and B that are 80 m apart. Vertical supporting wires are spread out in equal intervals hanging from cable.

The lowest point on the cable is 5 m above the roadbed. The height of the cable above the roadbed is given as y m and the horizontal distance from Tower A is given as x m.



(i) Find a quadratic function in the form

$y = a(x-h)^2 + k$ to model this situation.

(ii) Find the length of the vertical supporting wire that is 15 m horizontally from the origin.

At lowest point, (40,5)

So, $y = a(x-40)^2 + 5$

When $x = 0$, y= 25

$$25 = a(0-40)^2 + 5$$

$$a = \frac{1}{80}$$

When $x = 15$, $y = \frac{1}{80}(15-40)^2 + 5$
$y = 12.8125$

Length of the wire is 12.8125 m $\left(\text{Accept } 12\frac{13}{16}\text{m}\right)$

Important Concepts

Concept:
Many students struggle with this because it feels odd and challenging. However, this portion is just applying the concepts from Completing The Square.

Under Completing The Square, we learn a few things:

1) You can only complete the square if the coefficient of x^2 is +1.

2) Obtaining Turning Points (Line of Symmetry, Maximum and Minimum Value)

3) Solving Completing The Square via Square root Method and not Quadratic Formula

Sit down 15 minutes, internalise this and you will definitely get it right!

Validation

Validation of Completing The Square requires you to expand back to double check if it gives you the original equation. I typically will do this before continuing with the question because I don't want to risk redoing the whole question if I make a mistake in my completing the square steps.

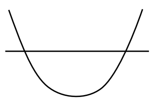
Quadratic Inequalities

Quadratic Inequalities

Solve $x^2 + 3x + 2 > 0$

$$x^2 + 3x + 2 > 0$$

$$(x + 2)(x + 1) > 0$$

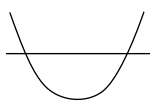


$$x < -2 \text{ or } x > -1$$

Solve $x^2 + 3x + 2 < 0$

$$x^2 + 3x + 2 < 0$$

$$(x + 2)(x + 1) < 0$$



$$-2 < x < -1$$

Solve $-x^2 + 3x - 2 > 0$ **

$$-x^2 + 3x - 2 > 0$$
$$x^2 - 3x + 2 < 0$$
$$(x - 2)(x - 1) < 0$$

$$1 < x < 2$$

Reverse Quadratic Inequalities

Find the value of b for which $-2 < x < \frac{1}{3}$ is the solution of $3x^2 + 5x < b$.

$$-2 < x < \frac{1}{3}$$

$$(x + 2)(3x - 1) < 0$$

$$3x^2 - x + 6x - 2 < 0$$

$$3x^2 + 5x - 2 < 0$$

$$3x^2 + 5x < 2$$

$$b = 2$$

Find the value of b for which $x < -2$ *or* $x > \frac{1}{3}$ is the solution of $3x^2 + 5x > b$.

$$x < -2 \text{ or } x > \frac{1}{3}$$

$$(x + 2)(3x - 1) > 0$$

$$3x^2 - x + 6x - 2 > 0$$

$$3x^2 + 5x - 2 > 0$$

$$3x^2 + 5x > 2$$

$$b = 2$$

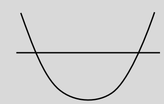
Important Concepts

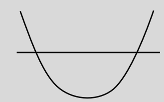
Concept:

In Inequalities, we need to be very careful with the signs.

The first thing to check is always the coefficient of x^2. Ensure that it is Positive. If it's negative, you have to Switch your sign when you divide by negative.

When solving quadratic inequalities, for a start, use a graph to help you in determining the range.





Validation

Validation is very straight forward over here.

Substitute the value in your range and prove that it satisfies the inequalities.

Quadratic Inequalities

Advance Quadratic Inequalities

Find the range of values of x for which $\frac{5}{6x^2-11x-35} < 0$.

For $\frac{5}{6x^2-11x-35} < 0$,

$$6x^2 - 11x - 35 < 0$$

$$(2x-7)(3x+5) < 0$$

$$-\frac{5}{3} < x < \frac{7}{2}$$

Find the range of the values of x $\frac{7x^2+7-14x}{3x^2+x-10} > 0$.

$$\frac{7(x^2-2x+1)}{3x^2+x-10} > 0$$

$$\frac{7(x-1)^2}{3x^2+x-10} > 0$$

Since $7(x-1)^2 > 0$,

$$3x^2 + x - 10 > 0$$

$$(3x-5)(x+2) > 0$$

$$x < -2 \text{ or } x > \frac{5}{3}$$

Important Concepts

Concept:
Questions look complicated but it's actually simple.

This is called Deduction whereby we are determining the appropriate range of values that will ensure your fraction becomes lesser or bigger than 0.

Look at the fraction (numerator & denominator). You will realise that either the numerator or denominator is ALWAYS Positive or ALWAYS Negative.

If you happen to see a quadratic equation that cannot be factorised, you may have to use completing the square to prove that it's always positive or negative.

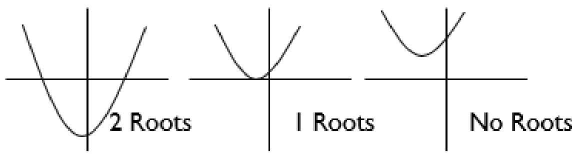
Validation

Substitute values in the range and prove that it satisfies the given inequality.

Finding Ranges

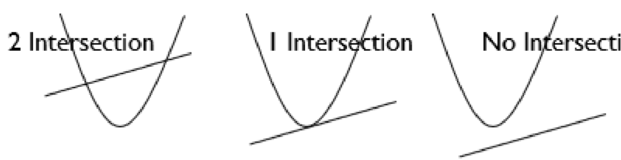
Determinants (Curve & Axis)

$b^2 - 4ac < 0$	**No Roots**	No Real Roots or Imaginary Roots or Graph is always positive (Completely Above x-axis) or Graph is always negative (Completely below x-axis)
$b^2 - 4ac = 0$	**1 Roots**	Real & Equal or Real & Repeated Roots
$b^2 - 4ac > 0$	**2 Roots**	Real & Distinct Roots or Different Roots
$b^2 - 4ac \geq 0$	**1 / 2 Roots**	Graph has real roots or Graph Intersects the x axis



Determinants (Curve & Line)

$b^2 - 4ac < 0$	**No Intersect**	Line Does Not Intersect The Curve
$b^2 - 4ac = 0$	**1 Intersect**	Line is tangent to Curve or Intersects Curve at one Point
$b^2 - 4ac > 0$	**2 Intersect**	Line Intersects curve at 2 points
$b^2 - 4ac \geq 0$	**1 / 2 Intersect**	Line Intersects / Meets Curve (May mean 1 or 2 points, consider both)



When to Reject?

We reject ranges on these scenarios:

1) Graph of $ax^2 + bx + c$, a cannot be 0.
2) Graph is always Positive, coefficient of x^2 cannot be Negative.
3) Graph is always Negative, coefficient of x^2 cannot be Positive.

Proving & Showing

Deduction

Apply this method when Conditions are given.

This allows you to break the equation into smaller pieces and explain step by step.

This is why it's called Proving through Deduction.

Completing The Square

Apply this method when you see a **Quadratic Equation.**

We are unable to prove whether $k^2 - 20k + 111$ is always Positive or Negative.

Through Completing The Square, we transform the equation into $(k - 10)^2 + 12$.

Now, we can easily explain ☺

Nature of Roots – Finding Unknown Values

Finding Ranges (Line & Axis)

If the equation $(k+1)x^2 + 4kx - 8x + 2k = 0$ has real roots, find the range of values of k.

$$b^2 - 4ac \geq 0$$
$$(4k-8)^2 - 4(k+1)(2k) \geq 0$$
$$(16k^2 - 64k + 64) - 8k^2 - 8k \geq 0$$
$$8k^2 - 72k + 64 \geq 0$$
$$k^2 - 9k + 8 \geq 0$$
$$(k-1)(k+8) \geq 0$$

$$k \leq 1 \text{ or } k \geq 8$$

Find the range of values of k for which the expression $3 - 4k - (k+3)x - x^2$ is negative for all real values of x.

$$-x^2 - (k+3)x + 3 - 4k$$
$$-x^2 + (-k-3)x + 3 - 4k$$
$$b^2 - 4ac < 0$$
$$(-k-3)^2 - 4(-1)(3-4k) < 0$$
$$k^2 + 6k + 9 + 12 + 16k < 0$$
$$k^2 + 22k + 21 < 0$$
$$(k+7)(k+3) < 0$$

$$3 < k < 7$$

Finding Ranges (with Rejections)

Find the range of values of a for which $ax^2 - 4x + a - 3$ is **positive** for all values of x.

$$b^2 - 4ac < 0$$
$$(-4)^2 - 4(a)(a-3) < 0$$
$$16 - 4a^2 + 12 < 0$$
$$-4a^2 + 12 + 16 < 0$$
$$a^2 - 3a - 4 > 0$$
$$(a-4)(a+1) > 0$$
$$a < -1\ (Reject)\ or\ a > 4$$
$$3x^2 + 5x < 2$$

Find the range of values of a for which $ax^2 - 4x + a - 3$ is **negative** for all values of x.

$$b^2 - 4ac < 0$$
$$(-4)^2 - 4(a)(a-3) < 0$$
$$16 - 4a^2 + 12 < 0$$
$$-4a^2 + 12 + 16 < 0$$
$$a^2 - 3a - 4 > 0$$
$$(a-4)(a+1) > 0$$
$$a < -1\ or\ a > 4\ (Reject)$$

Find the range of values of a for which $ax^2 - 4x + a - 3$ has 2 distinct roots for all values of x.

$$b^2 - 4ac > 0$$
$$(-4)^2 - 4(a)(a-3) > 0$$
$$16 - 4a^2 + 12 > 0$$
$$-4a^2 + 12 + 16 > 0$$
$$a^2 - 3a - 4 < 0$$
$$(a-4)(a+1) < 0$$
$$-1 < a < 4\ where\ a \neq 0$$

Important Concepts

Concept:

1) Always ensure that you rearrange

2) Determine the determinants
*Do make sure you are clear of all the phrases

3) Solve through Quadratic Inequality

Rejection usually happens

1) Unknown coefficient of x^2

2) Always Positive, Always Negative

The reason you must reject is because the coefficient of x^2 will change the shape of the graph if it's + or -. If it is 0, the quadratic graph will not exist.

Validation

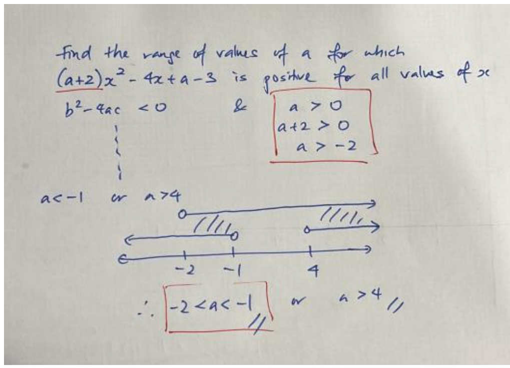
You can use your calculator Mode 3,3 to help you.

If you were to sub in a value in your range, it should fulfil the criteria that you are finding

e.g. You should obtain 2 roots from your calculator if you are finding range where graphs have 2 distinct points.

Nature of Roots – Finding Unknown Values

Finding Ranges for Positive and Negative Graphs



Important Concepts

Concept:

1) Always ensure that you rearrange

2) Determine the determinants
*Do make sure you are clear of all the phrases

3) Solve through Quadratic Inequality

Rejection usually happens

1) Unknown coefficient of x^2

2) Always Positive, Always Negative

The reason you must reject is because the coefficient of x^2 will change the shape of the graph if it's + or -. If it is 0, the quadratic graph will not exist.

Validation

You can use your calculator Mode 3,3 to help you.

If you were to sub in a value in your range, it should fulfil the criteria that you are finding

e.g. You should obtain 2 roots from your calculator if you are finding range where graphs have 2 distinct points.

Nature of Roots – Finding Unknown Values

Finding Ranges (Curve & Line)

Find the range of values of k for which the line $5y = k - x$ does not intersect the curve $5x^2 + 5xy + 4 = 0$.

$$5y = k - x$$
$$5x^2 + 5xy + 4 = 0$$
$$5x^2 + 5x\left(\frac{k-x}{5}\right) + 4 = 0$$
$$5x^2 + kx - x^2 + 4 = 0$$
$$4x^2 + kx + 4 = 0$$
$$k^2 - 4(4)(4) < 0$$

$$k^2 - 64 < 0$$
$$(k-8)(k+8) < 0$$
$$-8 < k < 8$$

The straight line $y - 1 = 2m$ does not intersect the curve $y = x + \frac{m^2}{x}$.
Find the largest integer value of m.

$$y = 2m + 1$$
$$y = x + \frac{m^2}{x}$$
$$(1) = (2): x + \frac{m^2}{x} = 2m + 1$$
$$x^2 - 2mx - x + m^2 = 0$$
$$x^2 - (2m+1)x + m^2 = 0$$

Line does not intersect curve, $b^2 - 4ac < 0$

$$[-(2m+1)]^2 - 4(1)(m^2) < 0$$
$$(2m + 1 + 2m)(2m + 1 - 2m) < 0$$
$$4m + 1 < 0$$
$$m < -\frac{1}{4}$$

The largest integer value of m is -1.

Find the values of p for which the line $y = 2x - 3$ is a tangent to the curve $y = px^2 + 6x + p - 6$.

$$2x - 3 = px^2 + 6x + p - 6$$
$$px^2 + 4x + p - 3 = 0$$
$$4^2 - 4(p)(p-3) = 0$$
$$4p^2 - 12p - 16 = 0$$
$$p^2 - 3p - 4 = 0$$
$$(p+1)(p-4) = 0$$
$$p = -1 \text{ or } p = 4$$

The line $y = \frac{1}{2}x + 6$ is a tangent to the curve $y^2 = kx$, where k is a constant.
Find the value of k.

$$\left(\frac{1}{2}x + 6\right)^2 = kx$$
$$\frac{1}{4}x^2 + 6x + 36 - kx = 0$$
$$\frac{1}{4}x^2 + (6-k)x + 36 = 0$$

Tangent to curve, 1 equal root

$$b^2 - 4ac = 0$$
$$(6-k)^2 - 4\left(\frac{1}{4}\right)(36) = 0$$
$$(6-k)^2 - 36 = 0$$

$$6 - k = 6 \text{ or } 6 - k = -6$$
$$k = 0 \text{ (NA) or } k = 12$$

Important Concepts

Concept:
1) Equate the Curve & Line

2) Rearrange

3) Determine the determinants

4) Solve through Quadratic Inequality

<u>**Rejection usually happens**</u>

1) Unknown coefficient of x^2

2) Always Positive, Always Negative

The reason you must reject is because the coefficient of x^2 will change the shape of the graph if it's + or -. If it is 0, the quadratic graph will not exist.

Validation

You can use your calculator Mode 3,3 to help you.

If you were to sub in a value in your range, it should fulfil the criteria that you are finding

e.g. You should obtain 2 roots from your calculator if you are finding range where graphs have 2 distinct points.

Nature of Roots – Proving/Showing/Explain Questions

Calculating

Show that $2x^2 + 3x + 5$ is always positive for all real values of x.

$$b^2 - 4ac$$
$$= (3)^2 - 4(2)(5)$$
$$= 9 - 22$$
$$= -13$$

Since $b^2 - 4ac < 0$ and $coefficient\ of\ x^2, a > 0$, the graph is always positive.

Show that $-2x^2 + 3x - 5$ is always negative for all real values of x.

$$b^2 - 4ac$$
$$= (3)^2 - 4(-2)(-5)$$
$$= 9 - 40$$
$$= -31$$

Since $b^2 - 4ac < 0$ and $coefficient\ of\ x^2, a < 0$, the graph is always negative.

Deduction

Show that the roots of the quadratic equation

$3(x+p)^2 - 1 = x - 1$ are not real if $p > \frac{1}{12}$.

$$3(x^2 + 2xp + p^2) - 1 - x + 1 = 0$$
$$3x^2 + 6px - x + 3p^2 = 0$$
$$3x^2 + (6p - 1)x + 3p^2 = 0$$
$$b^2 - 4ac = (6p-1)^2 - 4(3)(3p^2)$$
$$= (36p^2 - 12p + 1) - 36p^2$$
$$= -12p + 1$$
$$= -12\left(p + \frac{1}{12}\right)$$

Since $p > \frac{1}{12}$, $\left(p + \frac{1}{12}\right) > 0$

$$-12\left(p + \frac{1}{12}\right) < 0$$

$$b^2 - 4ac < 0$$

The roots of the equation are not real.

Show that the roots of the equation $6x^2 + 4(m-1) = 2(x+m)$ are real if $m \leq 2\frac{1}{12}$.

$6x^2 + 4(m-1) = 2(x+m)$

$6x^2 - 2x + 2m - 4 = 0$

Discriminant $= 100 - 48m$

Since $m \leq 2\frac{1}{12}$

$25 - 12m \geq 0$

$100 - 48m \geq 0$

Since discriminant ≥ 0, $6x^2 + 4(m-1) = 2(x+m)$ has real roots if $m \leq 2\frac{1}{12}$

Completing the Square

The equation of a curve is $y = 3x^2 - kx + 2k - 4$, where k is a constant. Show that the line $y = 2x + 5$ intersects the curve for all real values of k.

$y = 3x^2 - kx + 2k - 4 ---(1)$

$y = 2x + 5 ---(2)$

$(1) = (2): 3x^2 - kx + 2k - 4 = 2x + 5$

$3x^2 - kx - 2x + 2k - 9 = 0$

$3x^2 - (k+2)x + 2k - 9 = 0$

$b^2 - 4ac = [-(k+2)]^2 - 4(3)(2k-9)$

$= k^2 + 4k + 4 - 24k + 108$

$= k^2 - 20k + 111$

$= (k-10)^2 - 10^2 + 112$

$= (k-10)^2 + 12$

Since $(k-10)^2 + 12 > 0, b^2 - 4ac > 0$ and line intersects the curve for all real values of k.

The equation of a curve is $y = 3x^2 - kx + 2k - 4$, where k is a constant. Show that the line $y = 2x + 5$ intersects the curve for all real values of k.

$y = 3x^2 - kx + 2k - 4 ---(1)$

$y = 2x + 5 ---(2)$

$(1) = (2): 3x^2 - kx + 2k - 4 = 2x + 5$

$3x^2 - kx - 2x + 2k - 9 = 0$

$3x^2 - (k+2)x + 2k - 9 = 0$

$b^2 - 4ac = [-(k+2)]^2 - 4(3)(2k-9)$

$= k^2 + 4k + 4 - 24k + 108$

$= k^2 - 20k + 111$

$= (k-10)^2 - 10^2 + 112$

$= (k-10)^2 + 12$

Since $(k-10)^2 + 12 > 0, b^2 - 4ac > 0$ and line intersects the curve for all real values of k.

Simplifying

Multiplication:

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
$a \times \sqrt{b} = a\sqrt{b}$
$c\sqrt{a} \times d\sqrt{b} = cd\sqrt{ab}$
$\sqrt{a} \times \sqrt{a} = a$
$b\sqrt{a} \times b\sqrt{a} = b^2a$

Number × Number,
Surd × Surd

Division:

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Addition & Subtraction:

$2\sqrt{3} + 5\sqrt{3} = 7\sqrt{3}$
$4\sqrt{2} - \sqrt{2} = 3\sqrt{2}$

Similar Terms can be added or subtracted

Key to Solving Surds:
Simplify all Surds to their simplest forms
$\sqrt{50} = 5\sqrt{2}$
$\sqrt{27} = 3\sqrt{3}$

Rationalisation

$$\frac{2}{3\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$\frac{2}{\sqrt{5}-3} \times \frac{\sqrt{5}+3}{\sqrt{5}+3}$$

$$\frac{2}{\sqrt{5}+3} \times \frac{\sqrt{5}-3}{\sqrt{5}-3}$$

Train your speed in Surds Expansion.
It is back to Special Products.
$(a+b)^2 = a^2 + 2ab + b^2$
$(a-b)(a+b) = (a^2 - b^2)$

Surds

Simplifying

$\sqrt{4} = 2, \sqrt{9} = 3, \sqrt{16} = 4, \sqrt{25} = 5$

$\sqrt{32} = 4\sqrt{2}$

$\sqrt{75} = 5\sqrt{3}$

$\sqrt{18} = 3\sqrt{2}$

$\sqrt{50} = 5\sqrt{2}$

$\sqrt{72} = 6\sqrt{2}$

Train Your Speed in Simplifying Surds

1) Identify Perfect Squares
2) Square Root the Perfect Square
3) Leave your Prime Number inside the Root

Multiplying

Numbers X Numbers, Surds x Surds

$(4 + 2\sqrt{3})(5 + \sqrt{3}) = 20 + 10\sqrt{3} + 4\sqrt{3} + 6$

$(\sqrt{5} + \sqrt{3})(\sqrt{10} + \sqrt{6}) = \sqrt{50} + \sqrt{30} + \sqrt{30} + \sqrt{18}$

$(1 + \sqrt{3})^2 = 1 + 2\sqrt{3} + 3$

$(2\sqrt{5} - 3\sqrt{3}) = 4(5) - 12\sqrt{15} + 9(3)$

$(1 - \sqrt{3})(1 + \sqrt{3}) = 1 - 3$

$(\sqrt{10} - \sqrt{6})(\sqrt{10} + \sqrt{6}) = 10 - 6$

$(5 - \sqrt{5})\ (5 + \sqrt{5}) = 25 - 5$

$(2\sqrt{5} - 3\sqrt{3})(2\sqrt{5} + 3\sqrt{3}) = 20 - 27$

Rationalising

$$\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$\frac{2}{3\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$\frac{4\sqrt{3}}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$\frac{8\sqrt{3} - 4}{1 + 2\sqrt{3}} \times \frac{1 - 2\sqrt{3}}{1 - 2\sqrt{3}}$$

$$\frac{16 + 6\sqrt{5}}{7 + 3\sqrt{5}} \times \frac{7 - 3\sqrt{5}}{7 - 3\sqrt{5}}$$

Express $\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right)^2$ in the form $a + b\sqrt{15}$, where a and b are integers.

$$\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right)^2$$

$$= \frac{5+2\sqrt{15}+3}{5-2\sqrt{15}+3}$$

$$= \frac{8+2\sqrt{15}}{8-2\sqrt{15}} \times \frac{8+2\sqrt{15}}{8+2\sqrt{15}}$$

$$= \frac{64+32\sqrt{15}+60}{64-60}$$

$$= \frac{124 + 32\sqrt{15}}{4}$$

$$= 31 + 8\sqrt{15}$$

Important Concepts

Concept:

1. Always Simplify First
   This prevents your surds from getting too big during Expansion
2. Rationalise
   Be careful of the sign here
3. Multiplication

Numbers X Numbers, Surds x Surds

Train your speed for this chapter.

Surds is a fairly easy chapter so we shouldn't be spending too much time here.

Validation

We can use the calculator to validate our solution. Ensure that they are the same.

Surds

Solving

Without using a calculator, find the integer value of a and of b for which the solution of the equation $2x\sqrt{5} = x\sqrt{2} + \sqrt{18}$ is $\frac{\sqrt{a}+b}{3}$.

$$x\left(2\sqrt{5} - \sqrt{2}\right) = \sqrt{18}$$

$$x = \frac{\sqrt{18}}{2\sqrt{5}-\sqrt{2}} \times \frac{2\sqrt{5}+\sqrt{2}}{2\sqrt{5}+\sqrt{2}}$$

$$= \frac{2\sqrt{90}+6}{18}$$

$$= \frac{6\sqrt{10}+6}{18}$$

$$= \frac{\sqrt{10}+1}{3}$$

$$a = 10, b = 1$$

Given that $\sqrt{p + q\sqrt{8}} = \frac{9}{4-\sqrt{8}}$, where p and q are rational numbers, find the values of p and q.

$$\sqrt{p + q\sqrt{8}} = \frac{9}{4-\sqrt{8}}$$

$$p + q\sqrt{8} = \left(\frac{9}{4-\sqrt{8}}\right)^2$$

$$= \frac{81}{24-8\sqrt{8}}$$

$$= \frac{81}{24-8\sqrt{8}} \times \frac{24+8\sqrt{8}}{24+8\sqrt{8}}$$

$$= \frac{81(24+8\sqrt{8})}{64}$$

$$= \frac{243}{8} + \frac{81}{8}\sqrt{8}$$

$$p = \frac{243}{8} \quad q = \frac{81}{8}$$

Express $\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right)^2$ in the form $a + b\sqrt{15}$, where a and b are integers.

$$\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right)^2$$

$$= \frac{5+2\sqrt{15}+3}{5-2\sqrt{15}+3}$$

$$= \frac{8+2\sqrt{15}}{8-2\sqrt{15}} \times \frac{8+2\sqrt{15}}{8+2\sqrt{15}}$$

$$= \frac{64+32\sqrt{15}+60}{64-60}$$

$$= \frac{124+32\sqrt{15}}{4} = 31 + 8\sqrt{15}$$

Express $\frac{(12-3\sqrt{10})}{(2\sqrt{2}+\sqrt{5})}$ in the form of $\sqrt{a} - \sqrt{b}$, where a and b are integers.

$$\frac{(12-3\sqrt{10})}{(2\sqrt{2}+\sqrt{5})} \times \frac{2\sqrt{2}-\sqrt{5}}{2\sqrt{2}-\sqrt{5}}$$

$$= \frac{24\sqrt{2}-12\sqrt{5}-6\sqrt{20}+3\sqrt{50}}{4(2)-5}$$

$$= \frac{24\sqrt{2}-12\sqrt{5}-12\sqrt{5}+15\sqrt{2}}{3}$$

$$= \frac{39\sqrt{2}-24\sqrt{5}}{3}$$

$$= 13\sqrt{2} - 8\sqrt{5}$$

$$= \sqrt{338} - \sqrt{320}$$

Important Concepts

Concept:

1. Always Simplify First
   This prevents your surds from getting too big during Expansion

2. Rationalise
   Be careful of the sign here

3. Multiplication

Numbers X Numbers, Surds x Surds

Train your speed for this chapter.

Surds is a fairly easy chapter so we shouldn't be spending too much time here

Validation

We can use the calculator to validate our solution. Ensure that they are the same.

Surds

Word Problems (Mensuration)

A rectangle has a length of $(6\sqrt{3}+3)$cm and an area of 66 cm^2. Find the perimeter of the rectangle in the form $(a+b\sqrt{3})$ cm, where a and b are integers.

$$\text{Breadth} = \frac{66}{6\sqrt{3}+3}$$

$$= \frac{66}{6\sqrt{3}+3} \times \frac{6\sqrt{3}-3}{6\sqrt{3}-3} = \frac{66(6\sqrt{3}-3)}{99}$$

$$= 4\sqrt{3}-2 \text{ cm}$$

$$\text{Perimeter} = 2(6\sqrt{3}+3+4\sqrt{3}-2)$$

$$= 20\sqrt{3}+2 \text{ cm}$$

The volume of a right square pyramid of length $(3+\sqrt{2})$ cm is $\frac{1}{3}(29-2\sqrt{2})$ cm^3. Without using a calculator, find the height of the pyramid in the form $(a+b\sqrt{2})$ cm, where a and b are integers.

$$\frac{1}{3}(3+\sqrt{2})^2 h = \frac{1}{3}(29-2\sqrt{2})$$

$$h = \frac{29-2\sqrt{2}}{11+6\sqrt{2}}$$

$$= \frac{(29-2\sqrt{2})(11-6\sqrt{2})}{49}$$

$$= \frac{319-174\sqrt{2}-22\sqrt{2}+24}{49}$$

$$= \frac{343-196\sqrt{3}}{49}$$

$$= (7-4\sqrt{2})cm$$

Word Problems (Mensuration)

A cylinder has a radius of $(1+2\sqrt{2})$ cm and its volume is $\pi(84+21\sqrt{2})$ cm^3. Find, without using a calculator, the exact length of the height of the cylinder in the form $(a+b\sqrt{2})$ cm, where a and b are integers.

$$\pi(84+21\sqrt{2}) = \pi(1+2\sqrt{2})^2 \times h$$

$$h = \frac{84+21\sqrt{2}}{(1+2\sqrt{2})^2}$$

$$h = \frac{(84+21\sqrt{2})(4\sqrt{2}-9)}{(4\sqrt{2}+9)(4\sqrt{2}-9)}$$

$$h = \frac{756-336\sqrt{2}+189\sqrt{2}-168}{81-32}$$

$$h = \frac{588-147\sqrt{2}}{49}$$

$$h = (12-3\sqrt{2}) \text{ cm}$$

O Level*

Given that $3+2\sqrt{5}$ is a root of the equation $x^2+ax+b=0$, where a and b are integers, find the value of a and of b.

$$[x-(3+2\sqrt{5})[x-(3-2\sqrt{5})]$$

$$= (x-3-2\sqrt{5})(x-3+2\sqrt{5})$$

$$= x^2-3x+2\sqrt{5x}-3x+9-6\sqrt{5}-2\sqrt{5}x+6\sqrt{5}-20$$

$$= x^2-3x+2\sqrt{5x}-3x-2\sqrt{5x}+9-6\sqrt{5}+6\sqrt{5}-20$$

$$= x^2-6x-11$$

$$\therefore a=-6, b=-11$$

Finding Unknown Values

Substitution Method
Compare Coefficients

Remainder/Factor Theorem

By RT, $f(x) = R$
By FT, $f(x) = 0$

Forming Original Equation*

1. Check Degree
2. Check Coefficient
3. Formation of Equation

Solving Cubic Equation

1. Mode 3,4 (Casio Calc)
2. Factor Theorem
3. Long Division
4. Solve

Hence

1. Nature of Roots
2. Surds
3. Replacement Qn

Factorising Cubic Equation

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

1. Check Improper vs Proper Fraction
2. If Improper, do Long Division
3. Fully Factorise Denominator

Case 1: Linear

Case 2: Square

Case 3: Quadratic

Case	Fraction $\frac{N(x)}{D(x)}$	Form of denominator, D(x)	Partial Fraction Form (where A, B and C are unknown constants)
1	$\frac{N(x)}{(ax+b)(cx+d)}$	Linear Factors	$\frac{A}{ax+b} + \frac{B}{cx+d}$
2	$\frac{N(x)}{(ax+b)^2}$	Repeated Linear Factors	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
	$\frac{N(x)}{(ax+b)(cx+d)^2}$	Linear and Repeated Linear Factors	$\frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$
3	$\frac{N(x)}{(ax+b)(x^2+c^2)}$	Linear and Quadratic (which cannot be factorised) Factors	$\frac{A}{ax+b} + \frac{Bx+C}{x^2+c^2}$

$$\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$\frac{1}{(x+2)(x^2+3)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+3}$$

$$\frac{1}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

Polynomials

Long Division

Divide $6x^3 - 23x^2 - 20x + 9$ by $x + 1$.

$$
\begin{array}{r}
6x^2 - 29x + 9 \\
x + 1\overline{)\,6x^3 - 23x^2 - 20x + 9} \\
-(\,6x^3 + 6x^2) \\
\hline
-29x^2 - 20x + 9 \\
-(-29x^2 - 29x) \\
\hline
9x + 9 \\
-(9x + 9) \\
\hline
0
\end{array}
$$

Divide $3x^3 - 5x^2 + 10x - 3$ by $3x + 1$.

$$
\begin{array}{r}
x^2 - 2x + 4 \\
3x + 1\overline{)\,3x^3 - 5x^2 + 10x - 3} \\
-(\,3x^3 + x^2) \\
\hline
-6x^2 + 10x \\
-(-6x^2 - 2x\quad) \\
\hline
12x - 3 \\
-(12x + 4) \\
\hline
0
\end{array}
$$

Long Division (Missing Algebra)

Divide $x^3 - 5x^2 + 3x - 15$ by $x^2 + 3$.

$$
\begin{array}{r}
x - 5 \\
x^2 + 0x + 3\overline{)\,x^3 - 5x^2 + 3x - 15} \\
-(\,x^3 + 0x^2 + 3x\qquad) \\
\hline
-5x^2 + 0x - 15 \\
-(-5x^2 + 0x - 15\,) \\
\hline
0
\end{array}
$$

Divide $2x^3 - 9x^2 + 15$ by $2x - 5$.

$$
\begin{array}{r}
x^2 - 2x - 5 \\
2x - 5\overline{)\,2x^3 - 9x^2 + 0x + 15} \\
-(\,2x^3 - 5x^2\qquad) \\
\hline
-4x^2 + 0x \\
-(-4x^2 + 10x\quad) \\
\hline
-10x + 15 \\
-(-10x + 25) \\
\hline
-10
\end{array}
$$

Important Concepts

Concept:
There are two things to note in Long Division

1) Signs

2) Missing Algebra

Before you begin your long division, always check for missing algebra. When you are dividing, be very mindful of the brackets and the sign.

Validation

Re-expand to check if you obtain the original equation.

Polynomials – Question Type 1: Finding Unknown Values

Full Expansion & Compare Coefficients

Given the identity, $x^3 + 2x^2 + 2x - 3 = (Ax + B)(x - 1)(x + 1) + Cx - 1$ for all real values for x, find the value of A, of B and of C by comparing coefficients.

$$x^3 + 2x^2 + 2x - 3 = (Ax + B)(x^2 - 1) + Cx - 1$$
$$= Ax^3 - Ax + Bx^2 - B + Cx - 1$$
$$= Ax^3 + Bx^2 + Cx - Ax - B - 1$$

By comparing coefficients

$$A = 1, \quad B = 2, \quad C - A = 2$$
$$C - 1 = 2$$
$$C = 3$$

Ans: $A = 1, B = 2, C = 2$

Substitution

Given that the identity $3x^2 + x - 2 = A(x - 1)(x + 2) + B(x - 1) + C$, for all real values of x, find the value of A, of B and of C by substitution.

Sub $x = 1$

$$3(1)^2 + (1) - 2 = 0 + 0 + C$$
$$3 + 1 - 2 = C$$
$$C = 2$$

Sub $x = 0$

$$3(0) + 0 - 2 = A(-1)(2) + B(-1) + C$$
$$-2 = -2A - B + C$$
$$-2 = -2A + 2 + 2$$
$$-2A = -6$$
$$A = 3$$

Sub $x = -2$

$$3(-2)^2 + (-2) - 2 = 0 - 3B + C$$
$$12 - 4 = -3B + 2$$
$$-3B = 6$$
$$B = 2$$

Ans: $A = 3, B = 2, C = 2$

Important Concepts

Concept:

Most of the time, we will be using Method 2, Substitution as it is way faster and convenient as compared to Method 1.

In certain cases where we use Method 1, we will compare the coefficient of degree and the constant.

Validation

Substitute your values.

You can either

1) Re-expand your equation

2) Substitute random values on the left and right to make sure it tallies

Polynomials – Question Type 2: Remainder & Factor Theorem

Remainder & Factor Theorem

The expression $f(x) = x^3 + ax^2 + bx - 15$, where a and b are constants, has a factor $(x - 3)$ and leaves a remainder of -5 when divided by $(x + 2)$.

(i) Find the value of a and b.

By Factor Theorem,

$$f(3) = 0$$

$9a + 3b = -12\ldots$ (1)

By Remainder Theorem,

$$f(-2) = -5$$

$4a - 2b = 18\ldots$ (2)

Solving (1) and (2) using Simultaneous Eq,

$$a = 1$$

$$b = -7$$

The function $f(x) = x^3 + ax^2 + bx + 9$, where a and b are constants, is exactly divisible by $x + 1$ and leaves a remainder of 15 when divided by $x - 2$.

(i) Find the value of a and of b.

By Factor Theorem,

$$f(-1) = 0$$

$$(-1)^3 + a(-1)^2 + b(-1) + 9 = 0$$

$a - b = -8\ldots$ (1)

By Remainder Theorem,

$$f(2) = 15$$

$$(2)^3 + a(2)^2 + b(2) + 9 = 15$$

$$4a + 2b = -2$$

$$2a + b = -1\ldots \text{(2)}$$

Solving (1) and (2) using Simultaneous Eq,

$$a = -3$$
$$b = 5$$

Important Concepts

Concept:
If you substitute a solution into the equation, you will not have a remainder because it is a factor of the equation. This is Factor Theorem.

If you substitute any other values, you will have a remainder because it is not a factor. This is called Remainder Theorem.

Validation

With your answers, form the equation and conduct remainder and factor theorem, you will see that the answer should be the same.

Polynomials – Question Type 3: Formation of Polynomials

Forming Polynomial (Easy)

The coefficient of x^3 of a cubic polynomial, $f(x)$, is 4 and that the roots of the equation $f(x) = 0$ are $-1, 3$ and k.

Given that f(x) has a remainder of 60 when divided by -2 , find the value of k.

$$f(x) = 4(x+1)(x-3)(x-k)$$

$$f(2) = 60$$

$$4(2+1)(2-3)(2-k) = 60$$

$$-12(2-5) = 60$$

$$2-k = -5$$
$$k = 7$$

The term containing the highest power of x in the polynomial $f(x)$ is $2x^3$. Two of the roots of the equation $f(x) = 0$ are -4 and 2. It is given that $f(x)$ leaves a remainder of 5 when divided by $(x+3)$. Find $f(x)$.

$$f(x) = 2(x+4)(x-2)(x-a)$$

$$f(-3) = 5$$

$$2(1)(-5)(-3-a) = 5$$

$$a = -\frac{5}{2}$$

$$f(x) = 2(x+4)(x-2)\left(x - \left(-\frac{5}{2}\right)\right)$$

$$= (x+4)(x-2)(2x+5)$$

$$= 2x^3 + 9 - 6x - 40$$

Important Concepts

Concept:
There are two things to note when you are forming back your original polynomials

1) Coefficient of Highest Power (Degree)

2) Number of Roots

Always account for these 2 elements

Validation

With your answers, form the equation and conduct remainder and factor theorem, you will see that the answer should be the same.

Polynomials – Question Type 3: Formation of Polynomials

Forming Polynomial (Advance)

The term containing the highest power of x in the polynomial $f(x)$ is $2x^4$ and the roots of $f(x) = 0$ are 2 and -7. $f(x)$ has a remainder of -72 when divided by $(x+1)$, and a remainder of -80 when divided by $(x-1)$.
(i) Find the expression for $f(x)$ in descending power of x.

$$f(x) = 2(x-2)(x+7)(x-a)(x-b)$$

When $f(x)$ is divided by $x+1$,
Using Remainder Theorem,

$$f(-1) = 2(-3)(6)(-1-a)(-1-b)$$
$$-72 = -36(-1-a)(-1-b)$$
$$2 = 1 + b + a + ab$$

When $f(x)$ is divided by $x-1$,
Using Remainder Theorem,

$$f(1) = 2(-1)(8)(1-a)(1+b)$$
$$-80 = -16(1-a)(1+b)$$
$$5 = 1 + b - a - ab$$

$$7 = 2 + b$$
$$b = 5$$

$$2 = 1 + 5 + a + a(5)$$
$$6a = -4$$
$$a = -\frac{3}{2}$$

$$f(x) = 2(x-2)(x+7)(x+\frac{3}{2})(x-5)$$
$$f(x) = (x-2)(x+7)(2x+3)(x-5)$$
$$f(x) = 2x^4 + 13x^3 - 8x^2 - 17x - 70$$

The polynomial f(x) leaves a remainder of -5 and 7 when divided by $x+1$ and $x-2$ respectively. Find the remainder when f(x) is divided by $x^2 - x - 2$.

Let $f(x) = (x+1)(x-2)Q(x) + ax + b$

When $f(x)$ is divided by $x+1$,
Using Remainder Theorem,

$$f(-1) = -5$$
$$a(-1) + b = -5$$
$$-a + b = -5$$

When $f(x)$ is divided by $x-2$,
Using Remainder Theorem,

$$f(2) = 7$$
$$a(2) + b = 7$$
$$b = 7 - 2a$$

Sub (2) into (1), $\quad -a + 7 - 2a = -5$

$$-3a = -12$$
$$\therefore a = 4$$

Sub $a = 4$ into (2), $\quad \therefore b = -1$

$$\therefore f(x) = (x+1)(x-2)Q(x) + 4x - 1$$

The remainder is $4x - 1$.

Important Concepts

Concept:
There are two things to note when you are forming back your original polynomials

1) Coefficient of Highest Power (Degree)

2) Number of Roots

Always account for these 2 elements

For the Second Advance Question, you are dividing by a quadratic equation.

Do take note of this special case..

The remainder is always one degree lesser than the divisor, therefore the remainder will be $ax + b$.

Validation

With your answers, form the equation and conduct remainder and factor theorem, you will see that the answer should be the same.

Polynomials – Question Type 4: Solving Cubic Equations

Solving Cubic Equations

Factorise the cubic expression $6x^3 - 23x^2 - 20x + 9$

Let $f(x) = 6x^3 - 23x^2 - 20x + 9$
$f(-1) = 6(-1)^3 - 23(-1)^2 - 20(-1) + 9 = 0$
$\therefore$ By FT, $(x + 1)$ is a factor.

$$\begin{array}{r} 6x^2 - 29x + 9 \\ x + 1\,\overline{)\,6x^3 - 23x^2 - 20x + 9} \\ -(\,6x^3 + 6x^2 \qquad\qquad) \\ \hline -29x^2 - 20x + 9 \\ -(-29x^2 - 29x \quad) \\ \hline 9x + 9 \\ -(9x + 9) \\ \hline 0 \end{array}$$

$f(x) = (x + 1)(6x^2 - 29x + 9)$

$\therefore f(x) = (x + 1)(2x - 9)(3x - 1)$

... With Quadratic Formula

Given that g(x) = $3x^3 - 4x^2 - 18x + 9$,show that $(x - 3)$ is a factor of $g(x)$, hence solve the equation $g(x) = 0$.

(i) $g(3) = 3(3)^3 - 4(3)^2 - 18(3) + 9$
$= 81 - 36 - 48 + 9$
$= 0$
Since $g(3) = 0$, $x - 3$ is a factor by factor theorem.

(ii) $x = 3$ or $x = 0.468$ or $x = -2.14$

$$\begin{array}{r} 3x^2 + 5x - 3 \\ x - 3\,\overline{)\,3x^3 - 4x^2 - 18x + 9} \\ -3x^3 - 9x^2 \\ \hline 5x^2 - 18x \\ -5x^2 - 18x \\ \hline -3x + 9 \\ -\quad(-3x + 9\,) \\ \hline \end{array}$$

$\therefore g(x) = (x - 3)(3x^2 + 5x - 3)$

Given $g(x) = 0$

$(x - 3)(3x^2 + 5x - 3) = 0$

$x = 3$ or $x = \frac{-5 \pm \sqrt{5^2 - 4(3)(-3)}}{2(3)}$

$x = 0.468$ or $x = -2.14$

Important Concepts

Concept:
1) Use your calculator to find the first factor

2) Use Factor Theorem to prove that it's a factor

3) Long Division

Do take note of Sign and Missing Algebra

4) Fully Factorise the Equation

5) Solve if requested

Validation

Through the calculator, it will reveal all the solutions of the equation. No mistakes should be made here!

Polynomials – Question Type 4: Solving Cubic Equation (Hence Questions)

Level 1: Replacement

Factorise $f(x) = 2x^3 - x^2 - 5x - 2 = 0$
Hence, solve the equation
$2(y-1)^3 - (y-1)^2 - 5(y-1) - 2 = 0.$

$$f(x) = (x-2)(2x+1)(x+1)$$
$$x = 2, x = -\frac{1}{2}, x = -1$$

By Observation,

$$y - 1 = 2, y - 1 = -\frac{1}{2}, y - 1 = -1$$
$$y = 3, y = \frac{1}{2}, y = 0$$

Factorise $2x^3 - x^2 - 5x - 2 = 0$ completely.
Hence, solve the equation
$$16y^3 - 4y^2 - 10y - 2 = 0$$

$$f(x) = (x-2)(2x+1)(x+1)$$
$$x = 2, x = -\frac{1}{2}, x = -1$$

By Observation,

$$2y = 2, 2y = -\frac{1}{2}, 2y = -1$$
$$y = 1, y = -\frac{1}{4}, y = -\frac{1}{2}$$

Factorise $2x^3 - 9x^2 + x + 12 = 0$ completely.
Hence, solve the equation.
$$12y^3 + y^2 - 9y + 2 = 0$$

$$f(x) = (x+1)(x-4)(2x-3)$$
$$x = -1, x = 4, x = \frac{3}{2}$$

By Observation,

$$\frac{1}{y} = -1, \frac{1}{y} = 4, \frac{1}{y} = \frac{3}{2}$$
$$y = -1, y = \frac{1}{4}, y = \frac{2}{3}$$

Level 2: Nature of Roots

Determine the number of real roots of the equation
$f(x) = 2x^3 + 3x^2 + 2x + 8$, justifying your answer.

$f(x) = 2x^3 + 3x^2 + 2x + 8$
$\quad = (x+2)(2x^2 - x + 4)$ [Long Division]

For the factor $2x^2 - x + 4$,

Discriminant $= 1 - 4(2)(4)$
$\quad = -31 < 0$

Hence, the equation $2x^2 - x + 4 = 0$ has no real roots. Therefore $f(x) = 0$ has only 1 real root. The root is $x = -2$

Method 3: Quadratic Formula

Solve the equation $3x^3 - 8x^2 + 2x + 4 = 0$, expressing non-integer roots in the form $\frac{c \pm \sqrt{d}}{3}$, where c and d are integers.

$$f(x) = (x-2)(3x^2 - 2x - 2)$$

$$\begin{array}{r|l} & 3x^2-2x-2 \\ \hline x-2 & 3x^3-8x^2+2x+4 \\ & \underline{3x^3-6x^2} \\ & -2x^2+2x \\ & \underline{-2x^2+4x} \\ & -2x+4 \\ & \underline{-2x+4} \end{array}$$

$$3x^3 - 8x^2 + 2x + 4 = 0$$

$$(x-2)(3x^2 - 2x - 2) = 0$$

$$x - 2 = 0 \qquad 3x^2 - 2x - 2 = 0$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 3 \times -2}}{2 \times 3}$$

$$x = \frac{2 \pm \sqrt{28}}{6} = \frac{2(1 \pm \sqrt{7})}{6}$$

$$x = 2 \qquad x = \frac{1 \pm \sqrt{7}}{3}$$

Important Concepts

Concept:
There are 3 levels of Hence Questions in Solving Polynomials

1) Replacement

Level 1: By observation, you can clear see what is replaced

Level 2: Look at the coefficient of x so you can see that x is replaced wit $2x, 3x$

Level 3: The coefficient of the equation has been swapped. This means that x has been replaced by its reciprocal.

2) Nature of Roots

Whenever the question talks bout roots:

Determine the number of Roots,

Prove that it has 2 Roots,

Prove that it has only 1 Solution…

This indicates that you have to use Nature of Roots concept to explain

3) Surds (Quadratic Formula)

Whenever the questions talk about leaving your answer in 2dp, in the form of $\frac{c \pm \sqrt{d}}{3}$ or leave your answer in exact values. This indicates that you should be using quadratic formula.

Polynomials - Question Type 5: Factorising Cubic Equations

$$\boldsymbol{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$$
$$\boldsymbol{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$$

Factorise $x^3 - 27$

$$x^3 - 27$$
$$= (x-3)[(x)^2 + (x)(3) + (3)^2]$$
$$= (x-3)[(x^2 + 3x + 9]$$

Ans: $(x-3)(x^2+3x+9)$

Factorise $64x^3 + 1$

$$64x^3 + 1$$
$$= (4x+1)[(4x)^2 - (4x)(1) + (1)^2]$$
$$= (4x+1)[(16x^2 - 4x + 1]$$

Ans: $(4x+1)(16x^2 - 4x + 1)$

Factorise $8 - 27x^3$

$$8 - 27x^3$$
$$= (2-3x)[(2)^2 + (2)(3x) + (3x)^2$$
$$= (2-3x)(4 + 6x + 9x^2)$$

Ans: $(2-3x)(4+6x+9x^2)$

Factorise $64 + 27x^3$

$$64 + 27x^3$$
$$= (4+3x)[(4)^2 - (4)(3x) + (3x)^2]$$
$$= (4+3x)(16 - 12x + 9x^2)$$

Ans: $(4+3x)(16-12x+9x^2)$

Factorise $250x^3 - 54y^3$

$$250x^3 - 54y^3$$
$$= 2(125x^3 - 27x^3)$$
$$= 2(5x-3y)[(5x)^2 + (5x)(3y) + (3y)^2]$$
$$= 2(5x-3y)(25x^2 + 15xy + 9y^2)$$

Ans: $2(5x-3y)(25x^2+15xy+9y^2)$

Factorise $8x^3 - (x-1)^3$ completely.

$$8x^3 - (x-1)^3$$
$$= [(2x - (x-1)][(2x)^2 + (2x)((x-1) + (x-1)^2]$$
$$= [(x+1)][(4x^2 + 2x^2 - 2x + x^2 - 2x + 1)]$$
$$= [(x+1)][7x^2 - 4x + 1]$$

Ans: $(x+1)(7x^2 - 4x + 1)$

Important Concepts

Concept:
1. SOAP (Sign)
Same, Opposite, Always Positive

2. Do not confuse with $a^2 + 2ab + b^2$

3. Remember to apply powers for integers

$$(2x)^2 = 4x^2$$

Validation

Re-expand back to obtain original equation

Partial Fractions

Case	Fraction $\frac{N(x)}{D(x)}$	Form of denominator, D(x)	Partial Fraction Form (where A, B and C are unknown constants)
1	$\frac{N(x)}{(ax+b)(cx+d)}$	Linear Factors	$\frac{A}{ax+b} + \frac{B}{cx+d}$
2	$\frac{N(x)}{(ax+b)^2}$	Repeated Linear Factors	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
	$\frac{N(x)}{(ax+b)(cx+d)^2}$	Linear and Repeated Linear Factors	$\frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$
3	$\frac{N(x)}{(ax+b)(x^2+c^2)}$	Linear and Quadratic (which cannot be factorised) Factors	$\frac{A}{ax+b} + \frac{Bx+C}{x^2+c^2}$

Case 1: Linear Factor (Improper Fractions)

Express $\frac{3x^3-5}{x^2-1}$ in partial fractions.

By Long division

$$\frac{3x^3-5}{x^2-1} = 3x + \frac{3x-5}{x^2-1}$$

$$\frac{3x-5}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x+1}$$

$$3x-5 = A(x-1) + B(x+1)$$

$x = 1$: $3(1) - 5 = 2B$

$B = -1$

$x = -1$: $-3 - 5 = -2A$

$A = 4$

$$\therefore \frac{3x^3-5}{x^2-1} = 3x + \frac{4}{x+1} - \frac{1}{x-1}$$

Case 1: Linear Factor (Improper Fractions)

Express $\frac{3x^3+2x^2+4x-1}{x^3+x^2}$ in partial fractions.

$$x^3 - x^2 \;\overline{\big)\,3x^3+2x^2+4x-1}^{\;3}$$
$$\underline{3x^3+3x^2)}$$
$$-x^2+4x-1$$

$$\frac{3x^3+2x^2+4x-1}{x^2+x^3} = 3 + \frac{-x^2+4x-1}{x^2(x+1)}$$

$$\frac{-x^2+4x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$-x^2 + 4x - 1 = Ax(x+1) + B(x+1) + cx^2$$

Let $x = -1$ $\quad -1 - 4 - 1 = c$

$c = -6$

Let $x = 0$ $\quad B = -1$

$$-x^2 + 4x - 1 = Ax(x+1) - 1(x+1) - 6x^2$$

Let $x = 1$ $\quad -1 + 4 - 1 = 2A - 2 - 6$

$A = 5$

$$\frac{3x^3 + 2x^2 + 4x - 1}{x^2 + x^3} = 3 + \frac{5}{x} - \frac{1}{x^2} - \frac{6}{x+1}$$

Important Concepts

Concept:
1) Check whether it's improper or proper fraction

2) If it is improper, conduct long division

Do take note of Signs and Missing Algebra

3) Fully Factorise Denominator

4) Decide the case to apply

5) Cross Multiply and Solve

Validation

Substitute a value to make sure both left and right side balance

Partial Fractions

Case 2: Repeated Linear Factor

Express $\frac{16x^2-9x+18}{x^3+3x^2}$ in partial fractions.

Answer: $\frac{16x^2-9x+18}{x^3+3x^2} = \frac{-5}{x} + \frac{6}{x^2} + \frac{21}{x+3}$

$\frac{16x^2-9x+18}{x^3+3x^2} = \frac{16x^2-9x+18}{x^2(x+3)}$

Let $\frac{16x^2-9x+18}{x^3+3x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$

$16x^2 - 9x + 18 = Ax(x+3) + B(x+3) + Cx^2$

Let $x = -3$, $16(-3)^2 - 9(-3) + 18 = 9C$

$9C = 189$

$C = 21$

Let $x = 0$, $18 = 3B$

$B = 6$

Comparing x^2 term, $16x^2 = Ax^2 + Cx^2$

$A + C = 16$

$A + 21 = 16$

$A = -5$

$\frac{16x^2-9x+18}{x^3+3x^2} = \frac{-5}{x} + \frac{6}{x^2} + \frac{21}{x+3}$

Express $\frac{10x^2-7x+10}{(3x-2)(x^2+2)}$ in partial fractions.

Answer: $\frac{10x^2-7x+10}{(3x-2)(x^2+2)} = \frac{4}{3x-2} + \frac{2x-1}{x^2+2}$

Let $\frac{10x^2-7x+10}{(3x-2)(x^2+2)} = \frac{A}{3x-2} + \frac{Bx+C}{x^2+2}$

$10x^2 - 7x + 10 = A(x^2+2) + (Bx+C)(3x-2)$

Sub $x = \frac{2}{3}$ to get $A = 4$

Sub $x = 0$ to get $C = -1$

Sub $x = 1$ (or any other value) to get $B = 2$

$$\frac{10x^2 - 7x + 10}{(3x-2)(x^2+2)} = \frac{4}{3x-2} + \frac{2x-1}{x^2+2}$$

Case 3: Quadratic Factor

Express $\frac{8x^2-2x+19}{(1-x)(4+x^2)}$ in partial fractions.

$$\frac{8x^2 - 2x + 19}{(1-x)(4+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{4+x^2}$$

$8x^2 - 2x + 19 = A(4+x^2) + (Bx+C)(1-x)$

Sub $x = 1$, $8 - 2 + 19 = 5A$ $\quad A = 5$

Sub $x = 0$, $19 = 4(5) + C$ $\quad C = -1$

Compare coeff of x^2, $8 = A - B$ $\quad B = -3$

$$\frac{8x^2 - 2x + 19}{(1-x)(4+x^2)} = \frac{5}{1-x} - \frac{3x+1}{4+x^2}$$

Express $\frac{4x^3+x^2+6}{(x-2)(x^2+2)}$ in partial fractions.

$$\frac{4x^3 + x^2 + 6}{(x-2)(x^2+2)} = 4 + \frac{A}{x-2} + \frac{Bx+C}{x^2+2}$$

Multiplying by $(x-2)(x^2+2)$, we obtain

$4x^3 + x^2 + 6$

$= 4(x-2)(x^2+2) + A(x^2+2) + (Bx+C)(x-2)$

Sub $x = 2$: $\quad 4 \times 8 + 4 + 6 = A(4+2)$

$42 = 6A \Rightarrow A = 7$

Sub $x = 0$: $\quad 6 = -16 + 2(7) + C(-2)$

$-2C = 8 \Rightarrow C = -4$

Compare x^2: $1 = -8 + 7 + B \Rightarrow B = 2$

$$\therefore \frac{4x^3 + x^2 + 6}{(x-2)(x^2+2)} = 4 + \frac{7}{x-2} + \frac{2-4x}{x^2+2}$$

Important Concepts

Concept:

1) Check whether it's improper or proper fraction

2) If it is improper, conduct long division

Do take note of Signs and Missing Algebra

3) Fully Factorise Denominator

4) Decide the case to apply

5) Cross Multiply and Solve

Validation

Substitute a value to make sure both left and right side balance

Simplifying

Laws of Indices

Basic Rules

$y^a \times y^b = y^{a+b}$

$y^a \div y^b = y^{a-b}$

$(y^n)^m = y^{nm}$

Negative Powers

$y^{-1} = \frac{1}{y}$

$\left(\frac{x}{y}\right)^{-1} = \left(\frac{y}{x}\right)$

Fractional Powers

$y^{\frac{1}{2}} = \sqrt{y}$

$y^{\frac{1}{3}} = \sqrt[3]{y}$

Zero Powers

$y^0 = 1$

Word Problems

1. Standard Exponential Solving Questions

Advance Questions

1. Take note of Inequality Question
2. ROUND UP/DOWN
3. Infinity/Long Run Questions

Solving

Compare **P**ower

1. Change Base
2. Combine to Single Base
3. Compare **Powers**

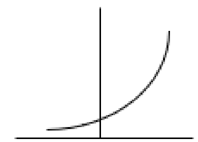
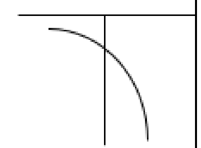
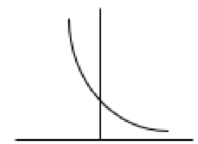
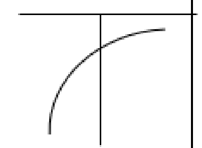
Add **L**n

1. Change Base
2. Combine Base
   OR
   Combine Powers (Adv.)
3. Add Ln and Solve

Substitution

1. Change Base
2. Split Powers
3. EXACT SAME Exponential, apply Substitution

Graphs

	$y = a^x$	$y = -a^x$
$a > 1$ $\log_a x = y$ $x = a^y$		
$0 < a < 1$		

Exponential – Solving Question Type 1

Powers

Given that $\sqrt{256^x} = \frac{4^{3-x}}{16}$, find the value of x.

$$\sqrt{256^x} = \frac{4^{3-x}}{16}$$

$$\sqrt{4^{4x}} = \frac{4^{3-x}}{4^2}$$

$$4^{\frac{4x}{2}} = 4^{3-x-2}$$

$$2x = 1 - x$$

$$x = \frac{1}{3}$$

Solve the following simultaneous equations

$$2^x \times 4^{y-1} = 32$$

$$27 \times 4^{y-\frac{1}{2}} = \frac{3^{2x+1}}{9\sqrt{3}}$$

$2^x \times 4^{y-1} = 32,$

$27 \times 4^{y-\frac{1}{2}} = \frac{3^{2x+1}}{9\sqrt{3}}.$

$2^x \times 2^{2(y-1)} = 2^5$

$x + 2(y-1) = 5$

$x + 2y = 7$

$3^3 \times 3^{y-\frac{1}{2}} = \frac{3^{2x+1}}{3^{2+\frac{1}{2}}}$

$3 + y - \frac{1}{2} = 2x + 1 - \frac{5}{2}$

$y - 2x = -4$

$(1) \times 2 \quad 2x + 4y = 14$

$(2) + (3) \quad 5y = 10$

$y = 2$

Substitute (4) into (1)

$x + 2(2) = 7$

$x = 3$

Powers (Advance)

Solve the equation $3(9^k) + 2(4^k) = 5(6^k)$

$$3(9^k) + 2(4^k) = 5(6^k)$$

$$3(3^k)^2 + 2(2^k)^2 = 5(3^k 2^k)$$

Let $x = 3^k$ and $y = 2^k$,

$$3x^2 + 2y^2 = 5xy$$

$$3x^2 - 5xy + 2y^2 = 0$$

$$(3x - 2y)(x - y) = 0$$

$3x = 2y$ or $x = y$

$3(3^k) = 2(2^k)$ or $3^k = 2^k$

$$\left(\frac{3}{2}\right)^k = \frac{2}{3}$$

$k = -1$ or $k = 0$

Important Concepts

Concept:
How to recognise:

- You can make them to similar bases
  This allows you to compare the powers.

Steps:

1) Change Base

2) Combine Powers

3) Compare Powers

Validation

Substitute final answer back and make sure it tallies

Exponential – Solving Question Type 2

Sub Ln on both side

Solve the following equations.

$7^x = e^{3x+5}$.

$$7^x = e^{3x+5}$$
$$x\ln 7 = 3x + 5$$
$$x(\ln 7 - 3) = 5$$
$$x = \frac{5}{\ln 7 - 3}$$
$$= -4.74\ (3sf)$$

Solve the equation $10^{x+1} = 2$.

$$10^{x+1} = 2$$
$$\lg(10^{x+1}) = lg2$$
$$(x+1)lg10 = lg2$$
$$x + 1 = lg2$$
$$x = \lg(2) - 1$$
$$x = -0.699\ (3sf)$$

Sub Ln on both side (Advance)

It is given that $4^{2x+3} = 7^{3-x}$.

Without using logarithms, find the exact value of 112^x.

Hence use your results in (i), solve $4^{2x+3} = 7^{3-x}$, giving your answer correct to 2 decimal places.

$$4^{2x+3} = 7^{3-x}$$

$4^{2x} \times 4^3 = 7^3 \times 7^{-x}$

$16^x \times 64 = 343 \times 7^{-x}$

$16^x \div 7^{-x} = \frac{343}{64}$

$16^x \times 7^x = \frac{343}{64}$

$(16 \times 7)^x = \frac{343}{64}$

$112^x = \frac{343}{64}$

$112^x = \frac{343}{64}$

$x\lg 112 = \lg\frac{343}{64}$

$x = \lg\frac{343}{64} \div \lg 112$

$x = 0.3558$

$x = 0.36\ (2\text{ d.p.})$

It is given that $2^{2x+1} + 4^{x-1} = 2(3^{1-x})$.

(i) Show that $12^x = 2\frac{2}{3}$.

(ii) Find the value of x, correct to 2 decimal places.

(i) $2^{2x+1} + 4^{x-1} = 2(3^{1-x})$ 8

$2^x \times 2^x \times 2^1 + 2^{2(x-1)} = 2 \times 3^1 \times 3^{-x}$

$2^x \times 2^x \times 2^1 + 2^{2x} \times 2^{-2} = 2 \times 3^1 \times 3^{-x}$

$2^{2x}\left(2 + \frac{1}{4}\right) = \frac{6}{3^x}$

$4^x \times 3^x = \frac{6}{2\frac{1}{4}}$

$12^x = 2\frac{2}{3}$

(ii) $12^x = 2\frac{2}{3}$

$\lg 12^x = \lg 2\frac{2}{3}$

$x = \frac{\lg 2\frac{2}{3}}{\lg 12}$

$x = 0.39$

Important Concepts

Concept:
How to recognise:

- There is no common bases so you are unable to compare powers, in this case, apply Ln to bring down power instead.

For advance questions,
you need to regroup your bases.

$$\frac{4^x}{3^x} = \left(\frac{4}{3}\right)^x$$

Or $(4^x)(3^x) = 12^x$

Validation

Substitute final answer back and make sure it tallies

Exponential – Solving Question Type 3

Substitution

Solve the following equations.

$$2(3^x) - 3^{2-x} = 3$$

$$2(3^x) - 3^{2-x} = 3$$

$$2(3^x) - \frac{9}{3^x} - 3 = 0$$

let $u = 3^x$,

$$2u - \frac{9}{u} - 3 = 0$$

$$2u^2 - 3u - 9 = 0$$

$$(2u + 3)(u - 3) = 0$$

$$u = 3 \quad \text{or} \quad u = -\frac{3}{2}(rej)$$

$$3^x = 3 \Longrightarrow x = 1$$

Solve the following equations.

$$25^{x+1} - 10 = 45(5^x).$$

$$25^{x+1} - 10 = 45(5^x)$$

$$25(5^{2x}) - 45(5^x) - 10 = 0$$

let $5^x = u$

$$25u^2 - 45u - 10 = 0$$

$$(5u - 10)\,(5u + 1) = 0$$

$$u = 2 \quad u = -\frac{1}{5}$$

$$5^x = 2 \quad 5^x = -\frac{1}{5}\text{ (reject)}$$

$$x = \frac{\lg 2}{\lg 5} = 0.431\ (3sf)$$

Solve the following equations.

$$2(3^x) - 8 = 6\sqrt{3^x}$$

Let $u = \sqrt{3^x}$

$$2u^2 - 8 - 6u = 0$$

$$u^2 - 3u - 4 = 0$$

$$(u - 4)(u + 1) = 0$$

$$u = 4 \quad \text{or} \quad u = -1$$

$$\sqrt{3^x} = 4 \qquad \sqrt{3^x} = -1$$

$$3^x = 16$$

$$x = \frac{\lg 16}{\lg 3} = 2.52$$

Important Concepts

Concept:
Split and Simplify the base, you will find that there are similar terms that you can substitute. This is different from Type 1 and Type 2.

You can't apply Type 1 because you won't get a result of $a^x = a^y$.

You can't apply Type 2 because there are multiple terms and substitute Ln does not help.

Validation

Substitute final answer back and make sure it tallies

Exponential – Word Problem

Word Problems

A man buys an antique porcelain at the beginning of 2015. After t years, it value, $\$V$, is given by $V = 15\,000 + 3000e^{0.2t}$.

(i) Find the value of the porcelain when the man first bought it.

(ii) Find the year in which the value of the porcelain first reaches \$50 000

(i) Sub $t = 0$,

$$V = 15\,000 + 3000e^{0} = 18\,000$$

(ii) $50\,000 = 15\,000 + 3000e^{0.2t}$

$35\,000 = 3000e^{0.2t}$

$\frac{35}{3} = e^{0.2t}$

$0.2t = \ln\left(\frac{35}{3}\right)$

$t = 12.283\ldots$

Ans: 2027

The mass, m grams, of a radioactive substance, present at time t days after being observed, is given by the formula $m = 30\,e^{-0.025t}$.

(i) Find the mass remaining after 30 days.

(ii) Find the number of days required for the mass to drop half of its initial value. Give your answer correct to the nearest integer.

(iii) State the value m approaches when t becomes large

(i) $m = 30e^{-0.025(30)}$ = 14.171 = 14.2

(ii) Sub $m = 15$

$$15 = 30e^{-0.025t}$$

$$e^{-0.025t} = \frac{1}{2}$$

$$-0.025t = \ln\frac{1}{2}$$

$$t = 27.726$$

$$t = 28$$

As $t \to \infty$, $30\,e^{-0.0125t} \to \infty$, $30e^{-0.0125t} \to 0$, the value m approaches to 0.

An grandfather clock had an initial value \$2000 in 1850. The clock appreciated in its value such that its value $\$V$ can be modelled by the equation $V = 20000 - Ae^{kt}$, where t is the number of years after its manufacture date.

(i) Find the value of A.

(ii) In the year 1880, the clock reached five times its initial value. Show that $k = -0.01959$ correct to 4 significant figures.

(iii) Explain why the value of the clock will not exceed \$20000.

(i) When $t = 0$, $V = 2000$

$2000 = 20000 - Ae^{k(0)}$

$A = 20000 - 2000 = 18000$

(ii) In the years 1880, $t = 30$, $V = 5(2000)$

$$20000 - 18000e^{30k} = 10000$$

$$-18000e^{30k} = -10000$$

$$e^{30k} = \frac{5}{9}$$

$$\ln e^{30k} = \ln\frac{5}{9}$$

$$30k = \ln\frac{5}{9}$$

$$k = \frac{\ln\frac{5}{9}}{30} = -0.01959\ (4sf)$$

(iii) Hence the value of the clock will not exceed \$20000.
For values of $t \geq 0$, $e^{-0.01959t} > 0$

$$-18000e^{-0.01959t} < 0$$

$$20000 - 18000e^{-0.01959t} < 20000$$

$$V < 20000$$

Hence the value of the clock will not exceed \$20000.

Exponential – Word Problems

Word Problems

A liquid is allowed to cool after being heated. The temperature, $\theta°C$ of the liquid, t seconds after being removed from the heat is given by $\theta = 25 + 80e^{-0.03t}$.

(i) Find the initial value of θ.

(ii) Find the time taken for the liquid to cool to $60°C$.

(iii) Explain why does not fall below $25°C$.

Ans:

(i) When $t = 0$, $\theta = 25 + 80e^0$

$$\theta = 150$$

(ii) $60 = 25 + 80e^{-0.03t}$

$$e^{-0.03t} = \frac{35}{80}$$

$$-0.03t = \ln\left(\frac{35}{80}\right)$$

$$t \approx 27.6s$$

(iii) Since $e^{-0.03t} > 0$

$$80e^{-0.03t} > 0$$

$$25 + 80e^{-0.03t} > 25$$

θ does not fall below 25°C.

The population of a town is given by $P = 250342e^{0.012t}$, where $t = 0$ represents the population in the year 2000.

(i) Find the new town's population in the year 2010. Round off the answer to the nearest whole number.

(ii) Find the year in which the population will be 320,000.

(iii) Find the minimum number of years required for the new town's population to be at least doubled from the year 2010.

Ans:

(i)

$$P = 282260$$
$$P = 250342e^{0.012t}$$
$$= 282259.82$$
$$= 282260$$

(ii)

$$320000 = 250342e^{0.012t}$$
$$In\left(\frac{320000}{250342}\right) = 0.012t$$
$$t = \frac{In\left(\frac{320000}{250342}\right)}{0.012}$$
$$= 20.46 \text{ (Year 2020)}$$

(iii)

$$282259.82 \times 2$$
$$= 564519.64$$
$$564519.64 = 250342e^{0.012t}$$
$$In\left(\frac{564519.64}{250342}\right) = 0.012t$$
$$t = \frac{In\left(\frac{564519.64}{250342}\right)}{0.012}$$
$$= 67.76$$
$$= 68$$

Important Concepts

Concept:

1) Initial means at the beginning, $t = 0$

2) Inequality vs Equation

Falling Below, Less Than, More Than, Exceeds

For Inequality, do take note of the negative signs.

Some questions may 'mask' a negative in Ln value and you will unknowingly forget to change the sign direction.

For example,

$$ln0.2\, x < 5$$
$$-1.61x < 5$$
$$x > \frac{5}{-1.61}$$
$$x > -3.1$$

3) Rounding Up/Rounding Down

On which day, In which year (Round Down)

Find the number of days/years (Round Up)

4) Infinity & Long Run

Take note of the presentation for this. Many times, students just sub t to be 100, 1000. That is wrong. Look at the answers on the right to see how I present.

Validation

Substitute final answer back and make sure it tallies

Exponential – Word Problems

Word Problems

The number of bacteria in a culture doubles every 3 hours. It is given that N_0 is the number of bacteria present at a particular time and that N is the number of bacteria present t hours later.

Calculate the value of the constant k in the relationship

$N = N_0 e^{kt}$.

$$N = N_0 e^{kt}$$

When $t = 0, N = N_0$

When $t = 0, N = N_0 e^{3k}$

$$\frac{N_0 e^{3k}}{N_0} = 2$$

$$e^{3k} = 2$$

$$3k = \ln 2$$

$$k = \frac{\ln 2}{3}$$

$= \mathbf{0.231}$(3 sig. fig)

The percentage, P, of carbon-14 remaining in a piece of fossilised wood is given by $P = 100e^{-kt}$, where k is a constant and t is measured in years. It takes 5730 years for the carbon-14 to be reduced to half of the original amount. Calculate

(i) the value of k,

(ii) the percentage of carbon-14 which would indicate a fossil age of 8000 years.

The size, S, and intensity, I of a naturally occurring event are connected by the formula $S = \lg\frac{I}{c}$, where c is a constant. Calculate, to 1 decimal place, the size of the event which has intensity 50 times that of an event of size 2.4.

(i) When $t = 0, P = 100$

When $t = 5730, P = \frac{100}{2}$

$$= 50$$

$$50 = 100e^{-5730k}$$

$$e^{-5730k} = \frac{1}{2}$$

$$-5730k = \ln\frac{1}{2}$$

$$k = \frac{\ln\frac{1}{2}}{-5730}$$

$$= 0.000\,120\,968$$

$$= 0.000\,121\ (3\ s.f)$$

(ii) When $t = 8000$,

$$P - 100e^{-8000(0.000\,120\,968)}$$

$$= 38.0\ (3\ s.f)$$

The percentage of carbon -14 which would indicate a fossil age of 8000 years is **38.0%**.

A manufactures produces a disinfectant that destroys 21% if all known germs within one minute of use. If N is the number of germs present when the disinfectant is first used, and assuming germs continue to be destroyed at the same rate, explain why the number of germs expected to be alive after n minutes is given by $(0.79)^n N$.

(i) The manufacturer decides to advertise by stating that the disinfectant destroys x% of all known germs within 20 minutes of use. Calculate, to 2 significant figures, the value of x.

(ii) Given that the number of germs expected to be alive after n minutes can be expressed as Ne^{kn},

find the value of the constant k.

(i) Number of germs expected after n minutes

$$= \underbrace{N \times (1-21\%) \times (1-21\%) \ldots \times (1-21\%)}_{n \text{ times}}$$

$$= \underbrace{N \times 0.79 \times 0.79 \times \cdots \times 0.79}$$

$$= (0.79)^n N \text{ (shown)}$$

(ii) Number of germs that are destroyed in 20 minutes

$$= N - (0.79)^n N$$

$$\frac{N -= (0.79)^{20} N}{N} \times 100\% = x\%$$

$$[1 - (0.79)^{20}] \times 100 = x$$

$$x = \mathbf{99}\ (2\text{ s.f.})$$

(iii) $(0.79)^n N = Ne^{kn}$

$(0.79)^n = (e^k)^n$

$e^k = 0.79$

$k = \ln 0.79$

$= 0.236\ (3\ s.f.)$

Simplifying

$\log_a y$ is read as logarithm of y to base a

$\log_a a = 1$ $\quad \log_{10} a = \lg a$

$\log_a 1 = 0$ $\quad \log_e e = \ln e = 1$

Laws of Logarithm

Product Law: $\log_a x + \log_a y = \log_a(xy)$

Quotient Law: $\log_a x - \log_a y = \log_a \frac{x}{y}$

Power Law: $r \times \log_a x = \log_a x^r$

Changing Base: $\log_a b = \frac{\log_c b}{\log_c a}$

New Base

Changing Log to Index Form: $\log_a x = y \quad \rightarrow \quad x = a^y$

Solving

1 Log

Log to Index

1. $\log_a y = x$
2. $y = a^x$

Multiple Logs

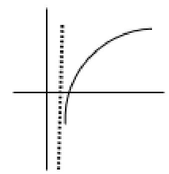
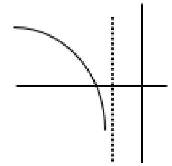
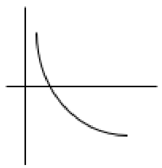
CCC

1. **C**hange to same base
2. **C**ombine to single Log
3. **C**ancel Log on both side

Substitution

1. Change to same base
2. EXACT SAME Log, apply Substitution

Graphs

$y = \log_a x$, where $a > 1$	$y = \log_a x$, where $a > 1$	$y = \log_a x$, where $0 < a < 1$
		
$y = \ln x$	$y = -\ln x$	$y = \log_{\frac{1}{3}} x$

Simplifying Logarithm

Given that $\log_x 2 = p$ and $\log_4 y = q$, express the following in terms of p and/or q.

(i) $\log_4 \frac{4x}{y}$, (ii) xy.

Ans:

(i) $\log_4 \frac{4x}{y}$

$= \log_4 4 + \log_4 x - \log_4 y$

$= 1 + \frac{\log_x x}{\log_x 2^2} - q$

$= 1 + \frac{1}{2\log_x 2} - q$

$= 1 + \frac{1}{2p} - q$

(ii) $x = 2^{\frac{1}{p}}$

$y = 2^{2q}$

$xy = 2^{\frac{1}{p}+2q}$

Given that $a = \log_2 x$ and $b = \log_4 y$, express in terms of a and/or b,

(i) $\log_2 64x^3$, (ii) $\log_y x$.

Ans:

(i) $\log_2 64x^3 = \log_2 64 + \log_2 x^3$

$= 3(2 + a)$

(ii) $\log_y x = \frac{\log_2 x}{\log_2 y}$

$= a \div \frac{\log_4 y}{\log_4 2}$

$= a \div \frac{b}{\frac{1}{2}}$

$= \frac{a}{2b}$

Given that $u = \log_3 z$, find, in terms of u,

(i) $\log_3 9z$,

(ii) $\log_3\left(\frac{z}{27}\right)$,

(iii) $\log_z 27$.

(i) $\log_3 9z = \log_3 9 + \log_3 z$

$= \log_3 3^2 + \log_3 z = 2 + u$

(ii) $\log_3\left(\frac{z}{27}\right) = \log_3 z - \log_3 27$

$= \log_3 z - \log_3 3^3 = u - 3$

(iii) $\log_z 27 = \frac{\log_3 27}{\log_3 z} = \frac{3}{u}$

Given that $log_2 a = b$, express

(i) a in terms of b,

(ii) $log_2\left(\frac{a^4}{32}\right)$ in terms of b,

(iii) $\left(\frac{1}{8}\right)^b$ in terms of a.

(i) $a = 2^b$

(ii) $log_2\left(\frac{a^4}{32}\right) = log_2(a^4) - log_2 32 = 4a - 5$

(iii) $\left(\frac{1}{8}\right)^b = 2^{-3b} = a^{-3} = \frac{1}{a^3}$

Important Concepts

Concept:

1) Remember all the properties in Log
2) Must be clear when to Split vs Merge
3) Must be strong is converting from Log to Number and Number to Log

Logarithm Properties:

Laws of Logarithm

Product Law: $\log_a x + \log_a y = \log_a(xy)$

Quotient Law: $\log_a x - \log_a y = \log_a \frac{x}{y}$

Power Law: $r \times \log_a x = \log_a x^r$

Changing Base: $\log_a b = \frac{\log_c b}{\log_c a}$ (New Base)

Changing Log to Index Form: $\log_a x = y \rightarrow x = a^y$

Validation

For simplifying here, you can let your algebra be any values and make sure that your final answer tallies with the values you substitute.

Logarithm – Solving Question Type 1 – One Log

Log to Index

Solve $\log_8 y + \log_2 y = 4$

$$\log_8 y + \log_2 y = 4$$

$$\frac{\log_2 y}{\log_2 8} + \log_2 y = 4$$

$$\frac{\log_2 y}{3} + \log_2 y = 4$$

$$\frac{4}{3}\log_2 y = 4$$

$$\log_2 y = 3$$

$$y = 8$$

Solve $\log_x(3x^2 + 10x - 28) = 2$

$$x^2 = 3x^2 + 10x - 28$$

$$2x^2 + 10x - 28 = 0$$

$$(x + 7)(x - 2) = 0$$

$$x = 2\,, x = -7\ (Reject)$$

Important Concepts

Concept:
How to recognise:

Log to Index is used when you have only one log term.

Steps:

Convert Log to Index & solve using algebra

Validation

Substitute final answer back and make sure it tallies

Logarithm – Solving Question Type 2 – Multiple Logs

CCC

Solve the equation $\log_2(2x+1) - \log_4(x+1) = 1$.

$\log_2(2x+1) - \log_4(x+1) = 1$

$\log_2(2x+1) - \frac{\log_2(x+1)}{\log_2 4} = \log_2 2$

$\log_2(2x+1) - \frac{1}{2}\log_2(x+1) = \log_2 2$

$\log_2 \frac{(2x+1)}{\sqrt{x+1}} = \log_2 2$

$2x + 1 = 2\sqrt{x+1}$

$4x^2 + 4x + 1 = 4x + 4$

$4x^2 = 3$

$x = \frac{\sqrt{3}}{2}$ or $x = -\frac{\sqrt{3}}{2}$ (reject)

Solve $\log_2(3x-5) + 3 = \log_2(4x+5)$,

$\log_2(3x-5) + 3 = \log_2(4x+5)$

$\log_2(3x-5) + \log_2 2^3 = \log_2(4x+5)$

$\log_2 8(3x-5) = \log_2(4x+5)$

$24x - 40 = 4x + 5$

Find the value(s) of y that satisfy the equation

$$\log_4(2y) = \log_{16}(y-3) + 3\log_9 3,$$

$\log_4(2y) = \log_{16}(y-3) + 3\log_9 3$

$\log_4(2y) = \frac{\log_4(y-3)}{\log_4 16} + 3\frac{\log_3 3}{\log_3 9}$

$\log_4(2y) = \frac{\log_4(y-3)}{2} + \frac{3}{2}$

$2\log_4(2y) = \log_4(y-3) + 3$

$\log_4(2y)^2 - \log_4(y-3) = 3$

$\log_4 \frac{(2y)^2}{y-3} = 3$

$\therefore \frac{4y^2}{y-3} = 4^3$

$4y^2 = 64(y-3)$

$y^2 = 16(y-3)$

$y^2 - 16y + 48 = 0$

$(y-4)(y-12) = 0$

$\therefore y = 4$ or $y = 12)$

Important Concepts

Concept:
Identification:

There will be multiple logarithm terms in the equation.

You will be able to change all of them to the same base

Steps:

1) Change Base

2) Combine Logarithm

3) Cancel Log on both sides

Validation

Substitute final answer back and make sure it tallies

Logarithm – Solving Question Type 3 – Multiple Logs with Similar Terms

Substitution

Solve the equation $3\log_x 3 = 8 - 4\log_3 x$.

$$3\log_x 3 = 8 - 4\log_3 x$$
$$\frac{3}{\log_3 x} = 8 - 4\log_3 x$$

Let $y = \log_3 x$

$$\frac{3}{y} = 8 - 4y$$
$$3 = 8y - 4y^2$$
$$4y^2 - 8y + 3 = 0$$
$$(2y-3)(2y-1) = 0$$
$$y = 1.5 \text{ or } 0.5$$
$$x = 3^{1.5} \text{ or } 3^{0.5}$$
$$= \sqrt{27} \text{ or } \sqrt{3}$$

Solve the equation $\lg x = \log_x 1000$, giving your answer to 2 significant figures.

$$\lg x = \log_x 1000$$
$$\lg x = \frac{\lg 10^3}{\lg x}$$
$$(\lg x)^2 = 3$$
$$\lg x = \sqrt{3} \text{ or } -\sqrt{3}$$
$$x = 10^{\sqrt{3}} \text{ or } 10^{-\sqrt{3}}$$
$$x = 54 \text{ or } 0.019 \text{ (2sf)}$$

Solve the equation $\log_4 x^2 - 3\log_x 4 = 1$.

$$\log_4 x^2 - 3\log_x 4 = 1$$
$$2\log_4 x - \frac{\log_4 4^3}{\log_4 x} = 1$$
$$2(\log_4 x)^2 - 3 = \log_4 x$$
$$2(\log_4 x)^2 - \log_4 x - 3 = 0$$

Let $\log_4 x$ be u.

$$2u^2 - u - 3 = 0$$
$$(2u-3)(u+1) = 0$$
$$u = \frac{3}{2} \text{ or } -1$$
$$\log_4 x = \frac{3}{2} \text{ or } \log_4 x = -1$$
$$x = 4^{\frac{3}{2}} \text{ or } x = 4^{-1}$$
$$x = 8 \text{ or } x = \frac{1}{4}$$

Solve the equation $4\log_6 x - 2\log_x 6 = 7$.

$$4\log_6 x - 2\log_x 6 = 7$$
$$4\log_6 x - \frac{2\log_6 6}{\log_6 x} = 7$$

let $u = \log_6 x$

$$4u - \frac{2}{u} = 7$$
$$4u^2 - 7u - 2 = 0$$
$$(4u+1)(u-2) = 0$$
$$u = -\frac{1}{4} \text{ or } 2$$
$$\log_6 x = -\frac{1}{4} \text{ or } \log_6 x = 2$$
$$x = 6^{-\frac{1}{4}} \qquad x = 36$$

Important Concepts

Concept:

Identification – you will realise that 90% of the times, your base has an algebra.

However, the key observation is once you change to common base, you will realise that they are similar terms. Just like in exponential, this indicates the substitution method.

From there, proceed to solve in algebraically and then find your final value.

Validation

Substitute final answer back and make sure it tallies

Binomial Expansion

$$(a+b)^n = \binom{n}{0}(a)^{n-0}(b)^0 + \binom{n}{1}(a)^{n-1}(b)^1 + \ldots$$

1. Second Term, b, include Sign
2. Apply Power to every value
3. Remember to add +…

Hence

Selective Expansion

Full Expansion

Appro-ximation

Finding Specific Term

$$T_{r+1} = \binom{n}{r}(a^{n-r})(b^r)$$

1. SPLIT every term
2. REARRANGE & COMBINE
3. Compare Powers to find r

Advance

Unknown Power

Decide Method

$$\binom{n}{0} = 1 \qquad \binom{n}{1} = n$$

$$\binom{n}{2} = \frac{(n)(n-1)}{2 \times 1}$$

$$\binom{n}{3} = \frac{(n)(n-1)(n-2)}{3 \times 2 \times 1}$$

Binomial Theorem – Question Type 1 Binomial Expansion

Binomial Expansion

Expand $(1-2x)^9$ in ascending powers of x up to the term in x^3.

$$(1-2x)^9 = 1 - 18x + 144x^2 - 672x^3 + \cdots$$

$$(1-2x)^9 = \binom{9}{0}(-2x)^0 + \binom{9}{1}(-2x)^1 + \binom{9}{2}(-2x)^2 + \binom{9}{3}(-2x)^3 + \cdots$$

$$= 1 - 18x + 144x^2 - 672x^3 + \cdots$$

Write down and simplify the first four terms in the expansion $\left(2x - \frac{p}{x^2}\right)^5$ in descending powers of x, where p is a non-zero constant.

$$\left(2x - \frac{p}{x^2}\right)^5$$

$$= (2x)^5 + 5(2x)^4\left(-\frac{p}{x^2}\right) + 10(2x)^3\left(-\frac{p}{x^2}\right)^2 + 10(2x)^2\left(-\frac{p}{x^2}\right)^3 + \cdots$$

$$= 32x^5 - 80px^2 + \frac{80p^2}{x} - \frac{40p^3}{x^4} + \cdots$$

Binomial Expansion

Find in descending powers of x, up to and including the x^3 term, the terms in the expansion of $\left(x - \frac{3}{x}\right)^7$.

$$\left(x - \frac{3}{x}\right)^7 = x^7 + \binom{7}{1}x^6\left(-\frac{3}{x}\right)^1 + \binom{7}{2}x^5\left(-\frac{3}{x}\right)^2 + \cdots$$

$$= x^7 - 21x^5 + 189x^3 \ldots$$

Write down and simplify the first 4 terms in the expansion of $\left(\frac{1}{2} + 2x\right)^8$ in ascending powers of x.

$$\left(\frac{1}{2} + 2x\right)^8$$

$$= \binom{8}{0}\left(\frac{1}{2}\right)^8 + \binom{8}{1}\left(\frac{1}{2}\right)^7(2x) + \binom{8}{2}\left(\frac{1}{2}\right)^6(2x)^2 + \binom{8}{3}\left(\frac{1}{2}\right)^5(2x)^3 + \cdots$$

$$= \frac{1}{256} + \frac{1}{8}x + \frac{7}{4}x^2 + 14x^3 + \cdots$$

Important Concepts

Concept:

Please remember the below common mistakes.

1) Correct Formula

2) Missing +…

3) Forgetting to input Signs

4) Apply Powers to Coefficient

Validation

You can use small approximation to validate.

If not, just double check your expansion manually.

Binomial Theorem – Question Type 1 Binomial Expansion (Hence Questions)

Selective Expansion

Expand $(1-2x)^9$ in ascending powers of x up to the term in x^3.
Find the value of k, given that the coefficient of x in the expansion of $\left(3x+\frac{1}{kx^2}\right)(1-2x)^9$ is -53.

$$(1-2x)^9 = 1 - 18x + 144x^2 - 672x^3 + \cdots$$

$$(1-2x)^9 = \binom{9}{0}(-2x)^0 + \binom{9}{1}(-2x)^1 + \binom{9}{2}(-2x)^2 + \binom{9}{3}(-2x)^3 + \cdots$$

$$= 1 - 18x + 144x^2 - 672x^3 + \cdots$$

$\left(3x+\frac{1}{kx^2}\right)(1-2x)^9$

$= \left(3x+\frac{1}{kx^2}\right)(1 - 18x + 144x^2 - 672x^3 + \cdots)$

Term in $x = 3x(1) + \frac{1}{kx^2}(-672x^3)$

coefficient of $x = -53$

$3 - \frac{672}{k} = -53$

$$k = 12$$

Given that the coefficient of x^{-1} in the expansion $(4x^3-1)\left(2x-\frac{p}{x^2}\right)^5$ is $-160p^2$, find the value of p.

$$(4x^3-1)\left(2x-\frac{p}{x^2}\right)^5$$

$$= (4x^3-1)\left(32x^5 - 80px^2 + \frac{80p^2}{x} - \frac{40p^3}{x^4} + \cdots\right)$$

Coefficient of $x^{-1} = 4(40p^3) + (-1)(80p^2)$

$$= -160p^3 - 80p^2$$

$$-160p^3 - 80p^2 = 160p^2$$

$$80p^2(2p-1) = 0$$

$$p = 0\ (NA) \text{ or } p = 0.5$$

Full Expansion

Given that the expansion of $(a-x)\left(\frac{1}{2}+2x\right)^8$ in ascending powers of x is $\frac{1}{128}+\frac{63}{256}x+bx^2+\cdots$, find the value of a & of b.

$$(a-x)\left(\frac{1}{2}+2x\right)^8 = \frac{1}{128}+\frac{63}{256}x+bx^2+\cdots$$

$$(a-x)\left(\frac{1}{256}+\frac{1}{8}x+\frac{7}{4}x^2+14x^3\right) = \frac{1}{128}+\frac{63}{256}x+bx^2+\cdots$$

Comparing x^0: $\frac{a}{256} = \frac{1}{128}$

$$a = 2$$

Comparing x^2: $\frac{7}{4}a - \frac{1}{8} = b$

$$b = \frac{27}{8}$$

Obtain the first three terms in the expansion of $\left(2-\frac{x}{3}\right)^5$, in ascending powers of x. Given that the first three terms in the expansion of $(1+hx+x^2)\left(2-\frac{x}{3}\right)^5$ are $32 - hx + 2hx^2$.

$$\left(2-\frac{x}{3}\right)^5 = 2^5 + \binom{5}{1}(2)^4\left(-\frac{x}{3}\right) + \binom{5}{1}\left(-\frac{x}{3}\right)^2$$

$$= 32 - \frac{80}{3}x + \frac{80}{9}x^2 + \cdots$$

$$(1+hx+x^2)\left(32-\frac{80}{3}x+\frac{80}{9}x^2+\cdots\right)$$

$$= \left[32 - \frac{80}{3}x + \frac{80}{9}x^2 + 32hx - \frac{80h}{3}x^2 + 32x^2\right]$$

$$= 32 - \frac{80}{3}x + 32hx + \frac{80}{9}x^2 + 32x^2 - \frac{80h}{3}x^2$$

$$-2\left[-\frac{80}{3} + 32h\right] = \frac{80}{9} + 32 - \frac{80h}{3}$$

$$\frac{160}{3} - 64h = \frac{80}{9} + 32 - \frac{80h}{3}$$

$$-\frac{112}{3}h = -\frac{112}{9} \Rightarrow h = \frac{1}{3}$$

Important Concepts

Concept:

3 Types of Hence – Binomial Expansion Question

1) Selective Expansion
This method is used when the question asked for the coefficient of a specific value. We do not need to expand every single term, therefore, we specifically expand the terms that will give us what we need.

2) Full Expansion

This method is used when the question asks for several unknown values. We have to fully expand and compare coefficients to obtain the answer.

Validation

Take a rough paper and expand out again.

Alternatively, sub a random value and make sure that every line has a similar value.

Binomial Theorem – Question Type 1 (Hence Questions)

Approximation

Write down the expansion of $(3-x)^3(3+2x)^8$ in ascending powers of x, up to x^2.

By letting $x = 0.01$ and your expansion in (iii), find the value of $2.99^3 \times 3.02^8$, giving your answer correct to 3 significant figures.

Show your workings clearly.

(i) $(3-x)^3 = 27 - 27x + 9x^2 - x^3$

(ii) $(3+2x)^8$

$= 3^8 + \binom{8}{1}(3)^7(2x) + \binom{8}{2}(3)^6(2x)^2 + \binom{8}{2}(3)^5(2x)^3$

$= 6\,561 + 34\,992x + 81\,648x^2 + 108\,864x^3 + \cdots$

(iii) $(3-x)^3(3+2x)^8$

$=$ their (i) $\times$ their (ii)

$= 177\,147 + 767\,637x + 2\,854\,035x^2 + \cdots$

(iv) $2.99^3 \times 3.02^8$

$= 177\,147 + 767\,637\,(0.01) + 2\,854\,035\,(0.01)^2$

$= 185108.7735$

$= 185\,000$

(v) For $2^3 \times 5^8$, need to use $x = 1$

Since 1 is large in comparison to 0.01, the value is inaccurate because a significantly large value is removed after the 3rd term.

Find the first four terms in the expansion of $(2x+3)\left(1-\frac{x}{2}\right)^{11}$, in ascending power of x.

Hence estimate the value of 3.2×0.95^{11}.

(i) $(2x+3)\left(1-\frac{x}{2}\right)^{11}$

$= (2x+3)\left(1 + \binom{11}{1}\left(-\frac{x}{2}\right) + \binom{11}{2}\left(-\frac{x}{2}\right)^2 + \binom{11}{3}\left(-\frac{x}{2}\right)^3 + \cdots.\right)$

$= (2x+3)\left(1 - \frac{11}{2}x + \frac{55}{4}x^2 - \frac{165}{8}x^3 + \cdots\right)$

$= 3 - \frac{33}{2}x + \frac{165}{4}x^2 - \frac{495}{8}x^3 + 2x - 11x^2 + \frac{55}{2}x^3 + \cdots$

$= 3 - \frac{29}{2}x + \frac{121}{4}x^2 - \frac{275}{8}x^3 + \cdots$

(ii) 3.2×0.95^{11}

Let x be 0.1.

$(2(0.1)+3)\left(1-\frac{(0.1)}{2}\right)^{11}$

$= 3 - \frac{29}{2}(0.1) + \frac{121}{4}(0.1)\text{^}2 - \frac{275}{8}(0.1)\text{^}3 + \cdots$

$= 1.818125$

Important Concepts

Concept:

3 Types of Hence – Binomial Expansion Question

3a) Replacement – Numbers

3b) Replacement - Algebra

Validation

Take a rough paper and expand out again.

Alternatively, sub a random value and make sure that every line has a similar value.

Binomial Theorem – Question Type 2 – Finding Specific Terms

Finding Specific Terms

Find the term independent of x in the binomial expansion of $\left(x-\frac{2}{x^2}\right)^9$.

$T_{r+1} = \binom{9}{r}(x)^{9-r}(-2x^{-2})^r$

$= \binom{9}{r}(-2)^r x^{9-r-2r}$

$= \binom{9}{r}(-2)^r x^{9-3r}$

$9-3r=0$ => r=3

$r=3$

$\therefore T_4$

$= \binom{9}{3}(-2)^3$

$= -672$

Given that the coefficient of x^8 in the expansion of $\left(2x^2-\frac{p}{x}\right)^{10}$ is negative $\frac{10}{3}$ times the coefficient of x^5. Show that the value of p is $\frac{1}{2}$.

For x^8, $x^{20-2r-r}=x^8$,

$20-3r=8$

$r=4$

For x^5, $x^{20-2r-r}=x^5$,

$20-3r=5$

$r=5$

$$\binom{10}{4}(2)^{10-4}\left(-\frac{1}{2}\right)^4 = -\frac{10}{3}\binom{10}{5}(2)^{10-5}\left(-\frac{1}{2}\right)^5$$

$$\frac{\binom{10}{4}2^6}{\binom{10}{5}2^5}\times\frac{3}{10}=p$$

$$p=\frac{1}{2}$$

In the expansion of $\left(2+\frac{4}{x^4}\right)\left(kx^3-\frac{2}{x}\right)^{13}$ where k is a constant and $k \neq 0$, find the value of k if there is no coefficient of $\frac{1}{x}$.

For x^{-1}, $x^{0+r}=x^{-1}$,

$0+r=-1$

$r=-1$

For x^5, $x^{-4+r}=x^{-1}$,

$-4+\text{r}=-1$

$r=3$

$$\left(kx^3-\frac{2}{x}\right)^{13} = \binom{13}{r}(kx^3)^{13-r}\left(-\frac{2}{x}\right)^r+\cdots$$

$$=\binom{13}{9}(kx^3)^4\left(-\frac{2}{x}\right)^9+\binom{13}{10}(kx^3)^3\left(-\frac{2}{x}\right)^{10}$$

$$=715k^4x^{12}\left(-\frac{512}{x^9}\right)+286k^3x^9\left(\frac{1024}{x^{10}}\right)+\cdots$$

$$=-366080k^4x^3+\frac{292864}{x}k^3+\cdots$$

$$=\left(2+\frac{4}{x^4}\right)\left(-366080k^4x^3+\frac{292864}{x}k^3+\cdots\right)$$

$$=\frac{585728k^3}{x}-\frac{1464320}{x}k^4+\cdots$$

$$585728k^3-1464320k^4=0$$

$$k^3(585728-1464320k)=0$$

$$k=\frac{2}{5}$$

Important Concepts

Concept:

1) Apply General Term Formula

2) SPLIT into individual terms

3) Rearrange (Numbers and Algebra)

4) Look at x term to compare powers and solve r

5) Substitute your r to find the coefficient

Validation

Take a rough paper and expand out again.

Alternatively, sub a random value and make sure that every line has a similar value.

Binomial Theorem – Question Type 3

Unknown Power

Given that the expansion of $\left(1+\frac{x}{2}\right)^n(3-2x)$ up to the first three terms, in ascending powers of x, is $h+10x+kx^2$, find the values of h, k and n.

$$\left(1+\frac{x}{2}\right)^n(3-2x)$$
$$=\left(1+\binom{n}{1}\left(\frac{x}{2}\right)+\binom{n}{2}\left(\frac{x}{2}\right)^2+\cdots\right)(3-2x)$$
$$=\left(1+\frac{1}{2}nx+\frac{1}{8}n(n-1)x^2+\cdots\right)(3-2x)$$

Comparing coefficient of x^0 : $h=3$

Comparing coefficient of x^1 : $\frac{3}{2}n-2=10 \Rightarrow n=8$

Comparing coefficient of x^2 :

$$\frac{3}{8}n(n-1)-n=k$$
$$k=\frac{3}{8}(8)(8-1)-8=13$$

Ans: $h=3 n=8 k=13$

The first 3 terms in the binomial expansion $(1+kx)^n$ are $1+5x+\frac{45}{4}x^2+\cdots$ Find the value of n and of k.

$$(1+kx)^n=1+\binom{n}{1}kx+\binom{n}{2}k^2x^2+\cdots$$
$$=1+nkx+\frac{n(n-1)k^2}{2}x^2+\cdots$$

Comparing coefficients : $nk=5$

$$\frac{n(n-1)k^2}{2}=\frac{45}{4}$$
$$2n^2k^2-2nk^2=45$$

Subs (1) in (2) : $50-10k=45$

$$\therefore k=\frac{1}{2} \text{ and } n=10$$

Find the value of n, given that the coefficients of x^4 and x^6 in the expansion of $\left(1+\frac{1}{3}x^2\right)^n$ are in the ratio of 3: 2.

$$\left(1+\frac{1}{3}x^2\right)^n$$
$$=1+\binom{n}{1}\left(\frac{1}{3}x^2\right)+\binom{n}{2}\left(\frac{1}{3}x^2\right)^2+\binom{n}{3}\left(\frac{1}{3}x^2\right)^3+..$$
$$=1+n\left(\frac{1}{3}x^2\right)+\binom{n}{2}\frac{1}{9}x^4+\binom{n}{3}\frac{1}{27}x^6+..$$

$\frac{\binom{n}{2}\frac{1}{9}}{\binom{n}{3}\frac{1}{27}}=\frac{3}{2}$ [Showing the coeff of $x^4=\binom{n}{2}\frac{1}{9}$]

$\frac{\binom{n}{2}}{\binom{n}{3}}=\frac{1}{2}$ [Showing the coeff of $x^6=\binom{n}{3}\frac{1}{27}$]

$$2\binom{n}{2}=\binom{n}{3}$$
$$\frac{2n(n-1)}{2}=\frac{m(n-1)(n-2)}{6}$$
$$n-2=6$$
$$n=8$$

Important Concepts

Concept:

Formula

With unknown powers,

You can be applying either binomial expansion or finding specific term method. Read the question to determine what they are looking for.

Validation

Take a rough paper and expand out again.

Alternatively, sub a random value and make sure that every line has a similar value.

Perpendicular Bisector

1. Gradient of line: $m = \frac{y_1 - y_2}{x_1 - x_2}$

2. Equation of line: $y - y_1 = m(x - x_1)$

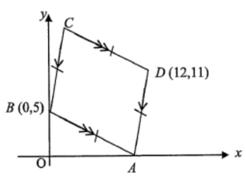
3. Length of line: $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

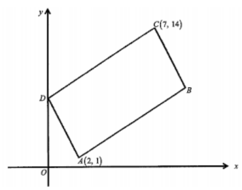
4. Midpoint Theory: $(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$

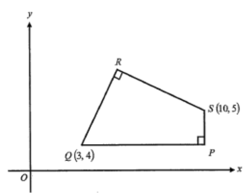
5. Gradient of Perpendicular Bisector = $-\frac{1}{Gradient\ of\ Tangent}$

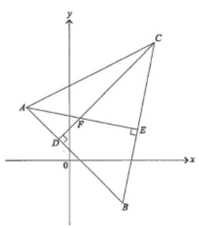
6. Perpendicular Bisector

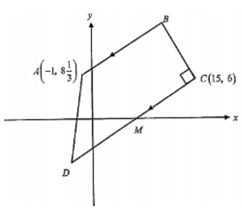
   -Find Midpoint of Line
   -Find Gradient of Line
   -Find Gradient of Perpendicular Bisector
   -Find Equation of Perpendicular Bisector (Midpoint & Grad)





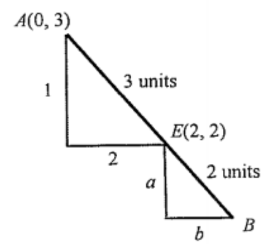






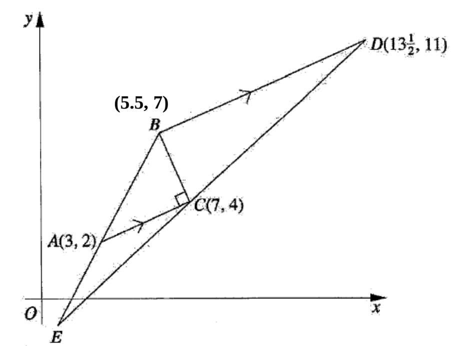
Collinear (Similar Triangle)

1. Ratio of Triangles
2. Method 1 vs Method 2



Area (Shoe-Lace Method)

1, Pick 1 coordinate
2. Anti Clockwise
3. Repeat First Coordinate



Area of $ABDC$

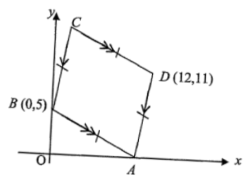
$$= \frac{1}{2}\begin{vmatrix} 3 & 7 & 13\frac{1}{2} & 5\frac{1}{2} & 3 \\ 2 & 4 & 11 & 7 & 2 \end{vmatrix}$$

= 22.5units

Coordinate Geometry

The diagram shows a rhombus $ABCD$. B and D are $(0, 5)$ and $(12, 11)$ respectively and A lies on the x-axis.

Show that the gradient of AC is -2.
Find,
the midpoint of BD,
the coordinates of A and C,



(i) $m_{BD} = \frac{11-5}{12-0}$
$= \frac{1}{2}$

Since BD is perpendicular to AC, $m_{BD} \times m_{AC} = -1$

$m_{AC} = -1 \div \frac{1}{2} = -2$ (shown)

(ii) Midpoint of $BD = \left(\frac{0+12}{2}, \frac{5+11}{2}\right)$
$= (6,8)$

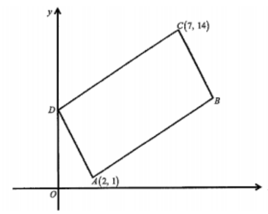
(iii) Equation of AC:
Since midpoint of BD is also the midpoint of AC,
$y - 8 = -2(x - 6)$
$y = -2x + 12 + 8$
$y = -2x + 20$
At A, $y = 0$
$-2x + 20 = 0$
$x = 10$
$\therefore A(10,0)$
Let the coordinates of C be (x_1, y_1)

$$(6,8) = \left(\frac{10 + x_1}{2}, \frac{10 + y_1}{2}\right)$$

$(x_1, y_1) = (2,16)$
$\therefore C(2,16)$

OR

By observation,
$A(10,0) \rightarrow D(12,11)$
$B(0,5) \rightarrow C(2,16)$



The diagram shows a parallelogram $ABCD$ in which the coordinates of the points A and C are $(2, 1)$ and $(7, 14)$ respectively. Given that the point D lies on the y-axis and that the gradient of AD is -3, find

(a) the coordinates of B and of D,

$D(0, d)$

Gradient $AD = \frac{d-1}{0-2} = -3$

$d = 7$

$\therefore D(0,7)$

$B(p, q)$

Midpoint AC = Midpoint BD

$$\left(\frac{2+7}{2}, \frac{1+14}{2}\right) = \left(\frac{0+p}{2}, \frac{7+q}{2}\right)$$

$p = 9$

$q = 8$

$B(9,8)$

Possible to solve by find equation of BC (M1) and equation of CD (M1) and get answer by solving simultaneously. (A1)

Important Concepts

Concept:

1.) Midpoint formula

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

2.) Gradient

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

3.) Equation of Line

$$y - y_1 = m(x - x_1)$$

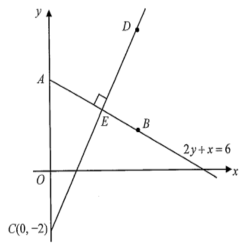
4) Gradient of Normal

$$m_{,1} \times m_2 = -1$$

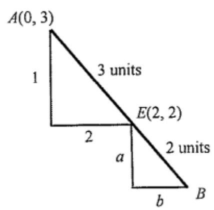
5) Area of Quadrilateral (Shoe Lace Method)

Coordinate Geometry – Involving Similar Triangles Method

In the diagram, the line $2y + x = 6$ cuts the y-axis at point A and passes through point B. The line CD cuts the y-axis at (0, -2) and intersects line AB at point E. the two lines AB and CD are perpendicular to each other.
(i)Find the coordinates of E.
(ii) The ratio of $AE : AB$ is 3 : 5. Find the coordinates of B.



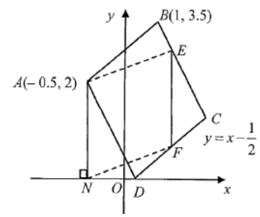
$2y + x = 6$
$2y = -x + 6$
$y = -\frac{1}{2}x + 3$ ------ (1)
Gradient of line AB is $-\frac{1}{2}$
Gradient of CD is 2.
Equation of CD is
$y = 2x - 2$ ----- (2)
Sub, (1) into (2),
$2(2x - 2) + x = 6$
$4x - 4 + x = 6$
$5x = 10$
$x = 2$ sub into (1)
$y = 2$



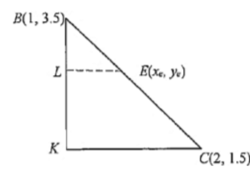
$2y + x = 6$
When $x = 0, 2y = 6$
$y = 3$
$A(0,3)$
Using similar triangles,
$\frac{a}{1} = \frac{2}{3}$
$a = \frac{2}{3}$ (Application of similar triangles)
$\frac{b}{2} = \frac{2}{3}$ (Application of similar triangles)
$b = \frac{4}{3}$

$$\therefore B\left(2 + \frac{3}{4}, 2 - \frac{2}{3}\right) = B\left(3\frac{1}{3}, 1\frac{1}{3}\right)$$

The points $A\ (-0.50, 2), B\ (1, 3.5), C$ and D are the four vertices of a parallelogram. The point E lies on BC such that $BE = \frac{1}{3}BC$. The line CD has the equation $y = x - \frac{1}{2}$. Lines are drawn, parallel to the y-axis, from A to meet the x-axis at N and from E to meet CD at F.
(i) Calculate the coordinates of the C and E.



Sub $y = 0, 0 = x - \frac{1}{2}$
$Coord\ of\ D = \left(\frac{1}{2}, 0\right)$
Mid Pt. AC = Mid Pt. BD
$\left(\frac{-0.5+x}{2}, \frac{2+y_C}{2}\right) = \left(\frac{1+0.5}{2}, \frac{3.5+0}{2}\right)$
$x_e = 2, y_e = 1.5\ Coord\ C = (2, 1.5)$

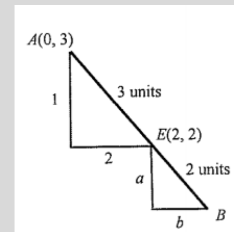


$\frac{BE}{BC} = \frac{BL}{Bk}$ $(similar\ triangles)$
$\frac{1}{3} = \frac{3.5 - y_e}{3.5 - 1.5}$
$y_e = \frac{17}{6}$

$\frac{BE}{BC} = \frac{BL}{KC}$ $(similar\ triangles)$
$\frac{1}{3} = \frac{1 - x_e}{1 - 2}$
$x_e = \frac{4}{3}$
$Coord\ E\ \left(1\frac{1}{3}, 2\frac{5}{6}\right)$

Important Concepts

Concept:
6) Collinear (Similar Triangle Method)



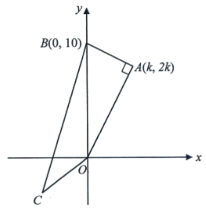
Identify the similar traingles

Since they are similar, if the hypothenuse has a ratio of 3:2, the base and the heights will also have a ratio of 3:2.

Use this ratio to find out the actual size of the triangles so you can obtain the coordinates.

Coordinate Geometry – Advance Questions

The diagram shows a quadrilateral with vertices O, (0, 0), A(k, $2k$), B((0,10) and C.



The length of OA is $4\sqrt{5}$ units and OA is perpendicular to AB. The line OC is parallel to the line 4y = 3x + 20 and the perpendicular bisector of AB passes through the point C. Find

(i) The value of k, explaining why the diagram is necessary
(ii) the coordinates of C.
(iii) Find the area of the quadrilateral OABC.

$$k^2 + (2k)^2 = \left(4\sqrt{5}\right)^2$$
$$5k^2 = 16(5)$$
$$k = 4 \text{ or } -4$$

The diagram is necessary because it **indicates the point *A* lies in quadrant I**, and so the **value of *k*** is **positive** therefore $k = 4$.

Mid-point of AB, $M = \left(\frac{0+4}{2}, \frac{10+8}{2}\right)$

$= (2, 9)$

Gradient of line $CM = -\frac{1}{\frac{8-10}{4-0}}$

$= 2$

Equation of CM : $y - 9 = 2(x - 2)$

$y = 2x + 5$

Gradient of $OC = \frac{3}{4}$

Equation of OC: $y = \frac{3}{4}x$

(1) = (2) $2x + 5 = \frac{3}{4}x$

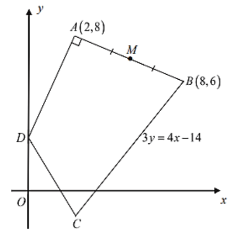
$\frac{5}{4}x = -5$

$x = -4$

$y = -3$

Coordinates of C are $(-4, -3)$

The diagram shows a quadrilateral $ABCD$. The coordinates of A and B are (2, 8) and (8, 6) respectively. M is the midpoint of AB and CM is perpendicular to AB. The equation of BC is $3y = 4x - 14$. The point D lies on the y-axis and $DAB = 90^\circ$.



(i) Find the coordinates of D.
(ii) Find the coordinates of C.
(iii) Find the area of $ABCD$.

Let D $(0, a)$

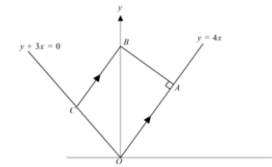
$$M_{AB} = -\frac{1}{3}$$
$$M_{AD} = 3$$
$$\frac{8-a}{2-0} = 3$$
$$a = 2$$
$$D(0,2)$$
$$y = 5x + c$$

At (5,7); $c = -8$

$$y = 3x - 8$$
$$y = \frac{4}{3}x - \frac{14}{3}$$
$$\frac{4}{3}x - \frac{14}{3} = 3x - 8$$
$$x = 2$$
$$y = -2$$
$$C(2, -2)$$

$$\frac{1}{2}\begin{vmatrix} 2 & 8 & 2 & 0 & 2 \\ 8 & 6 & -2 & 2 & 8 \end{vmatrix}$$
$$= \frac{1}{2}|(12 - 16 + 4) - (64 + 12 + 4)|$$
$$= 40 \text{units}^2$$

The diagram below shows a trapezium $OABC$, where O is the origin. The equation of AO is $y = 4x$ and the equation of OC is $y + 3x = 0$. The line through A perpendicular to OA meets y-axis at B and BC is parallel to AO. Given that the length of OA is $\sqrt{1700}$ units, find the coordinates of A, of B and of C



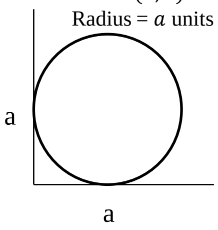
$$A(x_A, 4x_A)$$
$$\sqrt{(x_A - 0)^2 + (4x_A - 0)^2} = \sqrt{1700}$$
$$17{x_A}^2 = 1700$$
$$x_A = 10$$
$$\therefore A(10, 40)$$

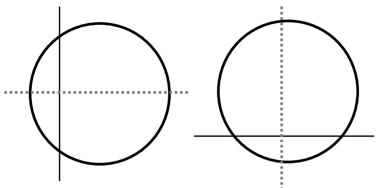
$$B(0, y_B)$$
$$\text{gradient of } AB = -\frac{1}{4}$$
$$\frac{y_B - 40}{0 - 10} = -\frac{1}{4}$$
$$y_B = 42.5$$
$$\therefore B(0, 42.5)$$

$$C(x_C, -3x_C)$$
$$\text{gradient of } CB = 4$$
$$\frac{42.5 - (-3x_C)}{0 - x_C} = 4$$
$$x_C = -\frac{85}{14}$$
$$\therefore C\left(-\frac{85}{14}, \frac{255}{14}\right)$$

$A(10{,}40)$, $B(0{,}42.5)$, $C\left(-\frac{85}{14}, \frac{255}{14}\right)$

Centre (a, a)
Radius = a units





(Vertical/Horizontal Chord)
Centre is Midpoint
of the 2 Intersections

Forming Equation

$$(x-a)^2 + (y-b)^2 = r^2$$

Centre (a, b) Radius: r

Finding Centre & Radius

$$x^2 + y^2 + 6x - 4y + 9 = 0$$

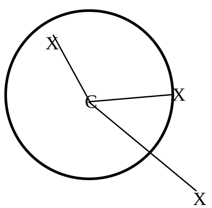
$x^2 + 6x + y^2 - 4y = -9$
$(x+3)^2 + (y-2)^2 = -9 + 3^2 + 2^2$
$\therefore (x+3)^2 + (y-z)^2 = 2^2$
Hence, centre = $(-3, 2)$, Radius = 2 units

Follow
Complete The Square Rules OR
'Shortcut Formula'

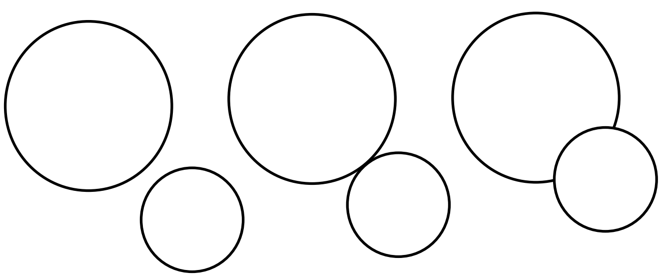
$x^2 + 6x + y^2 - 4y = -9$
$x^2 + 2gx + y^2 + 2fy + c = 0$

Centre $(-g, -f)$, Radius $\sqrt{g^2 + f^2 - c}$
Centre $(-3,2)$ Radius $\sqrt{9+4-9} = 2$

Position of Coordinates



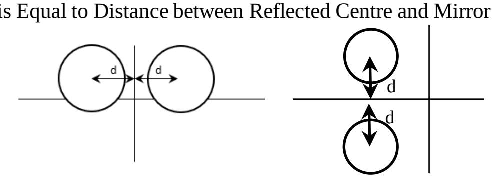
1. Find length of line (Centre to Coordinate)
2. If $l > r$, outside circle
3. If $l = r$, on circle
4. If $l < r$, inside circle



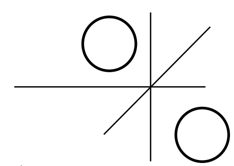
Reflection About Line

1. Always remember to sketch.
2. **Radius of the circle remains the same.**

Reflection about Vertical & Horizontal Lines
1. Remember that the Distance between Centre and Mirror is Equal to Distance between Reflected Centre and Mirror



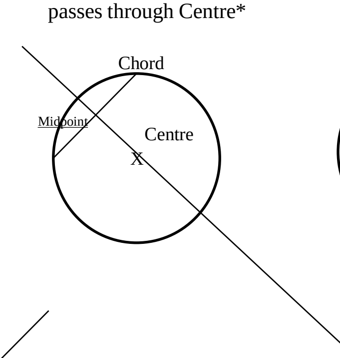
Reflection about Diagonal Lines

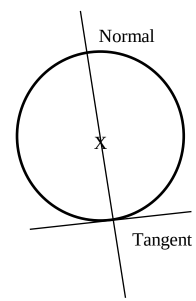


1. Find the Gradient of Diagonal Line (Mirror)
2. Find the Gradient of Perpendicular Line to the Mirror
3. Form Equation of Perpendicular Line passing through Centre
4. Simultaneous Equation between Equation of Perpendicular Line & Mirror
5. Use Midpoint Theory to find the Reflected Centre
6. Form the Reflected Circle Equation

Tangent, Chords & Perpendicular Bisector

Perpendicular Bisector of Chord passes through Centre*





Finding Equations of Circle – General and Standard Form

Write down the equation of the circle with centre $A(8,2)$ and radius $\sqrt{80}$. Find the length PQ.

Eqn. of circle: $(x-8)^2+(y-2)^2=80$
$x=0,\ 64+y^2-4y+4=80$
$y^2-4y-12=0$
$(y-6)(y+2)=0$
$y=6$ or -2
Length of PQ $=6-(-2)$
$=8$ units

A circle, centre C, has a diameter AB where A is the point $(-13,-4)$ and B is the point $(3,8)$.
(i) Find the coordinates of C and the radius of the circle.
Find the equation of the circle.

$$\text{centre}=\left(\frac{-13+3}{2},\frac{-4+8}{2}\right)$$
$$=(-5,2)$$
$$\text{radius}=\sqrt{(3+5)^2+(8-2)^2}$$
$$=10\text{ units}$$

$$(x+5)^2+(y-2)^2=100$$

A circle C_1, centre $C(3,-1)$, has a diameter AB where A is the point $(6,3)$.
Find the radius of the circle C_1 and the coordinates of B.
Find the equation of the circle C_1.

r = 5 units, $B=(0,-5)$
$r=\sqrt{(6-3)^2+(3+1)^2}=5$ units
Let $B=(p,q)$
Midpt. of $AB=C$,
$$\left(\frac{p+6}{2},\frac{q+3}{2}\right)=(3,-1)$$
$\therefore p+6=6$ & $q+3=-2$
$p=0$ $q=-5$
$B=(0,-5)$
(ii) Eqn C_1: $(x-3)^2+(y+1)^2=5^2$

The equations of the circles are

$C_1: x^2+y^2+6x-4y+9=0$,

$C_2: x^2+y^2-8y+15=0$.

(a) Find the centre and radius of the circle C_1.

(b) Find the centre and radius of the circle C_2.

C_1: $x^2+y^2+6x-4y+9=0$
$x^2+6x+y^2-4y=-9$
$(x+3)^2+(y-2)^2=-9+3^2+2^2$
$\therefore (x+3)^2+(y-z)^2=2^2$

Hence, centre = $(-3,2)$
Radius = 2 units

C_2: $x^2+y^2-8y+15=0$
$(x-0)^2+(y-4)^2=-15+4^2$
$(x-0)^2+(y-4)^2=1^2$

Hence, centre = $(0,4)$
Radius = 1 unit

The equation of a circle is $x^2+y^2-4x+2y-20=0$.

Find the coordinates of the centre, C and the radius of the circle.

$$x^2+y^2-4x+2y-20=0$$
$$(x-2)^2-4+(y+1)^2-1=20$$
$$(x-2)^2+(y+1)^2=25$$
Centre, C $=(2,-1)$
Radius = 5

Important Concepts

Concept:
1.) Equation of a circle
$$(x-h)^2+(y-k)^2=r^2$$
2.) Coordinates of the Center of the circle
$$(h,k)=\left[\frac{(x_1+x_2)}{2},\frac{(y_1+y_2)}{2}\right]$$
3.) Finding the radius of the circle
$$r=\sqrt{(x-h)^2+(y-k)^2}$$

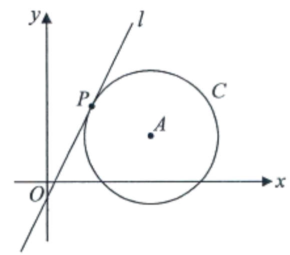
Variations of Questions to Obtain Equations of Circle

.A circle C_1 passes through the points P (1, 2) and Q (4, -1). The centre of the circle lies on the line y = $-\frac{1}{3}x + 2$.
(i) Find the equation of the perpendicular bisector of PQ.
(ii) Find the equation of circle C_1.

Midpoint PQ $= (2.5, 0.5)$
Gradient PQ $= \frac{2+1}{1-4} = -1$
Equation of perpendicular bisector of PQ:
Sub (2.5,0.5)
$0.5 = 2.5 + c$
$c = -2$
$y = x - 2$

$y = -\frac{1}{3}x + 2 \quad y = x - 2$
$x - 2 = -\frac{1}{3}x + 2$
$\frac{4}{3}x = 4$
$x = 3, y = 1$
centre (3,1)
Equation of circle : $(x-3)^2 + (y-1)^2 = r^2$
Sub P(1,2)
$(-2)^2 + (1)^2 = r^2$
Radius = $\sqrt{5}$
Equation of circle C_1: $(x-3)^2 + (y-1)^2 = 5$

The circle C has centre A with coordinates (7,3). The line l, with equation y=2x – 1, is the tangent to C at the point P.
(i) Find the equation of the line PA.
(ii) Find an equation of the circle C.



Gradient of PA $= -\frac{1}{2}$
Equation of PA
$y - 3 = -\frac{1}{2}(x - 7)$
$y = -\frac{1}{2}x + \frac{7}{2} + 3$
$y = -\frac{1}{2}x + \frac{13}{2}$

$2x - 1 = -\frac{1}{2}x + \frac{13}{2}$
$x = 3$
When $x = 3$, $y = 2(3) - 1 = 5$
Equation of circle
$(x-7)^2 + (y-3)^2 = (7-3)^2 + (3-5)^2$
$(x-7)^2 + (y-3)^2 = (4)^2 + (2)^2$
$(x-7)^2 + (y-3)^2 = 20$

A circle, centre A, passes through the points P (0, 8) and Q (8, 12). The y-axis is a tangent to the circle at P.
Find the equation of the circle.

Midpoint of PQ = $\left(\frac{0+8}{2}, \frac{8+12}{2}\right) = (4,10)$

Gradient of PQ = $\frac{12-8}{8-0} = \frac{1}{2}$
Gradient of perpendicular bisector of PQ $= -2$ Equation of perpendicular bisector of PQ is

$$y - 10 = -2(x - 4)$$
$$y = -2x + 18$$

y-coordinate of centre of circle = 8
Sub. y = 8, $8 = -2x + 18$ => x = 5
Circle of the circle, A is (5,8)

$$\text{Radius}^2 = (5-0)^2 = 25$$

Equation of the circle is $(x-5)^2 + (y-8)^2 = 25$

The positive x- and y-axes are tangents to a circle C.
The line T is tangent to C at the point (8, 1) on the circle.
Given that the centre of C lies above and to the right of (8, 1), find the equation of C.

The values of the x and y coordinates are the same.
Centre is on the line y = x,
Let centre of C be (a, a),

$$(x-a)^2 + (y-a)^2 = a^2$$

At (8,1),

$$(8-a)^2 + (1-a)^2 = a^2$$
$$64 - 16a + a^2 + 1 - 2a + a^2 = a^2$$
$$a^2 - 18a + 65 = 0$$
$$(a - 13)(a - 5) = 0$$

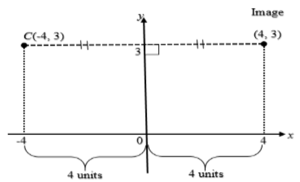
a = 13 or a = 5 (NA)
Equation of circle,
$(x-13)^2 + (y-13)^2 = 13^2$

Important Concepts

Circles – Reflections Concept (Vertical/Horizontal/Diagonal Lines)

Reflections

A circle with centre $C(-4,3)$ and radius 2 units is reflected in the y-axis.
(i) Sketch a diagram to locate the centre of the reflected circle.
(ii) Find the equation of the reflected circle.



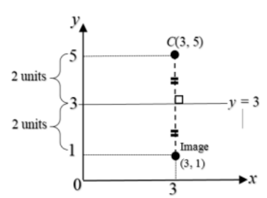
(ii) Centre of reflected circle
= (4,3)
Radius of reflected circle
= 2 units (radius remains the same)

∴ Equation of the circle is $(x-4)^2+(y-3)^2=2^2$

$$x^2-8x+16+y^2-6y+9-4=0$$
$$\therefore x^2+y^2-8x-6y+21=0$$

A circle with centre C(3,5) and radius 1 unit is reflected in the line y = 3.

(i) Sketch a diagram to locate the centre of the reflected circle.
(ii) Find the equation of the reflected circle.



(ii) Centre of reflected circle
= (3,1)
Radius of reflected circle
= 1 units (radius remains the same)

∴ Equation of the circle is $(x-3)^2+(y-1)^2=1^2$

$$x^2-6x+9+y^2-2y+1-1=0$$
$$\therefore x^2+y^2-6x-2y+9=0$$

Reflections about Line

A circle, C_1, has equation $2x^2-3x+2y^2-\frac{1}{2}(4y-3)=0$. Find the equation of another circle, C_2, which is a reflection of C_1 in the line $y-x-3=0$.

$$C_1: (x-0.75)^2+(y-0.5)^2=0.25^2$$

$y = x+3$
Gradient = 1
Perpendicular gradient = -1
Equation of the line joining the two centres:

$y-0.5=-(x-0.75)$
$y=x+1.25$ ….. (1)
$y=x+3$ ………. (2)

Sub (2) into (1),

$x+3=-x+1.25$
$x=-\frac{7}{8}$
Sub $x=-\frac{7}{8}$ into (2).
$y=2\frac{1}{8}$

Let centre of C_2 be (x, y)

$\left(-\frac{7}{8}, 2\frac{1}{8}\right)=\left(\frac{x+0.75}{2}, \frac{y+0.5}{2}\right)$
$x=-2.5$
$y=3\frac{3}{4}$

Equation of C_2:

$$(x+2.5)^2+\left(y-3\frac{3}{4}\right)^2=\frac{1}{16}$$

Important Concepts

Concept:

Under reflection, we must remember that the size of the circle (radius) is the same.

There are 2 type of reflection questions:

1) Reflection about vertical & horizontal line

Straightforward: Use midpoint theory to identify the location of the new centre.

2) Reflection about $y=mx+c$

a) Obtain the gradient of the line
b) Find the gradient of the Normal
c) Find the equation of the Normal that passes through the centre of the circle
d) Simultaneous Equation to find the
e) Midpoint theory to find the centre of the new circle

Validation

Sketch out the circle to check whether the graphs make sense

Paper 1

1. Equation of Line: $y - y_1 = m(x - x_1)$
We can replace the x and y base on the AXIS.
LINEARISING the equation.

2. Process of Linearizing Non-Linear Functions
Remember the generic formula:

$$y = mx + c$$

Gradient and y intercept MUST be a CONSTANT.

Paper 2

Are you in the Linear World or Non-Linear World (Curve)?

Remember that we sketch NEW y axis against NEW x axis.

1. Linearise Equation
2. Find New Coordinates
3. Draw your Line
4. Find Gradient & Y Intercept

Linear Law

Paper 1 – Linearisation Process

The variables x and y are such that when values of $\frac{y}{x}$ are plotted against x, a straight line passing through (1, 1.5) and (1.4, 2.5) is obtained.

(i) Find y in terms of x.

$$\text{Gradient} = \frac{2.5-1.5}{1.4-1} = \frac{5}{2}$$

$$Y = \frac{5}{2}X + c$$

Sub (1, 1.5),

$$1.5 = \frac{5}{2}(1) + c$$

$$c = -1$$

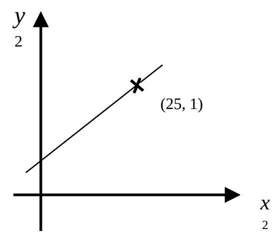
$$Y = \frac{5}{2}X + 1$$

$$\frac{y}{x} = \frac{5}{2}x - 1$$

$$y = \frac{5}{2}x^2 - x$$

Two variables x and y are related by the equation $\frac{x^2}{p^2} = 1 + \frac{3y^2}{q^2}$, where p and q are constants. When the graph of y^2 against x^2 is drawn, a straight line is obtained. Given that the line passes through the point (25, 1) and has a gradient $\frac{1}{15}$, find:

(i) The exact values of p and q



$$\frac{x^2}{p^2} = 1 + \frac{3y^2}{q^2}$$

$$\frac{3y^2}{q^2} = \frac{x^2}{p^2} - 1$$

$$y^2 = \left(\frac{q^3}{3p^2}\right)x^2 - \frac{q^2}{3}$$

Since gradient $= \frac{1}{15}$, $\frac{q^3}{3p^2} = \frac{1}{15}$

Sub $\frac{q^3}{3p^2} = \frac{1}{15}$ and (25, 1)

$$1 = \left(\frac{1}{15}\right)(25) - \frac{q^2}{3}$$

$$\frac{q^2}{3} = \frac{2}{3}$$

$$q = \pm\sqrt{2}$$

Sub $q^2 = 2$ into (1) :

$$\frac{2}{3p^2} = \frac{1}{15}$$

$$p = \pm\sqrt{10}$$

$\therefore p = \sqrt{10}, q = \sqrt{2}$,

$p = -\sqrt{10}, q = -\sqrt{2}$,

$p = \sqrt{10}, q = -\sqrt{2}$,

$p = -\sqrt{10}, q = \sqrt{2}$

Important Concepts

Concept:

1) Linearisation

Do remember that both the gradient and y intercept should not contain variables. They are constant values.

2) Finding Gradient and Y Intercept

Validation

Linear Law

Sub into Curve Equation

Variables x and y are related by the equation $y = ax^b + 3$ where a and b are

constants. When $\lg(y-3)$ is plotted against $\lg x$, a

straight line is obtained. The straight

line passes through $(-2.5, 8)$ and $(3.5, -4)$. Find

(i) the value of a and of b,

(ii) Find y when $x = 10$.

(i) $y = ax^b + 3$

$y - 3 = ax^b$

$\lg(y-3) = \lg a + b \lg x$

Gradient = $\frac{8-(-4)}{-2.5-3.5}$

$= -2$

$b = -2$

Sub $\lg x = -2.5, \lg(y-3) = 8$ and $b = -2$,

$8 = -2(-2.5) + \lg a$

$\lg a = 3$

$a = 10^3 = 1000$

(ii) $y = 1000x^{-2} + 3$

Sub $x = 10$

$y = 1000(10^{-2}) + 3 = 13$

Sub into Line Equation

The equation $y = \frac{x+c}{x+d}$, where c and d are constants, can

be represented by a straight line when $xy - x$ is plotted

against y. The line passes through the points $(0, 4)$ and

$(0.2, 0)$.

(i) Find the value of c and of d,

(ii) If $(2.5, a)$ is a point on the straight line, find the value of a.

(i) $y(x + d) = x + c$

$xy - x = -yd + c$

$\therefore c = 4$

Grad = $-\frac{4}{0.2} = -20$

$\therefore -d = -20$

$d = 20$

(ii) $\therefore xy - x = -20y + 4$

$a = -20(2.5) + 4 \quad = -46$

Important Concepts

Concept:

3) Substitution

We need to be clear which equation to use over here. Do we use the curve equation or line equation?
This depends on if they are finding the value on the Curve or on the Line.
Read the question to internalise this.

a) Sub into Curve Equation

b) Sub into Line Equation

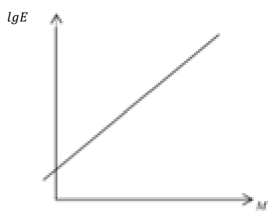
Linear Law Paper 2

The amount of energy, E erg, generated in an earthquake is given by the equation $E = 10^{a+bM}$, where a and b are constants and M is the magnitude of the earthquake. The table below shows some corresponding values of M and E.

M	1	2	3	4	5
E (erg)	2.0×10^{13}	6.3×10^{14}	2.0×10^{16}	6.3×10^{17}	2.0×10^{19}

(i) Plot $\lg E$ against M.
(ii) Using your graphs, find an estimate for the value of a and of b.
(iii) Using your answers from (ii), find the amount of energy generated, in erg, by an earthquake of magnitude 7.

M	1	2	3	4	5
ln E	13.3	14.8	16.3	17.8	19.3



$\lg E = a + bM$
$a = \text{vertical intercept} = 11.8$
$b = \text{gradient (their rise/run)}$
$= 1.5$
$\lg E = 11.8 + 1.5(7) = 22.3$
$E = 2.0 \times 10^{22}$ Erg

A cup of hot coffee was put on the table to cool at 3 pm. The temperature of the coffee, T °C, after x minutes, is given $T = 20 + ae^{-kx}$ where a and k are constants. The table shows that values of T and x taken at different timings. It is believed that an error was made in recording one of the values of T.

x	5	10	15	20
T	68.5	60.1	52.6	37.1

Using a scale of 4 cm to 5 minutes for x and 4 cm to 1 unit for $\ln(T-20)$, plot $\ln(T-20)$ against x and draw a straight-line graph.
Determine which value of T, in the table above, is the incorrect recording and use your graph to estimate its correct value.
Use your graph to estimate,
the value of a and the value of k.
the time when the temperature of the coffee is $50°C$.

Incorrect value of $T = 37.1$
$\ln(T-20) = 3.25$
Correct value: $T - 20 = 25.790$
$T = 45.79 \approx 45.8$

$T - 20 = ae^{-kx}$
$\ln(T-20) = \ln a - kx$
Gradient $= -k = \frac{4.15-3.25}{-20} = -0.045$
$k = 0.045[0.04 \le k \le 0.045]$
$\ln a = 4.15\ [4.05 \le \ln a \le 4.15]$
$a = 63.4\ [57.4 \le a \le 63.4]$

$\ln(50-20) = \ln 30 = 3.40119$
From the graph, $x = 16.25[16.25 \le x \le 16.9]$

Important Concepts

Concept:

1) Identify NEW Coordinates
2) Plot the line
3) Find the gradient and y intercept
4) Substitution
a) Line Equation vs Curve Equation
5) Missing Points
6) Drawing NEW Line

Validation

Linear Law Paper 2

Variables x and y are related by the equation $\frac{x+sy}{t} = xy$, where s and t are constants. The table below shows the measured values of x and y during an experiment

x	1.00	1.50	2.00	2.50	3.00
y	0.48	0.65	0.85	1.00	1.13

(i) On graph paper, draw a straight-line graph of $\frac{x}{y}$ against x, using a scale of *4* cm to represent *1* unit on the x-axis. The vertical $\frac{x}{y}$-axis should start at *1.5* and have a scale of *1* cm to *0.1* units.
(ii) Determine which value of y is inaccurate and estimate its correct value.
(iii) Use your graph to estimate the value of s and of t.
(iv) By adding a suitable straight line on the **same axes**, find the value of x and y which satisfy the following pair of simultaneous equations.

$$\frac{x+sy}{t} = xy$$
$$5y - 2x = 2xy$$

(i) $x + sy = xyt$
$\frac{x}{y} = tx - s$
Gradient $= t$ and $\frac{x}{y}$ − intercept $= -s$

(ii) Incorrect value of $y = 0.65$.
From graph, correct value of $\frac{x}{y} = 2.2$
Estimated correct value of $y = 0.68$.

(iii) From the graph,
$s = -1.75(-1.82 \sim -1.72)$
$t = 0.3(0.28 \sim 0.32)$

(iv) Draw the line : $\frac{x}{y} = -x + \frac{5}{2}$
From graph, $x = 0.575\ (0.55 \sim 0.60)$
and $\frac{x}{y} = 1.93\ (1.92 \sim 1.95) \Rightarrow y = 0.30$

The table below shows experimental values of two variables x and y obtained from an experiment.

x	1	2	3	4	5	6
y	5.1	17.5	37.5	60.5	98	137

It is also given that x and y are related by the equation $y = ax + bx^2$, where a and b are constants.

(i) Plot $\frac{y}{x}$ against x and draw a straight-line graph.
Use 2 cm to represent 1 unit on the horizontal axis and 4 cm to represent 10 units on the vertical axis.
(ii) Use the graph to estimate the value of a and of b.
(iii) By drawing a suitable straight line, estimate the value of x which $(b + 5)x = 38 - a$.

$a = \frac{y}{x}$ − intercept
$= 1.5$
$b =$ gradient
$= \frac{13.5}{3.8}$
$= 3.55$

$(b + 5)x = 38 - a$
$bx + 5x = 38 - a$
$bx + a = 38 - 5x$
Draw $\frac{y}{x} = 38 - 5x$, at point of intersection, $x = 4.25$

Important Concepts

Concept:

1) Identify NEW Coordinates

2) Plot the line

3) Find the gradient and y intercept

4) Substitution
a) Line Equation vs Curve Equation

5) Missing Points

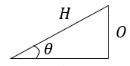
6) Drawing NEW Line

Validation

TRIGONOMETRY

Simplifying

1. Trigonometric Special Angles
2. Basic Angles
3. Trigonometric Identities
4. Addition Formula
5. Double Angle Formula
6. Half Angle Formula



Pythegoras Theorem, $H^2 = A^2 + O^2$

$tan\theta = \frac{O}{A}$ $cos\theta = \frac{A}{H}$ $sin\theta = \frac{O}{H}$

Angle Conversion
(Degree & Radian)
$180° = \pi\ rad$

Special Angle

	30	45	60
Sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
Cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
Tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Negative Angles

$\sin(-\theta) = -sin\theta$
$\cos(-\theta) = cos\theta$
$\tan(-\theta) = -tan\theta$

Trigo Functions

$sec\theta = \frac{1}{tan\theta}$
$cosec\theta = \frac{1}{sin\theta}$
$cot\theta = \frac{1}{tan\theta}$

Addition

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Double Angle

$\sin(2\theta) = 2sin\theta cos\theta$
$\cos(2\theta) = 2cos^2\theta - 1$
$= 1 - 2sin^2\theta$
$= cos^2\theta - sin^2\theta$
$\tan(2\theta) = \frac{2tan\theta}{1 - tan^2\theta}$

Half Angle

$\sin\left(\frac{\theta}{2}\right) = \pm * \sqrt{\frac{1 - cos\theta}{2}}$

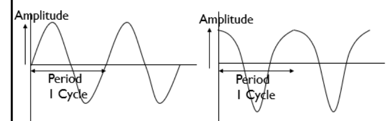
$\cos\left(\frac{\theta}{2}\right) = \pm * \sqrt{\frac{1 + cos\theta}{2}}$

The sign depends entirely on which quadrant $\frac{\theta}{2}$ lies in

Graphs

1, Basic Graph Shapes (sin, cos, tan)
2. Obtaining Amplitude, Period, Shifting
3. Application to Real World Context

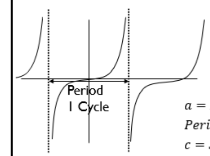
Graphs



$y = \text{asin}(b\theta) + c$ $y = \text{acos}(b\theta) + c$

$a = Amplitude\ (Height\ from\ Origin)$
$Period = \frac{360}{b}(Thinner\ or\ Fatter)$
$c = Shift\ Up\ or\ Down$

$y = atan(b\theta) + c$



$a = Amplitude(Infinity)$
$Period = \frac{180}{b}(Thinner\ or\ Fatter)$
$c = Shift\ Up\ or\ Down$

Quadrants

Step 1: Identify Quadrant
Step 2: Draw your Triangle
Step 3: Label the Sides of the Triangle
(Please be careful of the Signs)
Step 4: Find all the sides (Pythagoras)
Step 5: Solve

Solving

1. Simplifying
2. Basic Angle
   -Ensure it is Positive
   -Check Radian or Degree
3. Quadrant (ASTC)
4. Domain (Change Domain if required)
5. Solve

R Formula

1. Find Right Angle Triangle
2. Find more Theta, θ
3. Never CUT Theta, CUT 90°
4. Max/Min Value & it's θ
5. Solving

R Formula

$a \sin\theta \pm b \cos\theta = R \sin(\theta \pm \alpha)$
$a \cos\theta \pm b \sin\theta = R \cos(\theta \mp \alpha)$

$R = \sqrt{a^2 + b^2}$, $\alpha = tan^{-1}\left(\frac{b}{a}\right)$

Step 1: Prove Equation
Step 2: Apply R Formula
Step 3: Application Question
Max/Min Value, Find Value

Trigonometry – Question Type 1

Simplifying

Without using a calculator, find the **exact** value of $\tan 105°$.

$\tan 105° = \tan(60° + 45°)$

$= \frac{\tan 60° + \tan 45°}{1 - \tan 60° \tan 45°}$

$= \frac{\sqrt{3}+1}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$

$= \frac{\sqrt{3}+3+1+\sqrt{3}}{1+3} = \frac{2(\sqrt{3}+2)}{-2} = -\sqrt{3} - 2$

Without using a calculator,
show that $\sin 105° = \frac{1+\sqrt{3}}{2\sqrt{2}}$,
Hence, express $1 + \cot^2 105°$ in the form $a + b\sqrt{3}$, where a and b are integers.

$\sin 105° = \sin(45° + 60°)$
$= \sin 45° \cos 60° + \cos 45° \sin 60°$
$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)$
$= \frac{1+\sqrt{3}}{2\sqrt{2}}$

$1 + cot^2 105° = cosec^2 105°$
$= \frac{1}{sin^2 105°}$
$= \left(\frac{2\sqrt{2}}{1+\sqrt{3}}\right)^2$
$= \frac{8}{4+2\sqrt{3}}$
$= \frac{4}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$
$= 8 - 4\sqrt{3}$

It is given that $\tan(A + B) = 8$ and $\tan B = 2$. **Without using a calculator**, find the exact value of $\cot A$.

$$\tan(A+B) = \frac{\tan A + \tan B}{I - \tan A \tan B}$$
Since $\tan(A + B) = 8$ and $\tan B = 2$,
$$8 = \frac{\tan A + 2}{I - 2\tan A}$$
$$8 - 16\tan A = \tan A + 2$$
$$17\tan A = 6$$
$$\tan A = \frac{6}{17}$$
$$\cot A = \frac{17}{6}$$

Without the use of calculator, show that $\sin 75 = \frac{\sqrt{2}+\sqrt{6}}{4}$

$\sin 75^0 = \sin(30^0 + 45^0)$
$= \sin 30 \cos 45 + \cos 30 \sin 45$
$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right) + \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}}\right)$
$= \frac{1+\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{\sqrt{2}+\sqrt{6}}{4}$

Important Concepts

Concept:

There are 2 important concepts here.

Special Angles & Surds (Rationalise)

	$30° \left(\frac{\pi}{6}\right)$	$45° \left(\frac{\pi}{4}\right)$	$60° \left(\frac{\pi}{3}\right)$
sin	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Validation

Insert equation into calculator to obtain the value. Double check with your answer.

Trigonometry – Question Type

Principal Value

Without using a calculator, find, in radians, the principal value of

(i) $\tan^{-1}(\cos\pi)$,

(ii) $\cos^{-1}(\sin\frac{5\pi}{4})$,

(i) $\tan^{-1}(\cos\pi)$,

$\tan^{-1}(\cos\pi) = \tan^{-1}(-1)$

$= -\frac{\pi}{4}$

(ii) $\cos^{-1}\left(\sin\frac{5\pi}{4}\right)$

$\cos^{-1}\left(\sin\frac{5\pi}{4}\right) = \cos^{-1}\left(-\sin\frac{\pi}{4}\right)$

$= \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

$= \frac{3\pi}{4}$

State the principal value of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, giving your answer in radians in exact form.

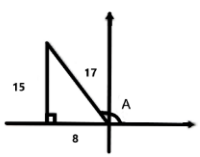
Principal value of $\cos^{-1} y$ are $0 \le \cos^{-1} y \le \pi$

principal values of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Evaluate, without using a calculator, $\tan\left[\cos^{-1}\left(-\frac{8}{17}\right)\right]$.

Let $A = \cos^{-1}\left(-\frac{8}{17}\right)$

$\cos A = -\frac{8}{17}$



$\tan A = -\frac{15}{8}$

$\therefore \tan\left[\cos^{-1}\left(-\frac{8}{17}\right)\right] = -\frac{15}{8}$

Principal Range

State the values between which each of the following must lie:

(i) the principal value of $\tan^{-1} x$,

(ii) the principal value of $\cos^{-1} x$.

(iii) the principal value of $\sin^{-1} x$,

(a)(i) $-90° < \tan^{-1} x < 90°$ or

$-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$

(ii) $0° \le \cos^{-1} \le 180°$ or

$0 \le \cos^{-1} x \le \pi$

(iii) $-90° \le \sin^{-1} x \le 90°$

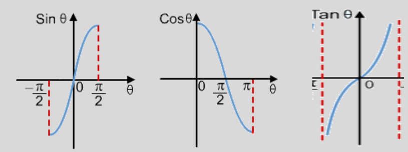
$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

Important Concepts

Concept:

Principal Value is the range of angles where it will give you all the possible solutions of a trigonometry.

Step 1: Understand range of principal values



Step 2: Special Angle Concept

If $sin\theta = \frac{1}{2}$, we know that $sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

Step 3: Double check

Ensure that it is in the principal range

Validation

Insert equation into calculator to obtain the value. Double check with your answer.

Trigonometry – Question Type 2

Quadrants

Given that $\sin A = \frac{3}{5}$ and $\tan B = \frac{12}{5}$ where A and B lie between 90° and 270°, find without using the calculator

(i) The value of $\tan(A + B)$

(ii) The value of $\sin 2A$

(iii) The value of $\cos\frac{B}{2}$.

(i) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{\left(-\frac{3}{4}\right)+\left(\frac{12}{5}\right)}{1-\left(-\frac{3}{4}\right)\left(\frac{12}{5}\right)}$

$= \frac{33}{56}$

(ii) $\sin 2A = 2\sin A\cos A$

$= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)$

$= -\frac{24}{25}$

(iii) $\cos B = 2\cos^2\frac{B}{2} - 1$

$-\frac{5}{13} = 2\cos^2\frac{B}{2} - 1$

$\cos^2\frac{B}{2} = \frac{4}{13}$

$\cos\frac{B}{2} = \pm\sqrt{\frac{4}{13}}$

Since $180° < B < 270°$, $90° < \frac{B}{2} < 135°$

$\cos\frac{B}{2} < 0$

$\therefore \cos\frac{B}{2} = -\frac{2}{\sqrt{13}}$

$= -\frac{2\sqrt{13}}{13}$

Given that $\cos x = -\frac{3}{5}$ and $\tan x < 0$, find $\sin\frac{x}{2}$.

$\cos x = 1 - 2\sin^2\frac{x}{2}$

$-\frac{3}{5} = 1 - 2\sin^2\frac{x}{2}$

$\sin^2 x = \frac{4}{5}$

$\sin x = \pm\sqrt{\frac{4}{5}}$

$\sin x = \sqrt{\frac{4}{5}}, -\sqrt{\frac{4}{5}}$ (reject)

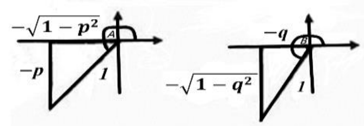
$= \frac{2\sqrt{5}}{5}$

Given that $\sin A = -p$ and $\cos B = -q$, where A and B are in the same quadrant and p and q are positive constants, find the value of

(i) $\sin(-A)$,

(ii) $\tan(45° - A)$,

(iii) $\sec(2B)$.



$\sin(-A) = -\sin A$

$= p$

$\tan(45° - A) = \frac{\tan 45° - \tan A}{1 + \tan 45° \tan A}$

$= \frac{1 - \tan A}{1 + \tan A}$ since $\tan 45° = 1$

$= \frac{1 - \frac{-p}{-\sqrt{1-p^2}}}{1 + \frac{-p}{-\sqrt{1-p^2}}}$

$= \frac{\sqrt{1-p^2} - p}{\sqrt{1-p^2} + p}$

$\sec 2B = \frac{1}{\cos 2B}$

$= \frac{1}{2\cos^2 B - 1}$

$= \frac{1}{2(-q)^2 - 1}$

$= \frac{1}{2q^2 - 1}$

Important Concepts

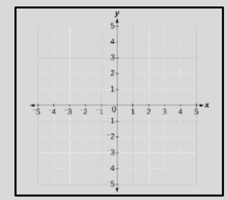
Concept:

1) Identify Quadrant

2) Draw and Label Triangle

3) Check the **SIGNS**

Signs follow the coordinate geometry concept



4) Solve

Validation

You can find the EXACT angle from the question and use that the see if it tallies with your answers!

Trigonometry – Question Type 3

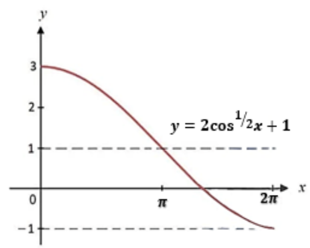
Trigonometry Graphs Sketching

It is given that $y = p\cos\frac{1}{2}x + q$, where p and q are positive integers.
State the period of y.
Given that the maximum and minimum values of y are 3 and –1 respectively, find the value of p and q.
Using the values of p and q found in part (ii), sketch the graph of y for $0 \leq x \leq 2\pi$.

(i) Period of $y = \frac{2\pi}{\frac{1}{2}} = 4\pi$ or 720°

(ii) $3 = p(1) + q$
$-1 = p(-1) + q$

$p = 2$
$q = 1$



The function f is defined by f(x) = $-3\cos(0.5x)$. Write down the period and amplitude of f(x). Sketch the graph of f(x) for $0° \leq x \leq 360°$.

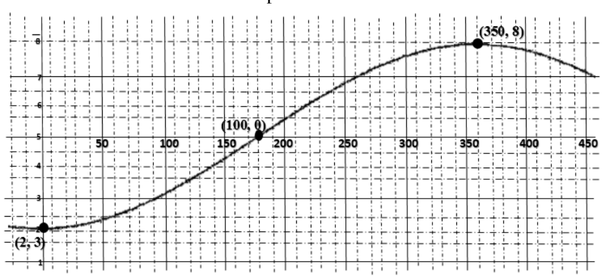
Period = 4π or 720°

Amplitude = 3

Period $= \frac{2\pi}{0.5}$

$= 4\pi$ or 720°

Amplitude $= 3$



Given that $y = a + b\cos 4x$, where a and b are integers, and x is in radians, state the period of y.
Given that the maximum and minimum values of y are 3 and -5 respectively, find the amplitude of y, the value of a and of b.
Using the values of a and b found in part **(iii)**,
sketch the graph of $y = a + b\cos 4x$ for $0 \leq x \leq \pi$.

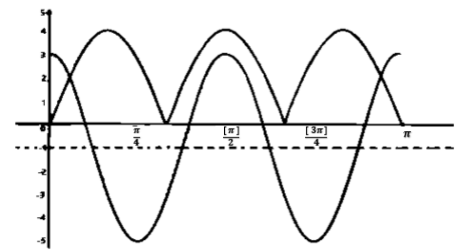
Answers:

(i) $\frac{\pi}{2}$

(ii) amplitude = 4 amplitude $= \frac{3-(-5)}{2} = 4$

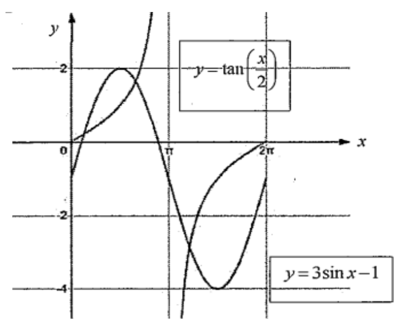
(iii) $b = 4$
$a = -1$

(iv)



Sketch $y = \tan\frac{x}{2}$ for $0 \leq x \leq 2\pi$

Period $-\frac{\pi}{\frac{1}{2}} = 2\pi$



Important Concepts

Concept:
1) The **amplitude** of a trigonometric function is half the distance from the highest point of the curve to the bottom point of the curve:

$$(Amplitude) = \frac{(Maximum) - (Minimum)}{2}$$

2) The **period** is the minimum interval it takes to capture an interval that when repeated over and over gives the complete function. The periods of the basic trigonometric functions are as follows:

Validation

Trigonometry – Question Type 3

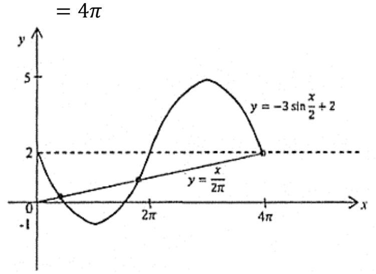
Trigonometry Graphs (Hence)

The function f is given by $f(x) = -3\sin\frac{x}{2} + 2$.

(i) State the amplitude and period of f.
(ii) Sketch the graph of $y = f(x)$ for $0 \le x \le 4\pi$.
By drawing a suitable straight line on the same axes, state the number of solutions to the equation
$4\pi - x - 6\pi \sin\frac{x}{2} = 0$ for $0 \le x \le 4\pi$.

(i) Amplitude = 3, Period = 4π
Amplitude = 3
Period = $2\pi \div \frac{1}{2}$
$= 4\pi$

(ii)



$4\pi - x - 6\pi\sin\frac{x}{2} = 0$

$\frac{4\pi - x - 6\pi\sin\frac{x}{2}}{2\pi} = \frac{0}{2\pi}$

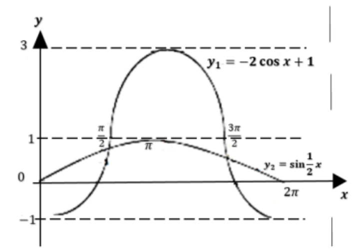
$2 - \frac{x}{2\pi} - 3\sin\frac{x}{2} = 0$

$-3\sin\frac{x}{2} + 2 = \frac{x}{2\pi}$

Since there are 3 points of intersection between the graphs $y = -3\sin\frac{x}{2} + 2$ and $y = \frac{x}{2\pi}$, there are 3 solutions.

It is given that $y_1 = -2\cos x + 1$ and $y_2 = \sin\frac{1}{2}x$.
For the interval $0 < x < 2\pi$.
(i) State the amplitude and period of y_1 and of y_2,
(ii) Sketch, on the same diagram, the graphs of y_1 and y_2,
(iii) Find the x-coordinate of the points of intersection of the two graphs drawn in (ii),
(iv) Hence, find the range of values of x for which $y_1 \le y_2$.

Amplitude of $y_1 = 2$, Period of $y_1 = 2\pi$
Amplitude of $y_2 = 1$, Period of $y_2 = 4\pi$



$-2\cos x + 1 = \sin\frac{1}{2}x$

$-2\left(1 - 2\sin^2\frac{x}{2}\right) + 1 = \sin\frac{1}{2}x$

$4\sin^2\frac{x}{2} - \sin\frac{1}{2}x - 1 = 0$

$\sin\frac{1}{2}x = 0.6403882$

$a = 0.69500$

$\frac{1}{2}x = 0.69500, \pi - 0.69500$

$= 0.695 \text{ or } 2.4466$

$x = 1.39, 4.89$

$0 < x \le 1.39$ or $4.89 \le x < 2\pi$

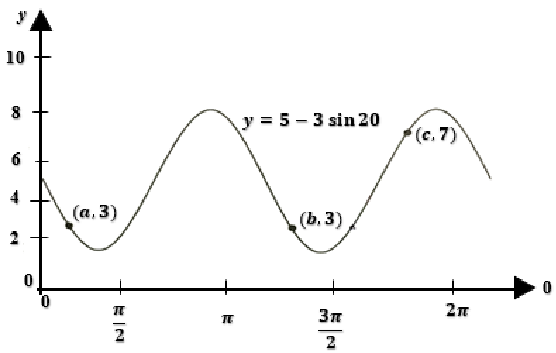
Important Concepts

Validation

Trigonometry – Question Type 3

(Symmetry Properties)

The diagram shows part of the graph of $y = 5 - 3\sin 2\theta$, passing through the points $(a, 3), (b, 3)$ and $(c, 7)$, where a, b and c are constants.

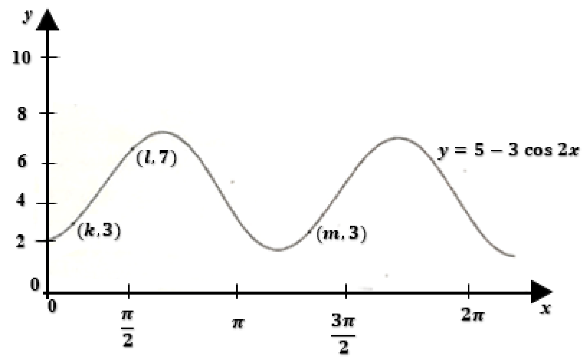


Using the symmetry of the graph, or otherwise, find an equation connecting

π, a and b [1]

π, b and c. [2]

Ans: $a + \pi = b$, $b + \frac{\pi}{2} = c$.



The diagram above shows part of a graph of $y = 5 - 3\cos 2x$, passing through the points $(k, 3), (l, 7)$ and $(m, 3)$, where k, l and m are constants. Using the symmetry of the graph, or otherwise, find an equation connecting

π, k and m, [1]

π, k and l, [1]

Ans: (i) $k + \pi = m$ (ii) $\text{k} + l = \frac{\pi}{2}$

Important Concepts

Validation

Trigonometry – Question Type 3

Trigonometry Graphs Application

A buoy floats and its height above the seabed, h m, is given by $h = a\cos bt + c$, where t is time measured in hours from 0000 hours and a, b and c are constants. The least height of the buoy above seabed is 180 metres and is recorded at 0000 hours. The greatest height of the buoy above seabed is 196 metres and is first recorded at 0600 hours.

(i) Find the values of a, b and c.

(ii) Using values found in (i), sketch the graph of $h = a\cos bt + c$ for $0 \le t \le 24$.

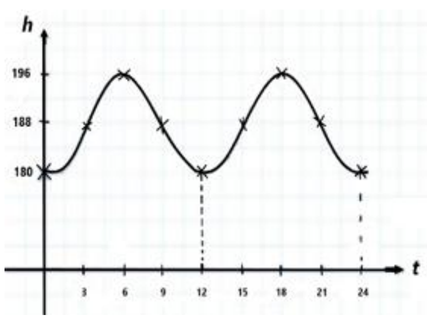
(iii) The buoy floats above the top of a huge rock first at 0500 hours. State the number of hours in each day that the buoy is above the rock.

Ans:

$$\frac{196-180}{2} = 8 \Rightarrow a = -8$$

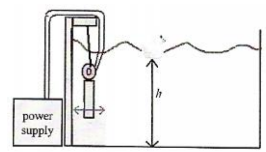
$$c = \frac{196-180}{2} = 188$$

$$b = \frac{2\pi}{12} = \frac{\pi}{6}$$



4 hours

To study the effects of water waves, a wave generator and a rubber duck were placed in a water tank as shown in the diagram above. The height, h metres, from the bottom of water tank was modelled by $h = a\sin(kt) + b$, where t is the time in hours after midnight and a, b and k are constants. The motion of the rubber duck was observed for 36 hours. The minimum height of 1.5 m from bottom of water tank was first recorded at 06 00. The maximum height of 2.5 m was first recorded at 18 00.



(i) Find the values of a, b and k.

(ii) Using the values found in (i), sketch the graph of $h = a\sin(kt) + b$ for $0 \le t \le 36$.

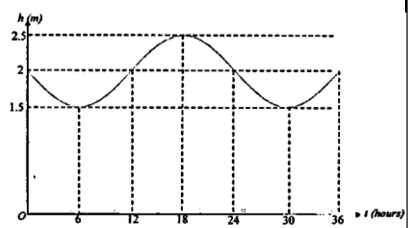
$$a = \frac{1.5-2.5}{2} = -0.5$$

$$-0.5 + b = 1.5$$

$b = 2$

Period $= \frac{2\pi}{k} = 12 \times 2$

$k = \frac{2\pi}{24} = \frac{\pi}{12}$



$y = 2.1$

$$-0.5\sin\left(\frac{\pi}{12}t\right) + 2 = 21$$

$$\sin\left(\frac{\pi}{12}t\right) = \frac{21-2}{-0.5}$$

$= -0.2$

Basic $\angle = \sin^{-1}(0.2)$

$= 0.201358$ (6 s.f.)

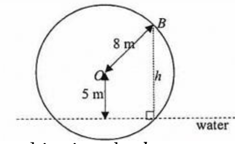
$\frac{\pi}{12}t = \pi + 0.201358, 2\pi - 0.201358$

$t = 12.769$ or 23.2309

$= 12.8$ or 23.2 (3s.f.)

$\therefore$ range of t is $12.8 < t < 23.2$

A waterwheel rotates 5 revolutions anticlockwise in 1 minute. Tom starts a stopwatch when the bucket B is at its highest height above water level. The radius of the waterwheel is 8 m and its centre is 5 m above the water level.



The height of bucket B above water level is given by $h = a\cos bt + c$, where t is the time, in seconds, since Tom started the stopwatch.

Determine the value of each of the constant a, b and c.

For how long is $h < 0$?

(i) $a = 8, b = \frac{\pi}{6}, c = 5$

Given: $h = a\cos bt + c$

Starting point is when B is at its highest point, i.e., when $t = 0, h = 13$

$\therefore 13 = a(1) + c \Rightarrow a + c = 13$ … (1)

and lowest point is when B is 3 m below water level.

$\therefore -3 = a(-1) + c \Rightarrow -a + c = -3$ … (2)

Solving (1) and (2),

$a = 8, c = 5$

Given: 5 revolutions take 1 minute

$\therefore$ 1 revolution take $\frac{1}{5}$ minute (=12 seconds)

So, period: $\frac{2\pi}{b} = 12 \Rightarrow b = \frac{\pi}{6}$

$\therefore h = 8\cos\left(\frac{\pi}{6}t\right) + 5$

(ii) Duration = 3.42 seconds

$h < 0 \Rightarrow 8\cos\left(\frac{\pi}{6}t\right) + 5 < 0$

When $h = 0$, $8\cos\left(\frac{\pi}{6}t\right) + 5 = 0$

$\cos\left(\frac{\pi}{6}t\right) = -\frac{5}{8}$

Basic angle $= \cos^{-1}\left(\frac{5}{8}\right) = 0.895\ 66$

The variable angle $\frac{\pi}{6}t$ lies in the 2nd and 3rd quadrants,

$\therefore \frac{\pi}{6}t = \pi - 0.895\ 66$ or $\pi + 0.895\ 66$ in the 1st revolution

$t = 4.289$ or 7.710

Duration = 7.710 – 4.289

= 3.42 seconds

Trigonometry – Question Type 3

Trigonometry Graphs Application

(a) Find, in radians, the two principal values of y for which $2\tan^2 y + \tan y - 6 = 0$.

(b) The height, h m, above the ground of a carriage on a carnival ferris wheel is modelled by the equation $h = 7 - 5\cos(8t)$, where t in the time in minutes after the wheel starts moving.
State the initial height of the carriage above ground.
Find the greatest height reached by the carriage.
Calculate the duration of time when the carriage is 9 m above the ground.

(a) $2\tan^2 y + \tan y - 6 = 0$

$(2\tan y - 3)(2\tan y + 2) = 0$

$\tan y = \frac{3}{2}$ or $\tan y = -2$

$y = \tan^{-1}\left(\frac{3}{2}\right)$ $\quad$ $y = \tan^{-1}(-2)$

$= 0.9827$ $\quad$ $= 1.1071$

≈ 0.983 (3s.f.) $\quad$ ≈ 1.11 (3s.f.)

(bi) Initial height = 2 m

(ii) Greatest height = $7 - 5(-1)$

=12 m

(iii) $7 - 5\cos 8t = 9$

$\cos 8t = -\frac{2}{5}$

$\alpha = 1.1592$

$8t = 1.9823, 4.300$

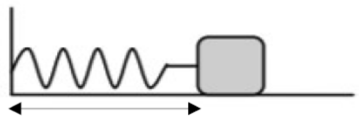
$t = 0.2477, 0.5375$

Duration = $0.5375 - 0.2477$

= 0.2898

$\approx$ 0.290 minutes (3s.f.)

An object is connected to the wall with a spring that has a original horizontal length of 20 cm. The object is pulled back 8 cm past the original length and released. The object completed 4 cycles per second.



(i) Given that the function $x = 8\cos(a\pi t) + b$, where x is the horizontal distance, in centimetres, of the object from the wall and t is the time in seconds after releasing the object, find the values of a and b.

(ii) Find the duration of time for each cycle such that the object is more than 27 cm from the wall.

(i) $b = 20$

Period = $\frac{2\pi}{a\pi}$

$\frac{1}{4} = \frac{2\pi}{a\pi} \Rightarrow a = 8$

(ii) $27 = 8\cos(8\pi t) + 20$

$\cos(8\pi t) = \frac{7}{8}$

$\alpha = 0.50536$

$8\pi t = 0.50536$

$t = 0.020107$

Duration of time = 0.020107×2

= 0.0402 s

The Ultraviolet Index describes the level of solar radiation on the earth's surface. The Ultraviolet Index, U, measured from the top of a building is given by $U = 6 - 5\cos qt$, where t is the time in hours, $0 \le t \le 20$, from the lowest value of Ultraviolet Index and q is a constant. It takes 10 hours for the Ultraviolet Index to reach its lowest value again.

(i) Explain why it is impossible to measure an Ultraviolet Index of 12.

(ii) Show that $q = \frac{\pi}{5}$.

(iii) The top of the building is equipped with solar panels that supply power to the building when the Ultraviolet Index is at least 3.5. Find the duration, in hours and minutes, that the building is supplied with power by the solar panels.

(i) Since max value of $U = 11$, we cannot measure a Ultraviolet Index of 12.
Max $U = 6 + 5 = 11$
Since max value of $U = 11$, we cannot measure a Ultraviolet Index of 12.

(ii) $10 = \frac{2\pi}{q}$ $q = \frac{2\pi}{10} = \frac{\pi}{5}$

(iii) Duration = 13 hours 20 mins

$6 - 5\cos\frac{\pi}{5}t = 3.5$

$\cos\frac{\pi}{5}t = \frac{1}{2}$

$\alpha = \frac{\pi}{3}$

$\frac{\pi}{5}t = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, \frac{\pi}{3} + 2\pi, 2\pi - \frac{\pi}{3} + 2\pi$

$t = 1.6666, 8.3333, 11.66, 18.33$

Duration = $(8.3333 - 1.6666) + (18.33 - 11.66)$

$= 13.3367$

$=$ 13 hours 20 mins

Trigonometry – Question Type 4

Proving (Simple Trigo)

Prove that $\frac{\tan A-\cot A}{\tan A+\cot A} = 2\sin^2 A$
Prove that $\frac{\tan^2 x-1}{\tan^2 x+1} = 1-2\cos^2 x$
Prove that $\frac{1+\tan^2 x}{1-\tan^2 x} = \sec 2x$
Prove that $\frac{\sin x}{\sec x+1} + \frac{\sin x}{\sec x-1} = 2\cot x$
Prove that $(\sec x - \tan x)(\operatorname{cosec} x + 1) = \cot x$
Prove that $(\tan x + \sec x)^2 = \frac{1+\sin x}{1-\sin x}$
Prove that $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = 2\tan x\sec x$.
Prove that $(\sec x - \tan x)(\operatorname{cosec} x + 1) = \cot x$.

Proving (Further Trigo)

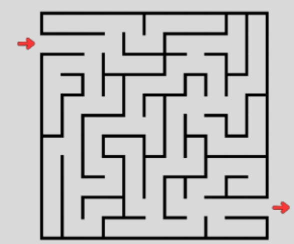
Prove $\sec 3x(\sin 3x - 2\sin^3 3x) = \tan 3x\cos 6x$.
Prove that $\operatorname{cosec} 2x + \cot 2x = \cot x$.
Prove that $\sin 3x = 3\sin x - 4\sin^3 x$
Prove that $\sin 3x = 3\sin x - 4\sin^3 x$
Prove that $\frac{1-\cos 2x+\sin x}{\sin 2x+\cos x} = \tan x$
Prove that $\frac{1-\cos 2x+\sin x}{\sin 2x+\cos x} = \tan x$
Prove $\tan(45° + A) + \tan(45° - A) = \frac{2}{\cos 2A}$
Prove $\frac{1+\sin 2A-\cos 2A}{1+\sin 2A+\cos 2A} = \tan A$

Important Concepts

Concept:

Like walking through a maze, you want to move forward from origin, and trace backward from destination until you MEET.

Start from the more complex side and simplify the expression.



Simple

$$\sec\theta = \frac{1}{\cos\theta}$$
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$
$$\cot\theta = \frac{1}{\tan\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

Addition

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

Double Angle

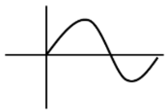
$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

Trigonometry – Question Type 5

Solving

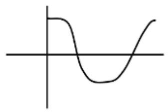
Solve $(sin\theta - 1)(cos\theta) = 0$ for $0 \le \theta \le 360$

$\sin\theta = 1$



$\theta = 90°$

$\cos\theta = 0$



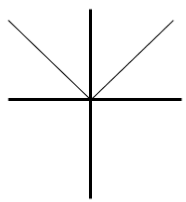
$\theta = 90°; 270°$

$\theta = 90° \quad or \quad 270°$

Solve $2sin\theta - \sqrt{3} = 0$ for $0 < \theta < 2\pi$.

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{2}, \quad \frac{2\pi}{3}$$



Solve $tan\theta = \tan(2)$ for $0 < \theta < 2\pi$.

(iii) $\tan\theta = \tan 2$

$\tan\theta = -2.185039863$ (Basic Form)

B.A = $tan^{-1}(2.185039863) = 1.1415$ rad.

$\theta = \pi - \boldsymbol{B.A.}, \quad 2\pi - \boldsymbol{B.A.}$

$\therefore \theta = 2.00$ rad., 5.14. rad. (Round off to 3sf)

Important Concepts

Left Column are questions

Validation

Substitute your answer back into the equation to make sure it tallies.

Trigonometry – Question Type 5

Solving

Solve the equation
$2\sin^2 A = 5\sin A\cos A$ for $0° < A < 360°$

$$2\sin^2 A = 5\sin A\cos A$$
$$\sin A(2\sin A - 5\cos A = 0)$$
$$\sin A = 0 \text{ or } 2\sin A = 5\cos A$$
$$A = 180° \text{ or } \tan A = \frac{5}{2}$$
$$\textit{basic angle}, \alpha = \tan^{-1}\frac{5}{2} = 68.2°$$
$$A = 68.2°, 180°, 248.2°$$

Find, for $0 \le x \le 4$, the exact solutions of the equation $2\cot x = \frac{2\tan x}{3}$

$$2\cot x = \frac{2\tan x}{3}$$
$$\tan^2 x = 3$$
$$\tan x = \pm\sqrt{3}$$
$$\text{Basic angle} = \frac{\pi}{3}$$
For $0 \le x \le 4$,
$$x = \frac{\pi}{3}, \frac{2\pi}{3}$$

Solve the equation $2\tan^2 y + 5\sec y - 1 = 0$ for $0 \le y \le 2\pi$.

$$2\tan^2 y + 5\sec y - 1 = 0$$
$$2(\sec^2 y - 1) + 5\sec y - 1 = 0$$
$$2\sec^2 y + 5\sec y - 3 = 0$$
$$(\sec y + 3)(2\sec y - 1) = 0$$
$$\sec y = -3 \quad \text{or} \quad \frac{1}{2}$$
$$\cos y = -\frac{1}{3} \quad \text{or} \quad 2 \text{ (rejected)}$$
$$\text{Basic angle} = \cos^{-1}\frac{1}{3}$$
$$= 1.23096$$
$$y = \pi - 1.23096, \pi + 1.23096$$
$$y = 1.91, 4.37$$

Solve the equation
$\sin 4x + 3\sin 2x = 0$ for $-180° \le x \le 180°$.

$$2\sin 2x\cos 2x + 3\sin 2x = 0$$
$$\sin 2x\,(2\cos 2x + 3) = 0$$
$$\therefore \sin 2x = 0 \text{ or } \cos 2x = -\frac{3}{2} \text{ (no solution)}$$
$$\text{basic angle} = 0°$$
$$\therefore 2x = -360°, 180°, 0°, 180° \text{ or } 360°$$
$$x = -180°, -90°, 0°, 90° \text{ or } 180°$$

Important Concepts

Concept:

Simplify

1) Basic Angle
-Must be Positive
-Check Radian or Degree Mode
-5SFs or 3DPs for intermediary Step

2) Quadrants (ASTC)

3) Domain (Remember to change Domain)

4) Brackets (Solve bracket)

Validation

Substitute your answer back into the equation to make sure it tallies.

Trigonometry – Question Type 5

Solving

Find all angles between 0° and 360° inclusively, for which $3\tan\theta = \frac{2}{cos^2\theta} - 4$.

$$3\tan\theta = \frac{2}{cos^2\theta} - 4 \;(\textbf{i.e.}\; 3\tan\theta = 2\times\frac{1}{cos^2\theta} - 4)$$

$$3\tan\theta = 2sec^2\theta - 4 \left(\frac{1}{cos^2\theta} = sec^2\theta\right)$$

$$3\tan\theta = 2(tan^2\theta + 1) - 4$$

$$3\tan\theta = 2(tan^2\theta + 2) - 4$$

$$0 = 2tan^2\theta - 3\tan\theta - 2$$

$(\tan\theta - 2)(2\tan\theta + 1) = 0$ **(Cross-Factorise)**

$2\tan\theta$	1	$\tan\theta$
$\tan\theta$	-2	$-4\tan\theta$
$2\tan^2\theta$	-2	$-3\tan\theta$

$$\tan\theta - 2 = 0$$

$\tan\theta = 2$ (Basic Form)

B.A.$= \tan^{-1}(2) = 63.434°$
(Set Calc. to **Deg.** Mode; Truncate to 3dp)

$\theta = 63.4°,\; 243.4°$ (Round off to 1dp)

OR

$$2\tan\theta + 1 = 0$$

$\tan\theta = -\frac{1}{2}$ (Basic Form)

B.A.$= tan^{-1}\left(\frac{1}{2}\right) = 26.565°$
(Omit "-" Sign; Truncate to 3dp)

$\theta = 153.4°,\; 333.4°$ (Round off to 1dp)

$\therefore \theta = 63.4°,\; 153.4°,\; 243.4°,\; 333.4°$ (Leave final Ans. In Ascending Order)

Find all angles between 0° and 360° inclusively, for which $\sec\theta(\tan\theta - 2) = -\cos ec\theta$.

$$\sec\theta(\tan\theta - 2) = -\cos ec\theta \left(\sec\theta = \frac{1}{\cos\theta}; \tan\theta = \frac{\sin\theta}{\cos\theta}; \cos ec\theta = \frac{1}{\sin\theta}\right)$$

$$\frac{1}{\cos\theta}\left(\frac{\sin\theta}{\cos\theta} - 2\right) = -\frac{1}{\sin\theta}$$

$$\frac{1}{\cos\theta}\left(\frac{\sin\theta - 2\cos\theta}{\cos\theta}\right) = -\frac{1}{\sin\theta}$$

$$\frac{\sin\theta - 2\cos\theta}{\cos^2\theta} = \frac{-1}{\sin\theta}$$

$\sin^2\theta$-2$\sin\theta\cos\theta = -\cos^2\theta$ (Cross multiply)

$\sin^2\theta - 2\sin\theta\cos\theta = -\cos^2\theta$ (Make RHS 0)

$(\sin\theta - \cos\theta)^2 = 0$ [Factorise using $a^2 - 2ab^2 = (a-b)^2$]

$$\sin\theta - \cos\theta = 0$$

$$\sin\theta = \cos\theta$$

$$\frac{\sin\theta}{\cos\theta} = 1$$

$\tan\theta = 1$ (Basic Form)

B.A.$= tan^{-1}(1) = 45°$ (Set Calc. to Deg. Mode)

$\therefore \theta = 45°,\quad 225°$

Important Concepts

Concept:

Simplify

1) Basic Angle
-Must be Positive
-Check Radian or Degree Mode
-5SFs or 3DPs for intermediary Step

2) Quadrants (ASTC)

3) Domain (Remember to change Domain)

4) Brackets (Solve bracket)

Validation

Substitute your answer back into the equation to make sure it tallies.

Trigonometry – Question Type 5

Solving

Find all the angles between 0° and 360° inclusive which satisfy the equation

$$\tan(2x + 60°) = 1.2,$$

$$\tan(2x + 60°) = 1.2$$

$$basic\ angle, \alpha\ = 50.19443$$

$$0° \leq x \leq 360°$$
$$0° \leq 2x \leq 720°$$
$$0° \leq 2x + 60 \leq 780°$$

$$2x + 60° = 230.194,\ \ 410.194,\ \ 590.194,\ 770.19443$$

$$x = 85.1°, 175.1°, 265.1°, 355.1°$$

Find all the values of x between 0 and 5 for which $\sin(2x - 1) = -0.75$.

$\sin(2x - 1) = -0.75$ (Basic Form)
B.A.$= sin^{-1}(0.75) = 0.84806\, rad.$

$$2x - 1 = 3.9896,\quad 5.4351,\quad -0.84806$$

$$2x = 4.9896,\quad 6.4351,\quad 0.15194$$
$$x = 2.49,\quad 3.22,\quad 0.0760$$ (Round off to 3sf)

$\therefore x = 0.0760$ rad., 2.49 rad., 3.22 rad.

Important Concepts

Concept:

Simplify

1) Basic Angle
-Must be Positive
-Check Radian or Degree Mode
-5SFs or 3DPs for intermediary Step

2) Quadrants (ASTC)

3) Domain (Remember to change Domain)

4) Brackets (Solve bracket)

Validation

Substitute your answer back into the equation to make sure it tallies.

Trigonometry – R Formula

R Formula

Express $12\sin\theta\cos\theta - 8\cos^2\theta + 7$ in the form $A\sin 2\theta + B\cos 2\theta + C$, where A, B and C ate constants.

Solve $12\sin\theta\cos\theta - 8\cos^2\theta + 7 = 0$ for

$0° < \theta < 180°$

$$12\sin\theta\cos\theta - 8\cos^2\theta + 7$$
$$= 6(2\sin\theta\cos\theta) - 8\cos^2\theta + 7$$
$$= 6\sin 2\theta - 8\left(\frac{1+\cos 2\theta}{2}\right) + 7$$
$$= 6\sin 2\theta - 4\cos 2\theta + 3$$

$$6\sin 2\theta - 8\cos^2\theta + 7 = 0$$
$$6\sin 2\theta - 4\cos 2\theta + 3 = 0$$

Let $6\sin 2\theta - 4\cos 2\theta = R\sin(2\theta - a)$

$$R = \sqrt{6^2 + 4^2} = \sqrt{52}$$
$$\tan a = \frac{4}{6}$$
$$a = 33.690°$$
$$\sqrt{52}\sin(2\theta - 33.690°) + 3 = 0$$
$$\sin(2\theta - 33.690°) = -\frac{3}{\sqrt{52}}$$

basic angle = 24.583°

$2\theta - 33.690° = -24.583°$ or $2\theta - 33.690° = 180° + 24.583°$

$\theta = 4.553°$ $\qquad$ $\theta = 119.137°$

$\theta = 4.6°$ $\qquad$ $\theta = 119.1°$

The expression $6\sin\theta - 7\cos\theta$ is defined for $0 \le \theta \le \pi$ radians.

(i) Using $R\sin(\theta - a)$, where $R > 0$ and $0 < a < \frac{\pi}{2}$,

solve the equation $6\sin\theta - 7\cos\theta = 8$.

(ii) Find the minimum value of $90 - (6\sin\theta - 7\cos\theta)^2$ and the corresponding value of θ

$$6\sin\theta - 7\cos\theta = 8.$$
$$R = \sqrt{6^2 + 7^2} = \sqrt{85}$$
$$a = \tan^{-1}\left(\frac{7}{6}\right)$$
$$\sqrt{85}\sin(\theta - 0.862170) = 8$$
$$\sin(\theta - 0.862170) = \frac{8}{\sqrt{85}}$$
$$\text{Basic angle} = \sin^{-1}\left(\frac{8}{\sqrt{85}}\right) = 1.050600$$
$$\theta - 0.862170 = 1.05060, 2.09099$$
$$\theta = 1.91, 2.95\ (3sf)$$

Minimum value $= 90 - \left(\sqrt{85}\right)^2 = 5$

Corresponding values of θ :

$$\sin(\theta - 0.862170) = 1$$
$$\theta = \frac{\pi}{2} + 0.862170 = 2.43 \text{ rad (3s.f.)}$$

Important Concepts

Concept:

1) Identify Triangles

2) Never Cut Angles

3) Use TOA CAH SOH to Prove

For Solving,

Do take note of BQDB (Concept)

Do take note that 95% of the time, your solution is between 0 to 90 because of the context of the question. You have to reject solutions.

For Maximum/Minimum & Corresponding Angles,

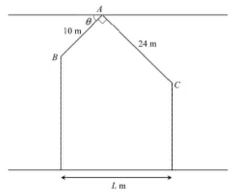
1) Apply Concept from Trigonometry Graphs

Validation

Trigonometry – R Formula

R Formula

The diagram shows the cross-section of a house with a rooftop BAC. The length of AB and AC are 10 m and 24 m respectively. The angle between AB and the horizontal through A is θ degrees and $\angle BAC = 90°$.



The base of the house is of length L m.

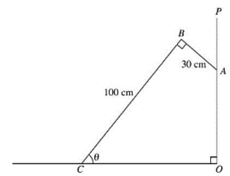
(i) Show that $L = 10\cos\theta + 24\sin\theta$.
(ii) Express L in the form $R\sin(\theta+\alpha)$, where $R > 0$ and α is an acute angle.
(iii) Find the longest possible base of the house and the corresponding value of θ.

(i) Let the point vertically above B and C be M and N respectively.
$\angle ACN = 90°$
$AM = 10\cos\theta$ and $AN = 24\sin\theta$
$L = MN = 10\cos\theta + 24\sin\theta$

(ii) $R = \sqrt{10^2 + 24^2}$
$= 26$
$\alpha = \tan^{-1}\left(\frac{10}{24}\right)$
$= 22.620°$ (3d.p.)
$L = 26\sin(\theta + 22.6°)$

(iii) Longest possible base is 26 m.
$\theta + 22.620° = 90°$
$\theta = 67.4°$ (1 d.p.)

The figure shows a stage prop ABC used by a member of the theatre, leaning against a vertical wall OP. It is given that $AB = 30$ cm, $BC = 100$ cm, $\angle ABC = \angle AOC = 90°$ and $\angle BCO = \theta$.



Show that $OC = (100\cos\theta + 30\sin\theta)$ cm.
Let D be foot of B on OC, let E be foot of A on BD.
Express OC in terms of $R\cos(\theta - \alpha)$, where R is a positive constant and α is an acute angle.
State the maximum value of OC and the corresponding value of θ. Find the value of θ for which $OC = 80$ cm.

(i) shown
$\cos\theta = \frac{CD}{100} \Longrightarrow CD = 100\cos\theta$
$\sin\theta = \frac{AE}{30} \Rightarrow AE = 30\sin\theta$
$OC = CD + AE = 100\cos\theta + 30\sin\theta$
(ii) $R = \sqrt{100^2 + 30^2} = 100\sqrt{109}$
$\alpha = \tan^{-1}\left(\frac{30}{100}\right)$
$= 16.7°(1dp)$
$\therefore OC = 10\sqrt{109}\cos(\theta - 16.7°)$
(iii) $OC_{max} = 10\sqrt{109}$
$\theta = 16.7°$
(iv) $80 = 10\sqrt{109}\cos(\theta - 16.7°)$
$\cos(\theta - 16.7°) = \frac{8}{\sqrt{109}}$
$\theta - 16.7° = 39.98°$ (θ is acute)
$\theta = 56.7°$

Important Concepts

Concept:

1) Identify Triangles

2) Never Cut Angles

3) Use TOA CAH SOH to Prove

For Solving,

Do take note of BQDB (Concept)

Do take note that 95% of the time, your solution is between 0 to 90 because of the context of the question. You have to reject solutions.

For Maximum/Minimum & Corresponding Angles,

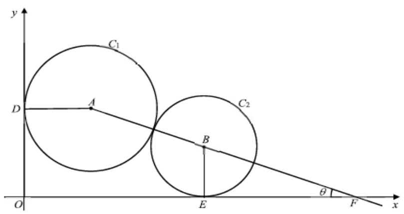
1) Apply Concept from Trigonometry Graphs

Validation

Trigonometry – R Formula

R Formula

The figure below shows two circles, C_1 and C_2, touching each other in the first quadrant of the Cartesian plane. C_1 has radius 5 and touches the y-axis at D. C_2 has radius 4 and touches the x-axis at E. The line AB joining the centre of C_1 and C_2, meets the x-axis at F. Angle BFO is θ.



(i) Find expressions for OD and OE in terms of θ and show that $DE^2 = 122 + 90\cos\theta + 72\sin\theta$.

(ii) Hence express DE^2 in the form $122 + R\cos(\theta - \alpha)$, where $R > 0$ and α is acute.

(iii) Calculate the greatest possible length of DE and state the corresponding value of θ.

(i) $OE = 5 + 9\cos\theta$

$OD = 4 + 9\sin\theta$

$DE^2 = OE^2 + OD^2$

$= (5 + 9\cos\theta)^2 + (4 + 9\sin\theta)^2$

$= 25 + 90\cos\theta + 81\cos^2\theta + 16 + 72\sin\theta + 81\sin^2\theta$

$= 41 + 81 + 90\cos\theta + 72\sin\theta$

$= 122 + 90\cos\theta + 72\sin\theta$

(ii) Let $90\cos\theta + 72\sin\theta = R\cos(\theta - \alpha)$

$R = \sqrt{90^2 + 72^2}$

$= \sqrt{13284}$

$= 115$ (3s.f.)

$\theta = \tan^{-1}\frac{72}{90}$

$= 38.65°$

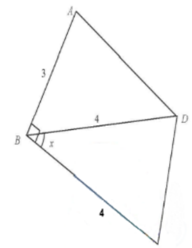
$DE^2 = 122 + 115\cos(\theta - 38.7°)$

(iii) DE is greatest when $\cos(\theta - 38.7°) = 1$

$DE = \sqrt{122 + 115} = 15.4$ units (3s.f.)

Corresponding θ is 38.7°.

The diagram below shows a quadrilateral $ABCD$ with AB = 3cm, $BC = BD$ = 4cm and $\angle ABC = 90°$. The acute angle DBC is x.



(i) Show that the area, A cm², of the quadrilateral is given by $A = 6\cos x + 8\sin x$.

(ii) Express A in the form $R\cos(x - a)$, where $R > 0$ and a is acute.

(iii) Hence state the maximum area of the quadrilateral.

(iv) Find x for which the area of $ABCD$ is 7cm²

(i) Area $= \frac{1}{2}(3)(4)\sin(90° - x) + \frac{1}{2}(4)(4)\sin x$

$= 6\cos x + 8\sin x$

(ii) Let $6\cos x + 8\sin x = R\cos(x - a)$

$= R\cos x\cos a + R\sin x\sin a$

Hence $R\cos a = 6$ (1)

$R\sin a = 8$ (2)

$\frac{(2)}{(1)}$: $\tan a = \frac{4}{3}$

$a = \tan^{-1}\left(\frac{4}{3}\right) = 53.13°$

$(1)^2 + (2)^2$: $R = \sqrt{8^2 + 6^2} = 10$

Therefore, $A = 10\cos(x - 53.1°)$

(iii) Max Area = 10 cm²

(iv) $10\cos(x - 53.13°) = 7$

$\cos(x - 53.13°) = 0.7$

$x - 53.13° = -45.57°, 45.57°$

$x = 7.6°$ or $x = 98.7°$ (rejected since x is acute)

Thus $x = 7.6°$

Important Concepts

Concept:

For more advance questions, you may need to apply

1) Area of Triangle

2) Pythagoras Theorem

3) Sine or Cosine Rule

Validation

Chain Rule

$$\frac{d}{dx}(ax+b)^n = (n)(ax+b)^{n-1}(a)$$

Product Rule

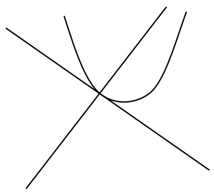
$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + g'(x)f(x)$$

Quotient Rule

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

Equation of Tangent & Normal

1. Gradient of Tangent
2. Gradient of Normal
3. Forming Equations



Increasing & Decreasing Functions

1. Finding Range
-Quadratic Inequalities
-Reverse Quadratic Inequalities
-Explanation

2. Proving Questions
-Prove by Deduction
-Prove by Completing The Square

(Connected) Rate of Change

1. Basic Questions

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

Decreasing Rate
*Put Negative

2. Advance Questions

$$\frac{dx}{dt} = k \times \frac{dy}{dt}$$

"Double Split"

Mensuration

*Similar Triangles
*Pythagoras Theorem
*TOA CAH SOH

Maxima & Minima

1. First Derivative Test (Box)
2. Second Derivative Test

Coordinate Geometry

Mensuration

*Similar Triangles
*Pythagoras Theorem
*TOA CAH SOH

Trigonometry

Differentiate $sinx, cosx, tanx$ only
Use Trigo Identities for the rest
Process: Power Trigo Bracket

Exponential

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

Recall Law of Indices

Logarithm

$$\frac{d}{dx}\ln\big(f(x)\big) = \frac{f'(x)}{f(x)}$$

Recall Law of Logarithm

Differentiation Techniques

Chain Rule

$$\frac{d}{dx}\left(\frac{2}{3\sqrt{6x+5}}\right) = \frac{2}{3}(6x+5)^{-\frac{1}{2}}$$
$$= \frac{2}{3}\left(-\frac{1}{2}\right)(6x+5)^{-\frac{3}{2}}$$
$$= -\frac{2}{\sqrt{(6x+5)^3}}$$

$$\frac{d}{dx}\left(\frac{(5x-3)^6}{8}\right) = \frac{1}{8}(6)(5x-3)^5(5)$$
$$= \frac{30}{8}(5x-3)^5$$
$$= \frac{15}{4}(5x-3)^5$$

$$\frac{d}{dx}\left[3\left(\frac{x}{6}-1\right)^4\right] = 3(4)\left(\frac{x}{6}-1\right)^3\left(\frac{1}{6}\right)$$
$$= 2\left(\frac{x}{6}-1\right)^3$$

Product Rule

$$\frac{d}{dx}\left(x\sqrt{3-x^2}\right) = x\left(\frac{1}{2}\right)(3-x^2)^{-\frac{1}{2}}(-2x) + (3-x^2)^{\frac{1}{2}}$$
$$= -x^2(3-x^2)^{-\frac{1}{2}} + (3-x^2)^{\frac{1}{2}}$$
$$=(3-x^2)^{-\frac{1}{2}}[-x^2+3-x^2]$$
$$= \frac{3-2x^2}{\sqrt{3-x^2}}$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{2x^2+x+1}{1+2x}\right) = \frac{(1+2x)(4x+1)-(2x^2+x+1)(2)}{(1+2x)^2}$$
$$= \frac{4x+1+8x^2+2x-4x^2-2x-2}{(1+2x)^2}$$
$$= \frac{4x^2+4x-1}{(1+2x)^2}$$

$$\frac{d}{dx}\left[\frac{2x-7}{\sqrt{x+1}}\right] = \frac{(2)\sqrt{x+1}-\left(\frac{1}{2}\right)(x+1)^{-0.5}(2x-7)}{(x+1)}$$
$$= \frac{(x+1)^{-0.5}[(2)(x+1)-\left(\frac{1}{2}\right)(2x-7)]}{(x+1)}$$
$$= \frac{4(x+1)-(2x-7)}{2(x+1)^{\frac{3}{2}}}$$
$$= \frac{2x+11}{2(x+1)^{\frac{3}{2}}}$$

Important Concepts

Differentiation Techniques

Trigonometry - Basic

Differentiate $y = \tan(2x + 1)$ with respect to x.

$$\frac{dy}{dx} = [\sec^2(2x+1)](2)$$
$$= 2\sec^2(2x+1)$$

Differentiate $y = \sin(x^3 + 1)$ with respect to x.

$$\frac{dy}{dx} = 3x^2[\cos(x^3+1)]$$

Differentiate $y = \sin(x^3 + 1)$ with respect to x.

$$\frac{dy}{dx} = -\frac{15}{2}\sin(\frac{5x}{2})$$

Trigonometry - Advance

Differentiate $y = \sin^3 x$ with respect to x.

$$\frac{dy}{dx} = 3\sin^2 x\cos x$$

Differentiate $y = 3\cos^2\left(2x + \frac{\pi}{6}\right)$ with respect to x.

$$\frac{dy}{dx} = -12\cos\left(2x+\frac{\pi}{6}\right)\sin\left(2x+\frac{\pi}{6}\right)$$

Differentiate $y = \tan^3\left(\frac{\pi}{8} - 2x\right)$ with respect to x.

$$\frac{dy}{dx} = -6\tan^2\left(\frac{\pi}{8}-2x\right)\sec^2\left(\frac{\pi}{8}-2x\right)$$

Important Concepts

Concept:
When you are differentiating Trigonometry,

Follow the flow of
1) Differentiate Power, Trigo, Bracket

Validation

Differentiation Techniques

Exponential - Basic

Differentiate $y = e^{f(x)}$

$$y = e^{f(x)}$$
$$\frac{dy}{dx} = f'(x)e^{f(x)}$$

Differentiate $y = e^{x}$

$$y = e^{x}$$
$$\frac{dy}{dx} = e^{x}$$

Differentiate $y = e^{3x+2}$

$$y = e^{3x+2}$$
$$\frac{dy}{dx} = 3e^{3x+2}$$

Differentiate $y = e^{2x^2+3x}$

$$y = e^{2x^2+3x}$$
$$\frac{dy}{dx} = (4x+3)e^{2x^2+3x}$$

Exponential - Advance

Differentiate $y = e^{x} \cdot e^{2x}$ with respect to x.

$$y = e^{x+2x} = e^{3x}$$
$$\frac{dy}{dx} = 3e^{3x}$$

Differentiate $y = e\sqrt{e^{x}} - e^{2x}$ with respect to x.

$$y = e^{1+\frac{x}{2}} - e^{2x}$$
$$\frac{dy}{dx} = \frac{1}{2}e^{1+\frac{x}{2}} - 2e^{2x}$$

Differentiate $y = 6e^{2x} + \frac{1}{e^{3x}}$ with respect to x.

$$y = 6e^{2x} + e^{-3x}$$
$$\frac{dy}{dx} = 12e^{2x} - 3e^{-3x}$$

Differentiate $y = \frac{4e^{3x}-3}{e^{x}}$ with respect to x.

$$y = \frac{4e^{3x}}{e^{x}} - \frac{3}{e^{x}}$$
$$y = 4e^{2x} - 3e^{-x}$$
$$\frac{dy}{dx} = 8e^{2x} + 3e^{-x}$$

Important Concepts

Concept:
Apply your Laws of Indices when dealing with Exponential

$$a^{x} \times a^{y} = a^{x+y}$$
$$a^{x} \div a^{y} = a^{x-y}$$

Validation

Differentiation Techniques

Ln- Basic

Differentiate $y = \ln f(x)$

$$y = \ln f(x)$$
$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

Differentiate $y = \ln x$

$$y = \ln x$$
$$\frac{dy}{dx} = \frac{1}{x}$$

Differentiate $y = \ln(3x + 4)$

$$y = \ln(3x + 4)$$
$$\frac{dy}{dx} = \frac{3}{3x + 4}$$

Differentiate $y = \ln x^2$

$$y = \ln x^2$$
$$y = 2\, lnx$$
$$y = \frac{2}{x}$$

Ln - Advance

Differentiate $y = \ln(4 - x)^7, x < 4$ with respect to x.

$$y = 7\ln(4 - x)$$
$$\frac{dy}{dx} = \frac{-7}{4 - x}$$

Differentiate $y = \ln\left(\sqrt{2x^2 + 1}\right)$ with respect to x.

$$y = \frac{1}{2}\ln(2x^2 + 1)$$
$$\frac{dy}{dx} = \frac{4x}{2(2x^2 + 1)} = \frac{2x}{(2x^2 + 1)}$$

Differentiate $y = \ln[(x)(x + 1)]$ with respect to x.

$$y = lnx + \ln(x + 1)$$
$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{x + 1}$$

Differentiate $y = \ln\left(\frac{x}{\sqrt{2x+1}}\right)$ with respect to x.

$$y = lnx - \frac{1}{2}\ln(2x + 1)$$
$$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{2x + 1}$$

Important Concepts

Concept:
Apply your Laws of Logarithm when dealing with Ln Integration.

$$\ln ab = lna + \ln b$$
$$\ln \frac{a}{b} = lna - \ln b$$
$$\ln a^x = x \ln a$$

Validation

Differentiation Techniques

Mixed

Show that $\frac{d}{dx}\left(\frac{\ln x}{4x}\right) = \frac{1-\ln x}{4x^2}$.

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\ln x}{4\mathrm{x}}\right) = \frac{4x\left(\frac{1}{x}\right) - 4\ln x}{(4x)^2}$$

$$= \frac{4-4\ln x}{16x^2}$$

$$= \frac{1-\ln x}{4x^2} \text{ (shown)}$$

Given that $y = \sin 4x$, show that $\frac{d^2y}{dx^2} \times \frac{dy}{dx} = -32\sin 8x$

$$\frac{dy}{dx} = 4\cos 4x$$

$$\frac{d^2y}{dx^2} = -16\sin 4x$$

$$\frac{d^2y}{dx^2} \times \frac{dy}{dx} = (-16\sin 4x)(4\cos 4x)$$

$$= -32(2\sin 4x\cos 4x)$$

$$= -32\sin 8x$$

Differentiate $\cos 2x\,(\tan^2 x - 1)$ with respect to x.
No simplification is required

$$\frac{d}{dx}[\cos 2x(\tan^2 x - 1)]$$

$$= \cos 2x(2\tan x\sec^2 x) + (\tan^2 x - 1)(-2\sin 2x)$$

$$= 2\cos 2x\tan x\ \sec^2 x - 2\sin 2x\,(\tan^2 x - 1)$$

Further Differentiation

If $y = (1+x)e^{3x}$, find the value of the constant k for which $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + ky = 0$.

$$\frac{dy}{dx} = 1(e^{3x}) + (1+x)(3)e^{3x}$$

$$= e^{3x}(3x+4)$$

$$\frac{d^2y}{dx^2} = 3e^{3x}(3x+4) + 3e^{3x}$$

$$= 3e^{3x}(3x+5)$$

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 3e^{3x}(3x+5) - 6e^{3x}(3x+4)$$

$$= e^{3x}\,(9x + 15 - 18x - 24)$$

$$= -9e^{3x}\,(x+1)$$

$$= -9y$$

Thus, $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$.

Given that $y = A - B\,cos\,4x - \frac{1}{2}sin\,2x$ and $\frac{d^2y}{dx^2} + 4y = 3\,cos\,4x + 1$, find the value of each of the following constants A and B.

$$y = A - B\cos 4x - \frac{1}{2}\sin 2x$$

$$\frac{dy}{dx} = 4\,B\sin 4x - \cos 2x$$

$$\frac{d^2y}{dx^2} = 16\,B\cos 4x + 2\sin 2x$$

$$\frac{d^2y}{dx^2} + 4y$$

$$= 16B\cos 4x + 2\sin 2x + 4[\,A - B\cos 4x - \frac{1}{2}\sin 2x]$$

$$= 12\,B\cos 4x + 4A$$

$$\therefore 12B\cos 4x + 4A = 3\cos 4x + 1$$

$B = \frac{1}{4}$, $A = \frac{1}{4}$

It is given that $f'(x) = \sin 3x - \frac{1}{2x+1}$ and $f(0) = \frac{2}{3}$. Find an expression for $6f(x) + f''(x)$.

$$f''(x) = 3\cos 3x + \frac{2}{(2x+1)^2}$$

$$f(x) = -\frac{1}{3}\cos 3x - \frac{1}{2}\ln(2x+1) + c$$

Sub $f(0) = \frac{2}{3}$,

$$\frac{2}{3} = -\frac{1}{3}\cos 3(0) - \frac{1}{2}\ln(1) + c$$

$$c = 1$$

$$f(x) = -\frac{1}{3}\cos 3x - \frac{1}{2}\ln(2x+1) + 1$$

$$6f(x) + f''(x)$$

$$= -2\cos 3x - 3\ln(2x+1) + 6 + 3\cos 3x + \frac{2}{(2x+1)^2}$$

$$= \cos 3x - 3\ln(2x+1) + 6 + \frac{2}{(2x+1)^2}$$

Differentiation Techniques – Equation of Tangent and Normal

Equation of Tangent Normal (Algebra)

The equation of a curve is $y = \frac{2x}{1+x}$.

(I) Find the equation of the tangent to the curve at point $P(1,1)$.

(ii) The tangent cuts the axes at Q and R respectively.
Find the triangle OPQ.

Ans:

(i) $\frac{dy}{dx} = \frac{(1+x)(2)-(2x)(1)}{(1+x)^2}$

$= \frac{2}{(1+x)^2}$

$\left.\frac{dy}{dx}\right|_{x=12} = \frac{1}{2}$

Equation of Tangent : $y - 1 = \frac{1}{2}(x-1) \Rightarrow y = \frac{1}{2}x + \frac{1}{2}$

(ii) $Q(-1,0)$ and $R\left(0,\frac{1}{2}\right)$

Area of Triangle = $\frac{1}{2}(1)\left(\frac{1}{2}\right) = \frac{1}{4}$ units^2

The equation of a curve is $y = (x-1)\ln(1-x)$. Find the exact x-coordinate of the point at which the normal is parallel to the y-axis.

$$y = (x-1)\ln(1-x)$$
$$\frac{dy}{dx} = (x-1)\frac{1}{1-x}(-1) + (1)\ln(1-x)$$
$$\frac{dy}{dx} = 1 + \ln(1-x)$$

Given that normal is parallel to the y-axis,

$$\frac{dy}{dx} = 0$$
$$1 + \ln(1-x) = 0$$
$$\ln(1-x) = -1$$
$$1 - x = e^{-1}$$
$$x = 1 - \frac{1}{e}$$
$$x = \frac{e-1}{e}$$

Equation of Tangent Normal (Trigonometry)

A curve C is such that $\frac{dy}{dx} = 8\cos 2x$ and $P\left(\frac{\pi}{3}, 2\sqrt{3}-3\right)$ is a point on C.

(i) The normal to the curve at P crosses the x-axis at Q.
Find the coordinates of Q.

(ii) Find the equation of C.

When $x = \frac{\pi}{3}, \frac{dy}{dx} = 8\cos\frac{2\pi}{3} = -4$

$\frac{0-(2\sqrt{3}-3)}{x-\frac{\pi}{3}} = \frac{1}{4}$

$Q\left(12 - 8\sqrt{3} + \frac{\pi}{3}, 0\right)$ or $(-0.809, 0)$

$y = 4\sin 2x + c$

Sub $\left(\frac{\pi}{3}, 2\sqrt{3}-3\right)$ $\quad 2\sqrt{3} - 3 = 4\sin\frac{2\pi}{3} + c$

$2\sqrt{3} - 3 = 4\left(\frac{\sqrt{3}}{2}\right)^3 + c \quad y = 4\sin 2x - 3$

Equation of Tangent Normal (Exponential)

A curve has equation given by $y = \frac{e^{4x-3}}{8e^{2x}}$.

(i) The curve passes through the y-axis at P.
Find the equations of the tangent and normal to the curve at point P.

$$y = \frac{e^{2x-3}}{8}$$
$$\frac{dy}{dx} = \frac{e^{2x-3}}{4}$$

When $x = 0, y = \frac{1}{8e^3}$

Gradient of tangent at $P = \frac{1}{4e^3}$

Equation of tangent at P:

$$y - \frac{1}{8e^3} = \frac{1}{4e^3}(x) \Rightarrow y = \frac{x}{4e^3} + \frac{1}{8e^3}$$

Gradient of normal at $P = -4e^3$

Equation of normal at P:

$$y - \frac{1}{8e^3} = -4e^3(x) \Rightarrow y = -4e^3x + \frac{1}{8e^3}$$

Important Concepts

Concept:

This component is closely related to Coordinate Geometry.

I highly recommend you to do a mini sketch so you can easily visualise the graph.

Validation

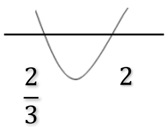
Differentiation Techniques – Increasing & Decreasing Functions

Increasing Decreasing Functions (Type 1)

The equation of a curve is $y = x^3 - 4x^2 + 4x + 14$.
Determine the values of x for which y is increasing function.

For y to be an increasing function,

$$\frac{dy}{dx} > 0$$
$$3x^2 - 8x + 4 > 0$$
$$(3x-2)\,(x-2) > 0$$



$x < \frac{2}{3}$ or $x > 2$

Reverse Inequalities

A curve has the equation $y = 2x^3 + ax^2 + 3bx + 11$.
The only values of x for which y is a decreasing function of x are those values for which $2 < x < 5$.
Find the value of a and b.

$$\frac{dy}{dx} = 6x^2 + 2ax + 3b$$

$$y \text{ is a decresing function}, \frac{dy}{dx} < 0$$

$$2 < x < 5$$
$$(x-2)(x-5) < 0$$
$$x^2 - 7x + 10 < 0$$
$$6x^2 - 42x + 60 < 0$$

Comparing terms,

$$2a = -42, a = -21$$
$$3b = 60, b = 20$$

The function f is defined, for all values of x, by
$f(x) = x^2e^{2x}$.

Find the values of x for which f is a decreasing function

$$f(x) = x^2e^{2x}$$
$$f'(x) = e^{2x}(2x) + x^2(2e^{2x})$$
$$f'(x) = 2xe^{2x}(1+x)$$

For increasing function,

$$f'(x) < 0$$
$$2xe^{2x}(1+x) < 0$$

Since $e^{2x} > 0$
$x(1+x) < 0$
Ans: $-1 < x < 0$

Given that $y = \frac{x^2}{e^x}$, find the range of values of x for which y is an increasing function.

$$y = \frac{x^2}{e^x}$$
$$\frac{dy}{dx} = \frac{e^x(2x) - x^2e^x}{(e^x)^2} = \frac{2x - x^2}{e^x}$$

Since y is an increasing function, $\frac{dy}{dx} > 0$

$$\frac{2x - x^2}{e^x} > 0$$

Since $e^x > 0$, $2x - x^2 > 0$

$$x(2-x) > 0$$
$$0 < x < 2$$

A curve has $y = \frac{x^2}{2-3x}$ where $x \neq \frac{2}{3}$.
Obtain an expression for $\frac{dy}{dx}$.
(ii) Find the values of x for which y is a decreasing function.

$$\frac{dy}{dx} = \frac{2x(2-3x) - (-3)x^2}{(2-3x)^2} = \frac{4x - 3x^2}{(2-3x)^2}$$

Since the curve is decreasing $\frac{dy}{dx} < 0$ and $x \neq \frac{2}{3}$

$$\frac{4x - 3x^2}{(2-3x)^2} < 0$$

Since $(2-3x)^2 > 0$

$$4x - 3x^2 < 0$$
$$x(4-3x) < 0$$

$\therefore x < 0$ or $x > \frac{4}{3}$

Important Concepts

Concept:

This component is closely related to Nature of Roots and Quadratic Inequalities. Study them together.

Take a look at questions in the middle, learn the presentation and explanation technique when you have to find a specific range of answers.

Validation

Substitute the values in your range back to your $\frac{dy}{dx}$

This allows you to validate whether your $\frac{dy}{dx}$ is Positive or Negative.

Differentiation Techniques - Increasing & Decreasing Functions

Increasing Decreasing Functions (Proving by Deduction)

A curve has an equation $y = \frac{3x}{2x-3}$. Show that, for all real values of x where $x \neq \frac{3}{2}$, y is a decreasing function of x.

$$\frac{dy}{dx} = \frac{(2x-3)(3) - 3x(2)}{(2x-3)^2} = \frac{-9}{(2x-3)^2}$$

For all real values of x where $x \neq \frac{3}{2}$,

$$(2x-3)^2 > 0$$
$$\frac{-9}{(2x-3)^2} < 0$$
$$\frac{dy}{dx} < 0$$

Since $\frac{dy}{dx} < 0$, y is a decreasing function.

A curve has equation $y = 3x - \frac{27}{x^2}$.

(i) Find $\frac{dy}{dx}$.

(ii) Show that $y = 3x - \frac{27}{x^2}$ is an increasing function for $x > 0$.

(iii) Show that the function $f(x) = \frac{2x}{x^2-1}$ is always decreasing for $x > 1$.

(i) $\frac{dy}{dx} = \frac{3x^3+54}{x^3}$

(ii) For $x > 0$, $x^3 > 0$ & $(3x^3 + 54) > 0 \Rightarrow \frac{3x^3+54}{x^3} > 0 \Rightarrow \frac{dy}{dx} > 0$;
Since $\frac{dy}{dx} > 0$, y is an increasing functions. [Shown]

(iii) $\frac{dy}{dx} = -\frac{2(1+x^2)}{(x^2-1)^2}$; For $x > 1$, $(x^2-1)^2 > 0 \Rightarrow \frac{2(1+x^2)}{(x^2-1)^2} > 0$
$\Rightarrow -\frac{2(1+x^2)}{(x^2-1)^2} < 0 \Rightarrow \frac{dy}{dx} < 0$; Since $\frac{dy}{dx} < 0$ when $x > 1$, *f(x)* is always decreasing for x>1. [Shown]

Proving by Completing the Square

A curve has equation $y = f(x)$, where $f(x) = \frac{1}{3}x^3 - 2x^2 + 13x + 5$. Determine, with explanation, whether f is an increasing or decreasing function.

$$f'(x) = x^2 - 4x + 13$$
$$= (x-2)^2 - 2^2 + 13$$
$$= (x-2)^2 + 9$$
$$(x-2)^2 \geq 0 \Rightarrow (x-2)^2 + 9 > 0$$

$\therefore$ $f'(x) > 0$, f is an increasing function.

Given that $y = -\frac{1}{3}x^3 + \frac{3}{2}x^2 - 5x + 1$, show that for all real values of x, y is a decreasing function of x.

$\frac{dy}{dx} = -(x-1.5)^2 - 2.75$; For all real values of x,
$(x-1.5)^2 \geq 0 \Rightarrow -(x-1.5)^2 < 0$

$\Rightarrow -(x-1.5)^2 - 2.75 < 0 \Rightarrow \frac{dy}{dx} < 0$; Since $\frac{dy}{dx} < 0$, y is a decreasing function of x. (shown)

Important Concepts

Concept:

This component is closely related to Nature of Roots: Proving and Showing Question.

We have to rely on either Deduction or Completing The Square to prove that the equation is always above or below 0.

We use Completing The Square when we see a quadratic equation because we are unable to explain the magnitude of the equation (even with factorisation).

Validation

Differentiation Techniques – Rate of Change

Connected Rate of Change (Easy)

The equation of a curve is $y = \frac{x-4}{\sqrt{2x+5}}$.

(i) Show that $\frac{dy}{dx}$ can be expressed in the form $\frac{ax+b}{(2x+5)^{\frac{3}{2}}}$ where a and b are constants.

(ii) Given that y is increasing at a rate of 0.4 units per second, find the rate of change of x when $x = 2$.

Ans:

(i) $\frac{dy}{dx} = \frac{(2x+5)^{\frac{1}{2}}(1) - \frac{1}{2}(x-4)(2x+5)^{-\frac{1}{2}}(2)}{2x+5}$

$= \frac{(2x+5)^{-\frac{1}{2}}(2x+5-x+4)}{2x+5}$

$= \frac{x+9}{(2x+5)^{\frac{3}{2}}}$

(ii) When $x = 2$, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$0.4 = \frac{2+9}{(4+5)^{\frac{3}{2}}} \times \frac{dx}{dt}$

$\frac{dx}{dt} = 0.4 \times \frac{27}{11}$

$= \frac{54}{55}$ or 0.982 unite per second

(i) Differentiate $y = 2e^{3x}(1-2x)$ with respect to x.

(ii) Given that x is decreasing at a rate of 5 units per second, find the rate of change of y at the instant when $x = -1.5$.

(i) $y = 2e^{3x}(1-2x)$

$\frac{dy}{dx} = 2e^{3x}(-2) + 6e^{3x}(1-2x)$

$= 2e^{3x}(1-6x)$

(ii) Given that $\frac{dy}{dx} = -5$ unit/s

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$= 2e^{3x}(1-6x)(-5)$

$= 2e^{3(-1.5)}(1+6\times1.5)(-5)$

$= 1.11$ units/sec

A particle moves along the curve in such as way that they y-coordinate of the particle is decreasing at a constant rate of 0.1 units per second. Find the rate of change of the x-coordinate at the instant when $x = 2$.

$$\frac{dy}{dx} = \frac{3x^2}{1+2x^3}$$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{3x^2}{1+2x^3}\frac{dx}{dt}$$

When $x = 2$,

$$-0.1 = \frac{3(2)^2}{1+2(2)^3}\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{17}{12}\left(-\frac{1}{10}\right)$$

$$\frac{dx}{dt} = \frac{17}{120}$$

Rate of change of the x-coordinate when $x = 2$, $-\frac{17}{120}$ unit/s.

OR x-coordinate decreases at a rate of $\frac{17}{120}$ unit/s when $x = 2$.

Important Concepts

Common Careless Mistake:

Forget to put Negative for decreasing Rate of Change

Validation

Differentiation Techniques – Rate of Change

Rate of Change (Advance Type 1)

Given that $y = \frac{4}{5}\left(\frac{x}{12} - 1\right)^6$ and that both x and y vary with time, find the value of y when the rate of change of y is $12\frac{4}{5}$ times the rate of change of x.

$$y = \frac{4}{5}\left(\frac{x}{12} - 1\right)^6$$
$$\frac{dy}{dx} = \frac{24}{5}\left(\frac{x}{12} - 1\right)^5\left(\frac{1}{12}\right) = \frac{2}{5}\left(\frac{x}{12} - 1\right)^5$$
$$\text{Given } \frac{dy}{dt} = \frac{64}{5} \times \frac{dx}{dt}$$
$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$
$$\frac{64}{5} \times \frac{dx}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$
$$\therefore \frac{dy}{dx} = \frac{64}{5}$$

$$\frac{2}{5}\left(\frac{x}{12} - 1\right)^5 = \frac{64}{5}$$
$$\left(\frac{x}{12} - 1\right)^5 = 32$$
$$x = 36$$
$$y = 51.2$$

The variables x and y increase in such a way that, when $x = 5$, the rate of increase of x with respect to time is thrice the rate of increase of y with respect to time. Given that $y = m\sqrt{2x - 1}$, where m is a constant, find the value of m.

$$\frac{dx}{dt} = 3 \times \frac{dy}{dt}$$
$$\frac{dx}{dt} = 3 \times \left(\frac{dy}{dx} \times \frac{dx}{dt}\right)$$
$$1 = 3 \times \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{3}$$

$$y = m(2x - 1)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{m}{\sqrt{2x - 1}}$$
$$Sub\ x = 5, \quad \frac{dy}{dx} = \frac{m}{\sqrt{2(5) - 1}} = \frac{m}{3}$$
$$\frac{dy}{dx} = \frac{m}{3} = \frac{1}{3} \Rightarrow m = 1$$

Rate of Change (Advance Type 2)

A rectangle has sides of length $2x$ cm and $3x$ cm.
Given that the area is increasing at a rate of $\frac{36cm^2}{s}$, Find the rate of increase of the perimeter when $x = 3$.

$$\frac{dP}{dt} = \frac{dP}{dA} \times \frac{dA}{dt}$$
$$\frac{dP}{dt} = \left(\frac{dP}{dx} \times \frac{dx}{dA}\right) \times \frac{dA}{dt}$$

Area of rectangle, $A = 2x \times 3x = 6x^2$
$$\frac{dA}{dx} = 12x$$
$$\text{Sub } x = 3, \frac{dA}{dx} = 12(3) = 36$$

Perimeter of a rectangle, $P = 2(2x) + 2(3x) = 10x$

$$\frac{dP}{dx} = 10$$
$$\frac{dP}{dt} = \left(\frac{dP}{dx} \times \frac{dx}{dA}\right) \times \frac{dA}{dt}$$
$$\frac{dP}{dt} = 10 \times \frac{1}{36} \times 36 = 10cm/s$$

When $x = 3$, Perimeter is increasing at a rate of $10cm/s$.

Some liquid is poured onto a flat surface and formed a circular patch. This circular patch is left to dry and its surface area decreases at a constant rate of 4 cm^2/s. The patch remains circular during the drying process. Find the rate of change of the circumference of the circular patch at the instant when the area of the patch is 400 cm^2

$$\frac{dC}{dt} = \frac{dC}{dr} \times \frac{dr}{dt}$$
$$\frac{dC}{dt} = \frac{dC}{dr} \times \left(\frac{dr}{dA} \times \frac{dA}{dt}\right)$$

$$A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r \qquad C = 2\pi r \Rightarrow \frac{dC}{dr} = 2\pi$$
$$Area: \pi r^2 = 400 \Rightarrow r = \frac{20}{\sqrt{\pi}}$$
$$\frac{dC}{dt} = \frac{dC}{dr} \times \left(\frac{dr}{dA} \times \frac{dA}{dt}\right)$$
$$\frac{dC}{dt} = 2\pi \times \left(\frac{1}{2\pi r} \times -4\right)$$
$$\frac{dC}{dt} = 2\pi \times \left(\frac{1}{2\pi\left(\frac{20}{\sqrt{\pi}}\right)} \times -4\right) = -0.354cm/s$$

Important Concepts

Concept:
Advance Type 1: $\frac{dy}{dx} = k \times \frac{dx}{dt}$

Advance Type 2: Double Chain Rule

Validation

Differentiation Techniques – Rate of Change

Rate of Change (Mensuration)

The volume of a cone of height h is $\frac{\pi h^3}{12}$. If h increases at a constant rate of 0.2 cm/s and the initial height is 2cm.
(i) Express V in terms of t
(ii) find the rate of change of V at time t.

(i) $Height = Initial\ Height + Increase\ in\ Height$

$$h = 2 + 0.2t$$

(ii) $V = \frac{\pi h^3}{12} = \frac{\pi(2+0.2t)^3}{12} = \frac{\pi(10+t)^3}{1500}$

$$\frac{dV}{dt} = \frac{\pi(10+t)^2}{500}$$

A right circular cone of depth 40 cm and radius 10 cm is held with vertex downwards. It contains water which leaks out through a hole at a rate of 8 cm^{3}s^{-1}. Find the rate at which the water level is decreasing when the radius of the surface of the water is 4 cm.

By similar triangles,

$$\frac{10}{40} = \frac{r}{h}$$

$r = \frac{h}{4}$

$$V = \frac{1}{3}\pi r^2 h$$

$= \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h$
$= \frac{1}{48}\pi h^3$

$$\frac{dV}{dh} = \frac{1}{16}\pi h^2$$

When $r = 4$, $h = 16$ cm.
Rate at which the volume is decreasing, $\frac{dV}{dt} = -8$
Using chain rule,

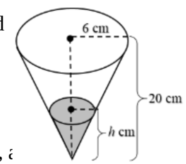
$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$-8 = \frac{1}{16}\pi(16)^2 \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{1}{2\pi}$$

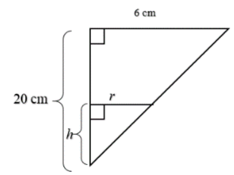
Rate at which the water level is decreasing is $\frac{1}{2\pi}$ cm s^{-1}.

A vessel filled with water is in the shaped of an inverted cone with radius 6 cm and height 20 cm. Water is leaking out from the vessel at a rate of 5 cm^3/s.



(i) Show that the volume of water, V cm^3, when the depth is h cm, is given by $V = \frac{3\pi h^3}{100}$.
(ii) Find the rate of decrease of the height when $h = 12$.

Given Info:
Conical Vessel;
Radius of Cone: 6;
Height of Cone = 20 $\frac{dV}{dt} = -5$



(i) Volume of Cone, $V = \frac{1}{3}\pi r^2 h$
Consider a Pair of Similar Δs, $\frac{r}{6} = \frac{h}{20}$

$$\therefore r = \frac{3h}{10}$$

Sub. $r = \frac{3h}{10}$ into (1), $V = \frac{1}{3}\pi\left(\frac{3h}{10}\right)^2 h$

$$V = \frac{1}{3}\pi\left(\frac{9h^2}{100}\right)h$$

$\therefore V = \frac{3\pi h^2}{100}$ (shown)

(ii) Using chain rule, $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$$V = \frac{3\pi}{100}h^3$$

$\frac{dh}{dV} = \frac{9\pi}{100}h^2$ (diff. using "Power Bring Down, Power -1")
Sub. $h = 12$ into $\frac{dh}{dV}$, $\frac{dh}{dV} = \frac{9\pi}{100}(12)^2 = \frac{324\pi}{25}$
Sub. $\frac{dh}{dV} = \frac{25}{324\pi}$ ("Flip Over") & $\frac{dV}{dt} = -5$ into (1),
$\frac{dh}{dt} = \frac{25}{324\pi} \times (-5) = -0.123$ (3sf)
$\therefore$When $h = 12$, Height is decreasing at a rate of 0.123 cm/s.

Important Concepts

Concept:

For Cone Questions, you can apply Similar Triangle

The concept is we cannot differentiate an equation with 2 variables. Therefore, we have to replace one of the variable.

In order for us to do that, we have to create an equation connecting the 2 variables.

Validation

Differentiation Techniques – Maxima and Minima

Nature of Stationary Point

The equation of a curve is $y = 2x(x-1)^3$.

(i) Find the coordinates of the stationary points of the curve.
(ii) Determine the nature of each of these points using the first derivative test.

$y = 2x(x-1)^3$
$\frac{dy}{dx} = 2x[3(x-1)^2] + 2(x-1)^3$
$= 6x(x-1)^2 + 2(x-1)^3$
$= 2(x-1)^2(3x + x - 1)$
$= 2(x-1)^2(4x-1)$

For $\frac{dy}{dx} = 0$

$2(x-1)^2(4x-1) = 0$

$x = 1$ or $x = \frac{1}{4}$
$y = 0$ or $y = -\frac{27}{128}$
$(1,0)$ and $\left(\frac{1}{4}, -\frac{27}{128}\right)$

By first derivative test,
$(1,0)$ is a point of inflexion and $\left(\frac{1}{4}, -\frac{27}{128}\right)$ is a min. point.

The equation of a curve is $y = e^x + 2e^{-x}$.

(i) Find the coordinates of the stationary point of the curve, leaving your answer in exact form.
(ii) Determine the nature of this point

(i) $\frac{dy}{dx} = e^x - 2e^{-x} = 0$
$e^{2x} = 2$
$x = \ln\sqrt{2}$
$y = e^{\ln\sqrt{2}} + 2e^{-\ln\sqrt{2}}$
$= \sqrt{2} + \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = 2\sqrt{2}$ Point is $(\ln\sqrt{2}, 2\sqrt{2})$

(ii) $\frac{d^2y}{dx^2} = e^x + 2e^{-x}$
$x = \ln\sqrt{2}, \frac{d^2y}{dx^2} = 2 + \frac{2}{\sqrt{2}} > 0$
Minimum point

The equation of the curve is $y = \frac{e^{2x}}{3+4x}$.

(i) Find the coordinates of the stationary point on the curve, leaving your answer in exact value.
(ii) Determine the stationary point.

$\frac{dy}{dx} = \frac{2e^{2x}(3+4x) - e^{2x}(4)}{(3+4x)^2}$

$= \frac{e^{2x}(6+8x-4)}{(3+4x)^2}$

$= \frac{e^{2x}(8x+2)}{(3+4x)^2}$

Stationary point, $\frac{dy}{dx} = 0$

$\frac{e^{2x}(8x+2)}{(3+4x)^2} = 0$

$e^{2x}(8x+2) = 0$

$e^{2x} > 0, 8x + 2 = 0$

$x = -\frac{1}{4}$

$y = \frac{e^{2(-1/4)}}{3 + 4(-1/4)}$

$y = \frac{e^{-1/2}}{2}$

$\left(-\frac{1}{4}, \frac{1}{2\sqrt{e}}\right)$

x	$-\frac{1^-}{4}$	$-\frac{1}{4}$	$-\frac{1^+}{4}$
Sign of dy/dx	-ve	0	+ve

Important Concepts

Concept:
First Derivative Test

x	−	x	+
Sign of dy/dx		0	

Substitute x value into $\frac{dy}{dx}$ to check the Gradient

Minimum, Maximum Point or Point of Inflexion

Second Derivative Test

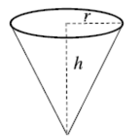
$\frac{d^2y}{dx^2} > 0$ refers to a Minimum Point

$\frac{d^2y}{dx^2} < 0$ refers to a Maximum Point

Validation

Differentiation Techniques – Maxima and Minima

Mensuration - Word Problems



The diagram shows a cone of radius r cm and height h cm. It is given that the volume of the cone is 10π cm^3

Show that the curved surface area, A cm^2, of the cone, is $A=\frac{\pi\sqrt{r^6+900}}{r}$.
Given that r can vary, find the value of r for which A has a stationary value.
Determine whether this value of A is a maximum or a minimum,

(i) Volume $=\frac{1}{3}\pi r^2 h = 10\pi$

$h=\frac{30}{r^2}$

$l^2 = r^2 + h^2$

$= r^2 + \left(\frac{30}{r^2}\right)^2$

$= \sqrt{r^2+\frac{900}{r^4}}$

$A = \pi r l = \pi r\sqrt{r^2+\frac{900}{r^4}}$

$A = \pi r\sqrt{\frac{(r^6+900)}{r^4}}$

$A = \frac{\pi r\sqrt{(r^6+900)}}{r^2}$

$A = \frac{\pi\sqrt{(r^6+900)}}{r}$

(ii) $u = \pi\sqrt{(r^6+900)}$, $v = r$

$\frac{du}{dr} = \frac{1}{2}\times\pi\times(r^6+900)^{-\frac{1}{2}}\times 6r^5$ $\quad\frac{dv}{dr}=1$

$\frac{du}{dr} = 3\pi r^5(r^6+900)^{-\frac{1}{2}}$

$\frac{dA}{dr} = \frac{3\pi r^6(r^6+900)^{-\frac{1}{2}} - \pi(r^6+900)^{\frac{1}{2}}}{r^2}$

When $\frac{dA}{dr}=0$ $\quad\frac{\pi(r^6+900)^{-\frac{1}{2}} - [3r^6 - r^6 - 900]}{r^2} = 0$

$\frac{\pi[3r^6-r^6-900]}{r^2(r^6+900)^{\frac{1}{2}}} = 0$

$2r^6 - 900 = 0$

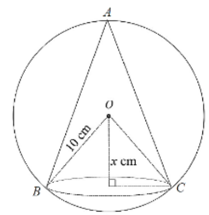
$r^6 = 450$

$r = 2.77$

(iii)

r	$r<2.768$	$r=2.768$	$r>2.768$
$\frac{dA}{dr}$	-	0	+
Sketch	\	-	/

A right circular cone, ABC, is inscribed in a sphere of radius 10 cm and centre O. The perpendicular distance from O to the base of the cone is x cm. [Volume of cone$=\frac{1}{3}\pi r^2 h$]



(i) Show that the volume, V, of the cone is
$V=\frac{1}{3}\pi(100-x^2)(10+x)$.
(ii) If x can vary, find the value of x for which V has stationary value.
(iii) Find this stationary volume.
(iv) Determine whether the volume is a maximum or minimum.

$\frac{dV}{dx} = \frac{1}{3}\pi[-2x(x+10)+100-r^2]$

$=\frac{1}{3}\pi[-20-2x^2+100-x^2]$

$=\frac{1}{3}\pi(-3x^2-20x+100)$

For stationary V, $\frac{dV}{dx}=0$

$\frac{1}{3}\pi(-3x^2-20x+100)=0$

$-3x^2-20x+100=0$

$(x+10)(3x-10)=0$

$x=-10$ (rejected), $x=\frac{10}{3}$

$V=\frac{1}{3}\pi\left(100-\frac{100}{9}\right)\left(\frac{10}{3}+10\right)$

$=1241.123$

$=1240$ cm^3 (3s.f.)

$\frac{d^2V}{dx^2}=\frac{1}{3}\pi(-6x-20)$

Since $\frac{d^2V}{dx^2}<0$, V is a maximum.

Important Concepts

Concept:

When proving in Mensuration, think of

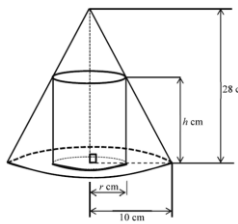
1) Pythagoras Theorem

2) Similar Triangles

Validation

Differentiation Techniques – Maxima and Minima

Mensuration - Word Problems



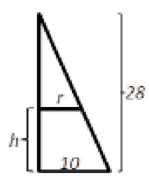
The diagram shows a cylinder of height h cm and base radius r cm inscribed in a cone of height 28 cm and base radius 10 cm.
Show that
(i) the height, h cm, of the cylinder is given by
$h = 28 - \frac{14}{5}r$.
(ii) the volume, V cm^3, of the cylinder is given by
$V = 14\pi r^2 \left(2 - \frac{r}{5}\right)$.
(b)(i) Given that r can vary, find the maximum volume of the cylinder.
(ii) Show that, in this case, the cylinder occupies $\frac{4}{9}$ of the volume of the cone.

(i)
Using similar triangles,

$\frac{28-h}{28} = \frac{r}{10}$

$28 - h = \frac{28}{10}r$

$h = 28 - \frac{17}{5}r$(shown)



(ii) Vol of cylinder $= \pi r^2 h$

$V = \pi r^2 \left(28 - \frac{14}{5}r\right)$

$V = 14\pi r^2 \left(2 - \frac{1}{5}r\right)$(shown)

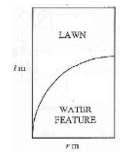
(bi) $\frac{dV}{dr} = 56\pi r - \frac{14}{5}\pi(3r^2)$

$= 14\pi r\left(4 - \frac{3}{5}r\right)$

At stat pt, $\frac{dV}{dr} = 0$

$14\pi r\left(4 - \frac{3}{5}r\right)$

As part of a garden design, there are plans to put inside a rectangular space which has sides of lengths r m and l m. This rectangular space is to include a quadrant-shaped water feature and a lawn. The area of the lawn is to be 360 m^2.



(i) Show that the perimeter, P m, of the lawn is given by $P = \frac{720}{r} + \pi r$.
(ii) A hedge is to be planted along the perimeter of the lawn.
Given that r can vary, find the dimensions of the rectangular space which can allow the shortest length of hedge to be planted along the perimeter of the lawn.

Answers:
(i) Area of lawn:

$lr - \frac{\pi r^2}{4} = 360$

$l = \frac{360 - \frac{\pi r^2}{4}}{r}$

$l = \frac{360}{r} + \frac{\pi r}{4}$---------(1)

Perimeter of lawn:

$P = l + r + (l - r) + \frac{1}{4}(2\pi r)$

$P = 2l + \frac{\pi r}{2}$ ------------(2)

Sub (1) into (2):

$P = 2\left(\frac{360}{r} + \frac{\pi r}{4}\right) + \frac{\pi r}{2}$

$P = \frac{720}{r} + \pi r$ (shown)

(ii) The shortest hedge can be planted when the rectangular space is 15.1cm by 35.7cm.

$\frac{dP}{dr} = -\frac{720}{r^2} + \pi$

$\frac{dP}{dr} = 0$ for stationary values

$-\frac{720}{r^2} + \pi = 0$

$r^2 = \frac{720}{\pi}$

$r = 15.1$ (3s.f.) ($rej. -15.1$)

$\frac{d^2P}{dr^2} = 1440r^{-3}$

$\left.\frac{d^2P}{dr^2}\right|_{r=15.138} = 0.415$(3s.f.)$> 0$

$\Rightarrow$ Shortest perimeter

$l = \frac{360}{15.138} + \frac{\pi(15.138)}{4}$

$= 35.7$ (3s.f.)

The shortest hedge can be planted when the rectangular space is 15.1cm by 35.7cm.

Important Concepts

Concept:

When proving in Mensuration, think of

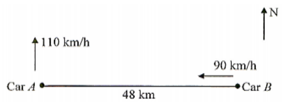
1) Pythagoras Theorem

2) Similar Triangles

Validation

Differentiation Techniques – Maxima and Minima

Mensuration - Word Problems



The diagram shows Car B, which is 48 km due east of Car A. Both cars start moving at the same time. Car A travels due north at a constant speed of 110 km/h while Car B travels due west at a constant speed of 90 km/h.

(i) The distance between Car A and Car B at time t hours after the cars started moving is denoted by L km. Express L in the form of $\sqrt{pt^2 + (q - rt)^2}$ where p, q and r are constants.

(ii) Given that t can vary, find the stationary value of L.

(iii) Determine whether this value stationary value of L gives the maximum or minimum distance between Car A and Car B.

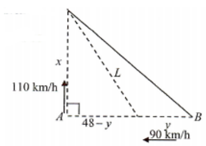
Ans:

(i) Let the distance that Car A travels be x km.
Let the distance that Car B travels be y km.

For Car A,

$$110 = \frac{x}{t}$$

$x = 100t$ km



For Car B,

$$90 = \frac{y}{t}$$

$y = 90t$ km

By Pythagoras Theorem,

$L = \sqrt{x^2 + (48 - y)^2}$ (since $L > 0$)

$= \sqrt{(110t)^2 + (48 - 90t)^2}$

$= \sqrt{12100t^2 + (48 - 90t)^2}$

$\frac{dl}{dt} \quad \frac{1}{2}[12100t^2 + (48 - 90t)^2]^{\frac{1}{2}} \quad [24200t + 2(48 - 90t)^2(-90)]$

$= \frac{40400t - 8640}{2\sqrt{12100t^2 + (48 - 90t)^2}}$

$\frac{dL}{dt} = 0$

$40400t - 8640 = 0$

$$t = \frac{108}{505}\text{ h} = 0.21386 \text{ hr}$$

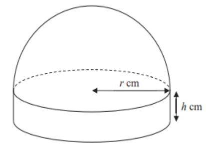
$$L = \sqrt{12100\left(\frac{108}{505}\right)^2 + \left[48 - 90\left(\frac{108}{505}\right)\right]^2}$$

$= 37.0621 = 37.1$ (3 s.f.)

(iii)

t	0.21386^-	0.21386	0.21386^+
Sign of $\frac{dL}{dt}$	-	0	t
Sketch of tangent	\	–	/

In the diagram, a solid model is made up of a hemisphere of radius r cm and a cylinder. The cylinder has a radius of r cm and a height of h cm.



(i) Given that the volume of the model is 650 cm^3, express h in terms of r.

(ii) Given that the total surface area of the model is A cm^2, show that $A = \frac{1300}{r} + \frac{5\pi r^2}{3}$

(iii) Given that r and h can vary, find the value of r for which A has a stationary value and determine whether this value of A is a maximum or a minimum.

$$\frac{2}{3}\pi r^3 + \pi r^2 h = 650$$

$$\pi r^2 h = 650 - \frac{2}{3}\pi r^3$$

$$h = \frac{650}{\pi r^2} - \frac{\frac{2}{3}\pi r^3}{\pi r^2}$$

$$h = \frac{650}{\pi r^2} - \frac{2r}{3}$$

$$A = 2\pi rh + \pi r^2 + 2\pi r^2$$

$$A = 2\pi r\left(\frac{650}{\pi r^2} - \frac{2r}{3}\right) + 3\pi r^2$$

$$A = \frac{1300}{r} - \frac{4\pi r^2}{3} 3\pi r^2$$

$$A = \frac{1300}{r} - \frac{5\pi r^2}{3}$$

$$A = \frac{1300}{r} - \frac{5\pi r^2}{3}$$

$$\frac{dA}{dr} = \frac{1300}{r^2} + \frac{10\pi r}{3}$$

$$-1300(3) + 10\pi r^3 = 0$$

$$r = 4.99$$

$$\frac{d^2A}{dr^2} = \frac{2600}{r^3} + \frac{10\pi}{3}$$

At $r = 4.99$; $\frac{d^2A}{dr^2} = \frac{2600}{(4.99)^3} + \frac{10\pi}{3} > 0$ A is minimum

Important Concepts

Concept:

When proving in Mensuration, think of

1) Pythagoras Theorem

2) Similar Triangles

Validation

Indefinite Integral

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)(a)} + C$$

Definite Integral

$$\int_b^a (ax+b)^n dx = \left[\frac{(ax+b)^{n+1}}{(n+1)(a)}\right]_b^a$$

Trigonometry

$$\int psin(qx+r)\,dx = \frac{-pcos(qx+r)}{q} + C$$

$$\int pcos(qx+r)\,dx = \frac{psin(qx+r)}{q} + C$$

$$\int psec^2(qx+r)\,dx = \frac{ptan(qx+r)}{q} + C$$

In our syllabus,
we only learn to integrate
$\sin x$, $\cos x$, $\sec^2 x$.
Apply identities if other trigo are tested.

Exponential

$$\int e^{f(x)} dx = \frac{e^{f(x)}}{f'(x)} + C$$

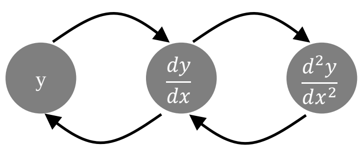
Recall Law of Indices

Logarithm

$$\int \frac{f'(x)}{f(x)} dx = lnf(x) + C$$

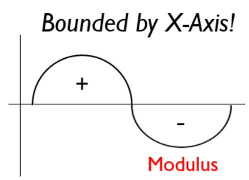
Recall Partial Fractions
If the Denominator is not Linear,
Apply Algebra Integration

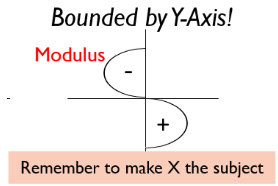
Curves



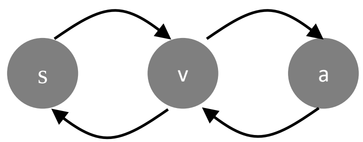
Rate of Change

Area (Region)





Kinematics



1. Find displacement, velocity, acceleration
2. Find time during Instantaneous Rest
3. Find minimum/maximum velocity
4. Total Distance (First 5s vs. Fifth Second)
5. Average Speed
6. 2 Collisions (Same distance travelled)

Algebra

Indefinite Integral

$$\int (3x+2)^4 dx = \frac{(3x+2)^5}{(5)(3)} + C$$
$$= \frac{(3x+2)^5}{15} + C$$

$$\int \frac{1}{(2-x)^3} dx = \int \frac{(2-x)^{-3}}{(1)} + C$$
$$= \frac{(2-x)^{-2}}{(-2)(-1)} + C$$
$$= \frac{1}{2(2-x)^2} + C$$

$$\int \frac{1}{\sqrt{x+1}} dx = \int \frac{(x+1)^{-0.5}}{(1)} dx$$
$$= \frac{(x+1)^{0.5}}{\frac{1}{2}} + c$$
$$= 2(x+1)^{0.5} + c$$
$$= 2\sqrt{x+1} + c$$

Definite Integral

$$\int_1^3 \sqrt{3x-1}\, dx = \left[\frac{(3x-1)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(3)}\right]_1^3$$
$$= \left[\frac{2(3x-1)^{\frac{3}{2}}}{9}\right]_1^3$$
$$= [\ 5.02\ ] - [\ 0.6285\ ]$$
$$= 4.40$$

$$\int_0^1 \frac{1}{3(x+1)^2} dx = \left[-\frac{1}{3}(x+1)^{-1}\right]_0^1$$
$$= \frac{1}{6} - 0$$
$$= \frac{1}{6}$$

Important Concepts

Validation

Algebra – Rules of Integration

Definite Integral

Given that $\int_0^3 f(x)dx = \int_3^4 f(x)dx = 5$.

(i) Evaluate $\int_0^4 2f(x)\,dx + \int_4^3 f(x)\,dx$.

(ii) Find the value of the constant k such that $\int_0^3 f(x) - kx\,dx = 8$.

(i)

$$\int_0^4 2f(x)\,dx + \int_4^3 f(x)\,dx$$
$$= 2\int_0^4 f(x)\,dx - \int_3^4 f(x)\,dx$$
$$= 2\left[\int_0^4 f(x)\,dx\right] - \int_3^4 f(x)\,dx$$
$$= 2(5+5) - 5$$
$$= 15$$

(ii)

$$\int_0^3 f(x) - kx\,dx = 8$$
$$\int_0^3 f(x)\,dx - \int_0^3 kx\,dx = 8$$
$$5 - \left[\frac{kx^2}{2}\right]_0^3 = 8$$
$$-\frac{9k}{2} = 3$$
$$-9k = 6$$
$$\therefore k = -\frac{2}{3}$$

Definite Integral

Given that $\int_1^3 \text{f(x)dx} = 2$ and $\int_3^7 \text{f(x) dx} = 5$, find

(i) $\int_1^7 \text{f(x)dx}$,

(ii) $\int_1^3 2\text{f(x)dx} - \int_7^3 \text{f(x)dx}$,

(iii) $\int_3^7 \text{f(x)} - 2\text{x dx}$

(i) $\int_1^7 \text{f(x)dx} = \int_3^7 \text{f(x) dx} + \int_1^3 \text{f(x)dx} = 5 + 2 = 7$

(ii) $\int_1^3 2\text{f(x)dx} - \int_7^3 \text{f(x)dx} = 2(2) - (-5) = 9$

(iii) $\int_3^7 \text{f(x)} - 2\text{x dx} = 5 - [\text{x}^2]_3^7 = -35$

Important Concepts

Validation

Trigonometry

Simple	Advance
$\int \cos 2x \, dx = \frac{\sin 2x}{2} + c$, where c is an arbitrary constant	$\int \frac{1}{\cos^2 4x} \, dx = \int sec^2 4x \, dx = \frac{\tan 4x}{4} + c$
$\int \sin\left(\frac{\pi}{4} - x\right) dx = \frac{-\cos\left(\frac{\pi}{4} - x\right)}{(-1)} + c$ where c is an arbitrary constant $= \cos\left(\frac{\pi}{4} - x\right) + c$	$\int 2\cos^2 x \, dx = \int \cos 2x + 1 \, dx = \frac{\sin 2x}{2} + x + c$
$\int sec^2\left(\frac{\pi}{2} - 2x\right) dx = \frac{\tan\left(\frac{\pi}{2} - 2x\right)}{(-2)} + c$ where c is an arbitrary constant. $= -\frac{1}{2}\tan\left(\frac{\pi}{2} - 2x\right) + c$	

Important Concepts

Validation

Exponential

Simple	Advance

$\int e^x\,dx$

$\int e^x\,dx = \frac{e^x}{1} + c$, where c is an arbitrary constant.

$$\int e^{x+2}.\sqrt{e^x}\,dx = \int e^{x+2}.(e^x)^{\frac{1}{2}}\,dx$$
$$= \int e^{x+2}.e^{\frac{1}{2}x}\,dx$$
$$= \int e^{x+2+\frac{1}{2}x}\,dx$$
$$= \int e^{2+\frac{3}{2}x}\,dx$$
$$= \frac{e^{2+\frac{3}{2}x}}{\left(\frac{3}{2}\right)} + c$$
$$= \frac{2e^{2+\frac{3}{2}x}}{3} + c$$

$\int e^{2x+1}\,dx$

$\int e^{2x+1}\,dx = \frac{e^{2x+1}}{2} + c$, where c is an arbitrary constant.

$$\int \frac{e}{e^{1-3x}}\,dx = \int e^{1-(1-3x)}\,dx$$
$$= \int e^{1-1+3x}\,dx$$
$$= \int e^{3x}\,dx$$
$$= \frac{e^{3x}}{3} + c$$

$\int 15e^{-x} - e^x dx$

$\int 15e^{-x} - e^x dx = \frac{15e^{-x}}{(-1)} - \frac{e^x}{1} + c$,

where c is an arbitrary constant.

$$= -\frac{15}{e^x} - e^x + c$$

$$\int (1+e^x)^2\,dx^* = \int 1 + 2e^x + (e^x)^2\,\,dx$$
$$= \int 1 + 2e^x + e^{2x}\,dx$$
$$= x + \frac{2e^x}{1} + \frac{e^{2x}}{2} + c$$
$$= x + 2e^x + \frac{e^{2x}}{2} + c$$

Important Concepts

Validation

Logarithm

Indefinite Integral

$$\int \frac{3}{x}\,dx = \int 3\left(\frac{1}{x}\right)dx$$
$$= 3\left(\frac{\ln x}{1}\right) + c$$
$$= \frac{3\ln x}{1} + c$$
$$= 3\ln x + c$$

$$\int \frac{1}{2x-1}\,dx = \frac{\ln(2x-1)}{2} + c$$

$$\int \frac{7}{5-x}\,dx$$
$$= \int 7\left(\frac{1}{5-x}\right)dx$$
$$= 7\left[\frac{\ln(5-x)}{(-1)}\right] + c$$
$$= \frac{7\ln(5-x)}{(-1)} + c$$
$$= -7\ln(5-x) + c$$

Definite Integral

(i) Express $\frac{4x^3+5x^2+x-1}{x^2(x+1)}$ in partial fractions.

(ii) Hence, find $\int \frac{4x^3+5x^2+x-1}{x^2(x+1)}dx$.

(i) $\frac{4x^3+5x^2+x-1}{x^2(x+1)} = 4 + \frac{x^2+x-1}{x^2(x+1)}$

$\frac{x^2+x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$x^2 + x - 1 = Ax(x+1) + B(x+1) + Cx^2$

When $x = 0$, $-1 = B$

When $x = -1$, $1 - 1 - 1 = C$

$C = -1$

When $x = 1$, $1 = 2A - (1+1) - 1$

$1 = 2A - 3$

$2A = 4$

$A = 2$

(ii) $\int \frac{4x^3+5x^2+x-1}{x^2(x+1)}$

$\int \left(4 + \frac{2}{x} - \frac{1}{x^2} - \frac{1}{x+1}\right)dx$

$4x + 2\ln x + \frac{1}{x} - \ln(x+1) + c$

Important Concepts

Validation

Hence

(i) Given that $y = x\sqrt{5x^2-6}$, find $\frac{dy}{dx}$.

(ii) Hence, evaluate $\int_2^4 \frac{5x^2-3}{\sqrt{5x^2-6}}\,dx$.

$y = x\sqrt{5x^2-6}$

$\frac{dy}{dx} = (5x^2-6)^{\frac{1}{2}} + x\left(\frac{1}{2}\right)(5x^2-6)^{-\frac{1}{2}}(10x)$

$= \frac{5x^2-6+5x^2}{\sqrt{5x^2-6}}$

$= \frac{2(5x^2-3)}{\sqrt{5x^2-6}}$

(ii) $\int_2^4 \frac{5x^2-3}{\sqrt{5x^2-6}}\,dx = 13.5$ (3s.f.)

$\int_2^4 \frac{5x^2-3}{\sqrt{5x^2-6}}\,dx$

$= \frac{1}{2}\int_2^4 \frac{2(5x^2-3)}{\sqrt{5x^2-6}}\,dx$

$= \frac{1}{2}\left[x\sqrt{5x^2-6}\right]_2^4$

$= \frac{1}{2}\left[4\sqrt{74} - 2\sqrt{14}\right]$

$= 13.5$ (3s.f.)

(i) Express $\frac{2x+16}{(x^2+4)(2x-1)}$ in partial fractions.

(ii) Differentiate $\ln(x^2+4)$ with respect to x.

(iii) Hence, using your results in (i) and (ii), find $\int \frac{x+8}{(x^2+4)(2x-1)}\,dx$.

(i) $\frac{2x+16}{(x^2+4)(2x-1)} = \frac{-2x}{x^2+4} + \frac{4}{2x-1}$

(ii) $\frac{d}{dx}\ln(x^2+4) = \frac{2x}{x^2+4}$

(iii) $\int \frac{x+8}{(x^2+4)(2x-1)}\,dx = \int \frac{-x}{x^2+4} + \frac{2}{2x-1}\,dx$

$= -\frac{1}{2}\int \frac{2x}{x^2+4}\,dx + \int \frac{2}{2x-1}\,dx$

$= -\frac{1}{2}[\ln(x^2+4)] + \ln(2x-1) + c$

(i) Show that $\frac{d}{dx}\left(\frac{e^{3x}}{\sqrt{x-1}}\right) = \frac{e^{3x}(6x-7)}{2\sqrt{(x-1)^3}}$.

(ii) Hence or otherwise, find $\int_2^3 \frac{e^{3x}(6x-7)}{\sqrt{(x-1)^3}}\,dx$.

(i) $\frac{d}{dx}\left(\frac{e^{3x}}{\sqrt{x-1}}\right) = \frac{3e^{3x}\sqrt{x-1} - \frac{1}{2}(x-1)^{-\frac{1}{2}}e^{3x}}{x-1}$

$\frac{e^{3x}}{x-1}\left(3\sqrt{x-1} - \frac{1}{2\sqrt{x-1}}\right)$

$\frac{e^{3x}}{x-1}\left(\frac{6(x-1)-1}{2\sqrt{x-1}}\right)$

$\frac{e^{3x}(6x-7)}{2\sqrt{(x-1)^3}}$

(ii) $\int_2^3 \frac{e^{3x}(6x-7)}{2\sqrt{(x-1)^3}}\,dx = \left[\frac{e^{3x}}{\sqrt{x-1}}\right]_2^3$

$\frac{1}{2}\int_2^3 \frac{e^{3x}(6x-7)}{\sqrt{(x-1)^3}}\,dx = \left[\frac{e^{3x}}{\sqrt{x-1}}\right]_2^3$

$\int_2^3 \frac{e^{3x}(6x-7)}{\sqrt{(x-1)^3}}\,dx = 2\left[\frac{e^{3x}}{\sqrt{x-1}}\right]_2^3$

$= 2\left[\frac{e^9}{\sqrt{2}} - e^6\right]$

$= 10652.6336$

Important Concepts

Validation

Hence

Differentiate $x\cos(2x+1)$ with respect to x.
Hence, find $\int 4x\sin(2x+1)\,dx$.

$$\frac{d}{dx}[x\cos(2x+1)] = x[-2\sin(2x+1)] + [\cos(2x+1)(1)]$$
$$= -2x\sin(2x+1) + \cos(2x+1)$$

$$\int[\cos(2x+1) - 2x\sin(2x+1)]dx = x\cos(2x+1) + c$$
$$\int \cos(2x+1)\,dx - \int 2x\sin(2x+1)\,dx = x\cos(2x+1) + c$$
$$\int 2x\sin(2x+1)dx = \int \cos(2x+1)\,dx - x\cos(2x+1) + c_1$$
$$\int 4x\sin(2x+1)dx = 2\int \cos(2x+1)dx - 2x\cos(2x+1) + c_2$$
$$= \frac{2\sin(2x+1)}{2} - 2x\cos(2x+1) + c_3$$

(i) Differentiate $x\cos 2x$ with respect to x.

(ii) Using your answer to part (i), find $\int_{\pi}^{2\pi} x\sin 2x\,dx$.

(i) $\frac{d}{dx}(x\cos 2x)$

$$= x(-2\sin 2x) + \cos 2x = -2x\sin 2x + \cos 2x$$

(ii) $\int_{\pi}^{2\pi}(-2x\sin 2x + \cos 2x)dx = [x\cos 2x]_{\pi}^{2\pi}$

$$\int_{\pi}^{2\pi}(-2x\sin 2x) = \int_{\pi}^{2\pi}\cos 2x dx + [x\cos 2x]_{\pi}^{2\pi}$$
$$\int_{\pi}^{2\pi}(x\sin 2x)dx = \frac{1}{2}\left[\frac{1}{2}\sin 2x\right]_{\pi}^{2\pi} - \frac{1}{2}[x\cos 2x]_{\pi}^{2\pi}$$

$$\int_{\pi}^{2\pi}(x\sin 2x)dx = \frac{1}{4}[\sin 4\pi - \sin 2\pi] - [2\pi\cos 4\pi - \pi\cos 2\tau$$
$$\int_{\pi}^{2\pi}(x\sin 2x)dx = \frac{1}{4}[0-0] - \frac{1}{2}[2\pi - \pi]$$
$$\int_{\pi}^{2\pi}(x\sin 2x)dx = -\frac{\pi}{2}$$

(i) Differentiate $\sin^3 2x$ with respect to x.
(ii) Hence evaluate the following

(a) $\int_0^{\frac{\pi}{8}} \sin^2 2x\cos 2x\,dx$

(b) $\int_0^{\frac{\pi}{8}} \cos^3 2x\,dx$

(i) $6\sin^2 2x\cos 2x$

(ii) $\frac{d}{dx}(\sin^3 2x) = 3\sin^2 2x(2\cos 2x)$

$\frac{d}{dx}(\sin^3 2x) = 6\sin^2 2x\cos 2x$

(ii)(a) 0.0589 (3sf)

$$\int_0^{\frac{\pi}{8}} \sin^2 2x\cos 2x\,dx$$
$$= \frac{1}{6}\int_0^{\frac{\pi}{8}} \frac{d}{dx}(\sin^3 2x)\,dx$$
$$= \frac{1}{6}[\sin^3 2x]_0^{\frac{\pi}{8}} = 0.0589 \text{ (3sf)}$$

(b) $\int_0^{\frac{\pi}{8}} \cos^3 2x\,dx$

$$= \int_0^{\frac{\pi}{8}} \cos^2 2x\cos 2x\,dx$$
$$= \int_0^{\frac{\pi}{8}} (1 - \sin^2 2x)\cos 2x\,dx$$
$$= \int_0^{\frac{\pi}{8}} \cos 2x\,dx - \frac{1}{6}\int_0^{\frac{\pi}{8}} 6\sin^2 2x\cos 2x\,dx$$
$$= \left[\frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x\right]_0^{\frac{\pi}{8}}$$
$$= 0.295 \text{ (3sf)}$$

Important Concepts

Validation

Equation of Curves

Algebra

A curve is such that $\frac{dy}{dx} = \frac{3}{k}x + 6$, where k is a constant. Given that the gradient of the normal at the point $(-2, 1)$ on the curve is $-\frac{1}{2}$, find
(i) The value of k,
(ii) The equation of the curve

(i) $\frac{3(-2)}{k} + 6 = 2$

$\frac{-6}{k} = -4$

$k = 1\frac{1}{2}$

(ii) $\frac{dy}{dx} = \frac{3}{\frac{3}{2}}x + 6 = 2x + 6$

$y = \frac{2x^2}{2} + 6x + c$

At $(-2,1)$; $1 = 4 + 6(-2) + c$

$c = 9$

$y = x^2 + 6x + 9$

A curve is such that $\frac{d^2y}{dx^2} = 2(1 - 2x)$. The equation of the normal to the curve at the point $(-1,7)$ is $9y = x + 64$
Find the equation of the curve.

$\frac{d^2y}{dx^2} = 2(1 - 2x)$

$\frac{dy}{dx} = -2x^2 + 2x + c$

Gradient of normal $= \frac{1}{9}$ Gradient of tangent $= -9$

Sub $\frac{dy}{dx} = -9, x = -1,$

$-9 = -2(-1)^2 + 2(-1) + C$

$C = -5$

$\frac{dy}{dx} = -2x^2 + 2x - 5$

$y = -\frac{2x^3}{3} + x^2 - 5x + D$

Sub $(-1, 7)$, ,1

$7 = -\frac{2(-1)^3}{3} + (-1)^2 - 5(-1) + D$

$D = \frac{1}{3}$

A curve which $\frac{d^2y}{dx^2} = 6x - 4$ has a minimum point at (1, 5). Find the equation of the curve.

$$\frac{d^2y}{dx^2} = 6x - 4$$

$$\frac{dy}{dx} = \int (6x - 4)dx$$

$$= \frac{6x^2}{2} - 4x + c_1$$

$$= 3x^2 - 4x + c_1$$

$$\text{At } (1, 5), \frac{dy}{dx} = 0$$

$$0 = 3(1)^2 - 4(1) + c_1$$

$$= -1 + c_1$$

$$c_1 = 1$$

$$\frac{dy}{dx} = 3x^2 - 4x + 1$$

$$y = \int (3x^2 - 4x + 1)dx$$

$$= 3\left(\frac{x^3}{3}\right) - 4\left(\frac{x^2}{2}\right) + x + c_2$$

$$= x^3 - 2x^2 + x + c_2$$

$$\text{At } (1, 5), 5 = (1)^3 - 1(1)^2 + (1) + c_2$$

$$\therefore c_2 = 5$$

$$\text{Hence, } y = x^3 - 2x^2 + x + 5$$

Important Concepts

Validation

Equation of Curves

Exponential

The curve $y = f(x)$ is such that $f''(x) = 3(e^x - e^{-3x})$ and the point $P(0, 2)$ lies on the curve. Given that the gradient of the curve at P is 5, find the equation of the curve.

$f'(x) = 3e^x + e^{-3x} + C$, where C is a constant

$$f'(0) = 5$$
$$3e^0 + e^0 + C = 5$$
$$C = 1$$
$$\therefore f'(x) = 3e^x + e^{-3x} + 1$$
$$f(x) = \int (3e^x + e^{-3x} + 1)dx$$

$= 3e^x - \frac{e^{-3x}}{3} + x + D$, where D is a constant

$$f(0) = 2$$
$$3 - \frac{1}{3} + 0 + D = 2$$
$$D = -\frac{2}{3}$$

Equation of curve : $y = 3e^x - \frac{1}{3e^{3x}} + x - \frac{2}{3}$

The gradient function of the normal to a curve is $\frac{e^{\frac{1}{2}x}}{3-2e^x}$. Given that the curve passes through the point (ln 4,10), find the equation of the curve.

Grad. Function of Normal = $\frac{e^{\frac{1}{2}x}}{3-2e^x}$

$\Rightarrow$ Grad. Function of Tangent, $\frac{dy}{dx} = -\left(\frac{3-2e^x}{e^{\frac{1}{2}x}}\right)$

$$= \frac{2e^x - 3}{e^{\frac{1}{2}x}}$$

$$y = \int \frac{2e^x - 3}{e^{\frac{1}{2}x}} dx$$
$$y = \int \frac{2e^x}{e^{\frac{1}{2}x}} - \frac{3}{e^{\frac{1}{2}x}} dx$$
$$y = \int 2e^{\frac{1}{2}x} - 3e^{-\frac{1}{2}x} dx$$
$$y = \frac{2e^{\frac{1}{2}x}}{\left(\frac{1}{2}\right)} - \frac{3e^{-\frac{1}{2}x}}{\left(-\frac{1}{2}\right)} + c$$
$$y = 4e^{\frac{1}{2}x} + 6e^{-\frac{1}{2}x} + c$$

$y = 4e^{\frac{1}{2}x} + \frac{6}{e^{\frac{1}{2}x}} + c$ ------- (1)

Sub. (ln4,10) into (1), $10 = 4e^{\frac{1}{2}\ln 4} + \frac{6}{e^{\frac{1}{2}\ln 4}} + c$

$$10 = 11 + c$$
$$\therefore c = -1$$

$\therefore$ Equation of curve is $y = 4e^{\frac{1}{2}x} + \frac{6}{e^{\frac{1}{2}x}} - 1$ or $y = 4\sqrt{e^x} + \frac{6}{\sqrt{e^x}} - 1$

Important Concepts

Validation

An ice cube is melting such that the side of the cube is decreasing at a rate of $\frac{1}{4}t^2$ cm/min. It is given that the side of the cube is 8 cm at the start of the experiment, calculate

(i) The length of the side of the cube when $t = 4$
(ii) the rate at which the total surface area of the cube is decreasing when $t = 4$

Ans:

(i) $\frac{dl}{dt} = -\frac{1}{4}t^2$

$l = \int -\frac{1}{4}\left(\frac{t^3}{3}\right) + c$, where c is an arbitrary constant

$l = -\frac{t^3}{12} + c$

Sub. $l = 8$ & $t = 0$ into $l = \frac{t^3}{12} + c$,

$$8 = -\frac{(0)^3}{12} + c$$
$$c = 8,$$
$$\therefore l = \frac{t^3}{12} + 8$$

Sub. $t = 4$ into $l = \frac{t^3}{12} + 8$,

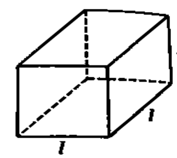
$$l = -\frac{(4)^3}{12} + 8$$
$$\therefore l = 2\frac{2}{3}$$

When $t = 4$, the side of the cube is $2\frac{2}{3}$ cm.

(ii) Using chain rule, $\frac{dA}{dt} = \frac{dA}{dl} \times \frac{dl}{dt}$

Sub. $t = 4$ into $\frac{dl}{dt} = -\frac{1}{4}t^2$

$$\frac{dl}{dt} = -\frac{1}{4}(4)^2$$
$$\frac{dl}{dt} = -4$$



Total surface area of cube, $A = 6t^2$

$$\frac{dA}{dl} = 12l$$

Sub. $l = 2\frac{2}{3}$ into $\frac{dA}{dl} = 12l$

$$\frac{dA}{dl} = 12 \times 2\frac{2}{3}$$
$$\frac{dA}{dl} = 32$$

Sub. $\frac{dA}{dl} = 32$ & $\frac{dl}{dt} = -4$ into $\frac{dA}{dt} = \frac{dA}{dl} \times \frac{dl}{dt}$,

$$\frac{dA}{dl} = 32 \times (-4)$$
$$\frac{dA}{dl} = -128$$

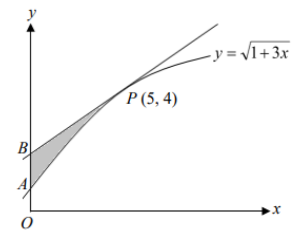
$\therefore$ The total surface area of the cube is decreasing at a rate of 128 cm/min.

Important Concepts

Validation

Area under Graph

Integrating w.r.t Y Axis



The diagram shows part of the curve $y = \sqrt{1+3x}$, intersecting the y-axis at A. The tangent to the curve at the point P(5,4) intersects the y-axis at B.

(i) Find the coordinates of A and B.

(ii) Calculate the area of the shaded region ABP.

i) $y = \sqrt{1+3x}$

Sub $x = 0, y = 1$

A(0, 1)

$\frac{dy}{dx}$

$= \frac{1}{2}(1+3x)^{-1/2}(3)$

$= \frac{3}{2\sqrt{1+3x}}$

When x = 5,

Gradient

$= \frac{dy}{dx}$

$= \frac{3}{2\sqrt{1+3(5)}}$

$= \frac{3}{8}$

Equation of tangent at P

$y - 4 = \frac{3}{8}(x-5)$

$y = \frac{3}{8}x + \frac{17}{8}$

$B(0, \frac{17}{8})$

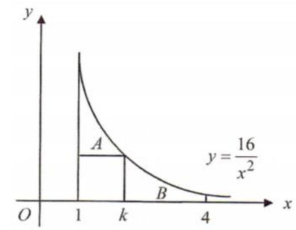
(ii) $= \int_0^5 \left[\frac{3}{8}x + \frac{17}{8} - (1+3x)^{1/2}\right] dx$

$$\left[\frac{3}{8}\left(\frac{x^2}{x}\right) + \frac{17}{8}x - \frac{(1+3x)^{3/2}}{3\left(3/2\right)}\right]_0^5$$

$$= \left[\frac{3}{8}\left(\frac{25}{2}\right) + \frac{17}{8}(5)\right] - \left[\frac{2(1-15)^{3/2}}{9} - \frac{2}{9}(1)\right]$$

$$= \frac{245}{16} - 14$$

$= 1\frac{5}{16}$ or 1.3125 units^2



The diagram shows part of the curve $y = \frac{16}{x^2}$. Also shown are lines perpendicular to the x-axis at the points with x-coordinates 1, k and 4.

Given that the areas of the regions marked A and B are equal, find the value of k.

When $x = k, y = \frac{16}{k^2}$

Area of A

$= \int_1^k \frac{16}{x^2} dx - (k-1)\left(\frac{16}{k^2}\right)$

$= \left[-\frac{16}{x}\right]_1^k - \left(\frac{16}{k} - \frac{16}{k^2}\right)$

$= -\frac{16}{k} - \left(-\frac{16}{1}\right) - \frac{16}{k} + \frac{16}{k^2}$

$= -\frac{32}{k} + 16 + \frac{16}{k^2}$

Area of B

$= \int_k^4 \frac{16}{x^2} dx$

$= \left[-\frac{16}{x}\right]_k^4$

$= -\frac{16}{4} - \left(-\frac{16}{k}\right)$

$= -4 + \frac{16}{k}$

Area of A = Area of B

$-\frac{32}{k} + 16 + \frac{16}{k^2} = -4 + \frac{16}{k}$

$\frac{16}{k^2} - \frac{48}{k} + 20 = 0$

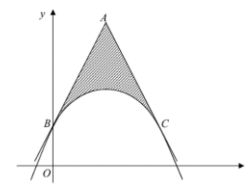
$16 - 48k + 20k^2 = 0$

$5k^2 - 12k + 4 = 0$

$(k-2)(5k-2) = 0$

$k = 2$ or $k = 0.4$ (N.A)

The diagram shows the graph of $y = -\frac{1}{2}(x-2)^4 + 16$.
AB and AC are tangents to the curve at B and C respectively.
B lies on the y-axis and AB = AC.



(i) Find the gradient function of the curve.

(ii) Find the equation of the tangent at B. Hence, state the coordinates of A.

(iii) Find the area of the shaded region.

(i) $y = -\frac{2}{3}(x-2)^4 + 16,$

$\therefore \frac{dy}{dx} = -2(x-2)^3$

(ii) Grad of AB $= -2(-8) = 16$

At B, $x = 0$, $\therefore y = 8$

Eqn AB: $y = 16x + 8$

$\therefore$ A is $(2, 40)$

(iii) Area OBACD = $(8 + 40) \times 2$

$= 96$ units^2

Area bounded by curve and axes

$= \int_0^4 \left(-\frac{1}{2}(x-2)^4 + 16\right) dx$

$= \left(-\frac{1}{10}(x-2)^5 + 16x\right)_0^4$

$= \left(-\frac{1}{10} \times 32 + 64\right) - \left(\frac{1}{10} \times 32\right)$

= 57.6

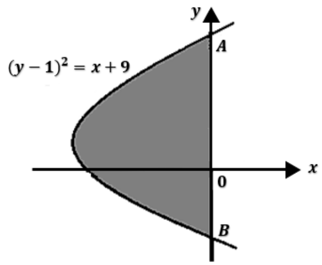
$\therefore$ shaded area $- 96 - 57.6 = 38.4$ units^2

Area under Graph

Integrating w.r.t Y Axis

The diagram shows the curve $(y-1)^2 = x+9$, which cuts the y-axis at A and B.

(i) Find the coordinates of A and B,
(ii) By integrating with respect to the y-axis, find the area of the region bounded by the curve and the y-axis.



(i) A and B on the y-axis, sub $x = 0$ into $(y-1)^2 = x+9$,

$$(y-1)^2 = 9$$
$$y-1 = \pm\sqrt{9}$$
$$y-1 = \pm 3$$
$$y-1 = \pm 3 + 1$$
$$y = -2 \quad or \quad y = 4$$

Judging from the graph, the coordinates of A is (0, 4) and B is (0, -2)

(ii) To integrate with respect to the y-axis

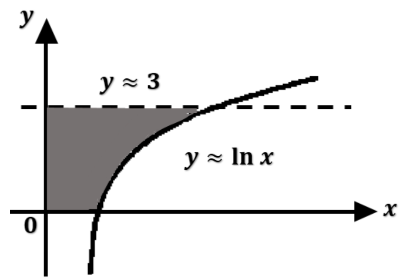
$$(y-1)^2 = x+9$$
$$x+9 = (y-1)^2$$
$$x = (y-1)^2 - 9$$

Area of region = $\left|\int_{-2}^{4} (y-1)^2 - 9\,dy\right|$

$$= \left|\left[\frac{(y-1)^2}{3\times 1} - 9y\right]_{-2}^{4}\right|$$
$$= \left|\left[\frac{(4-1)^3}{3} - 9(4)\right] - \left[\frac{(-2-1)^3}{3} - 9(4-2\right]\right|$$
$$= |-36|$$
$$= 36\,units^2$$

The diagram shows the curve $y = \ln x$.

Find the area of the region bounded by the curve, the x-axis, the y-axis and the line $y = 3$.



We have learnt in Topic 23 that we cannot integrate $y = \ln x$.

Thus, we have no choice but to integrate $x = e^y$ with respect to the y-axis instead.

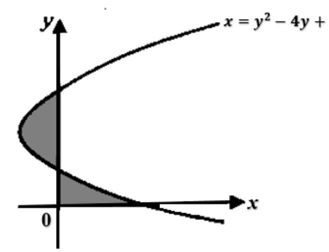
$$y = \ln x$$
$$\log_e x = y$$
$$x = e^y$$

Area of Region = $\int_0^3 e^y\,dy$

$$= [e^y]_0^3$$
$$= e^3 - e^0$$
$$= 19.1\,units^2\ (3sf)$$

The diagram shows part of the curve $x = y^2 - 4y + 3$.

Find the total area of the regions bounded by the curve, the x-axis and the y-axis



To Find y-intercepts
Cuts y-axis, sub $x = 0$ into $x = y^2 - 4y + 3$,
$y^2 - 4y + 3 = 0$
$(y-1)(y-3) = 0$
$y = 1$ or $y = 3$

Total area of regions

$$= \int_0^1 y^2 - 4y + 3\,dy + \left|\int_1^3 y^2 - 4y + 3\,dy\right|$$
$$= \left[\frac{y^3}{3} - \frac{4y^2}{2} + 3y\right]_0^1 + \left|\left[\frac{y^3}{3} - \frac{4y^2}{2} + 3y\right]_1^3\right|$$
$$= \left[\frac{y^3}{3} - 2y^2 + 3y\right]_0^1 + \left|\left[\frac{y^3}{3} - 2y^2 + 3y\right]_1^3\right|$$
$$= \left[\frac{1^3}{3} - 2(1)^2 + 3(1)\right] - 0 + \left|\left[\frac{3^3}{3} - 2(3)^2 + 3(3)\right] - \left[\frac{1^3}{3} - 2(1)^2 + 3(1)\right]\right|$$
$$= 1\frac{1}{3} + \left|-1\frac{1}{3}\right|$$
$$= 1\frac{1}{3} + 1\frac{1}{3} = 2\frac{2}{3}\ units^2$$

Kinematics

Displacement

Mr. Tan drives his car along a straight road. As he passes a point A he applies the brake and his car slows down, coming to a rest at point B. For the journey from A to B, the distance, s meters, of the car from A, t seconds after passing A, is given by

$$s = 600(1 - e^{6}) - 12t$$

(i) Find an expression, in terms of t, for the velocity of the car during the journey from A to B.
(ii) Find the velocity of the car at A.
(iii) Find the time taken for the journey from A to B.
(iv) Find the average speed of the car for the journey from A to B.

$v = 100e^{\frac{t}{6}} - 12$

$s = 600 - 600e^{\frac{t}{6}} - 12t$

$\frac{ds}{dt} = -600 \cdot e^{\frac{t}{6}} \cdot \left(-\frac{1}{6}\right) - 12$

$v = 100e^{\frac{t}{6}} - 12$

(ii) $v = 88 \text{ m/s}$

$v = 100e^{\frac{t}{6}} - 12$

$= 100 - 12$

$= 88 \text{ m/s}$

(iii) $t = 12.72\ s$

$0 = 100e^{\frac{t}{6}} - 12$

$100e^{\frac{t}{6}} = 12$

$-\frac{t}{6} = \ln\frac{12}{100}$

$t = 12.72\ s$

(iv) Ave speed $= 29.5 \text{ m/s}$

Ave speed $= \frac{\text{tot dist}}{\text{tot time}}$

$= \frac{600\left(1-e^{\frac{12.72}{6}}\right)-12(12.72)}{12.72}$

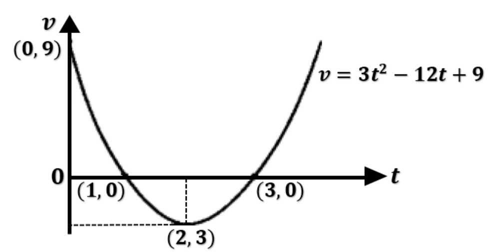
$= 29.5 \text{ m/s}$

A particle moves in a straight line so that its displacement, s m, from a fixed point O is given by $s = t^3 - 6t^2 + 9t + 18$, where t is the time in seconds after passing a point P on the line.

(i) Find the initial displacement of the particle from fixed point O.
(ii) Find the value(s) of t for which the particle is instantaneously at rest. Hence, show the that at one of the two instances of rest, the particle will return to its starting position.
(iii) Find the distance travelled by the particle during the first 4 seconds.
(iv) Calculate the minimum velocity of the particle.
(v) Sketch the velocity-time graph of the particle for $0 \le t \le 4$

Ans:

(i) 18 m
(ii) $t = 1$ *or* $t = 3$; Show that $s = 18$ when $t = 3$
(iii) 12 m
(iv)



Important Concepts

Validation

KINEMATICS

Velocity

A particle moves in a straight line so that, t seconds after passing a fixed-point O, its velocity, v m/s, is given by $v = 2t^2 - 16t + 30$.

(i) Find an expression, in terms of t, for the displacement of the particle.

(ii) Calculate the total distance travelled by the particle in the first 7 seconds.

$v = 2t^2 - 16t + 30$

$s = \frac{2t^3}{3} - \frac{16t^2}{2} + 30t + D$

$s = \frac{2t^3}{3} - 8t^3 + 30t + D$

Sub $t = 0, s = 0$:

$D = 0$

When $t = 3$,

$s = -8(3)^2 + \frac{2(3)^2}{3} + 30(3)$

$= 36$ m

When $t = 5$,

$s = -8(5)^2 + \frac{2(5)^2}{3} + 30(5) = 33\frac{1}{3}$ m

When $t = 7$,

$s = -8(7)^2 + \frac{2(7)^3}{3} + 30(7)$

$= 46\frac{2}{3}$ m

Total distance travelled in 1st 7 seconds

$= 36 + \left(36 - 33\frac{1}{3}\right) + \left(46\frac{2}{3} - 33\frac{1}{3}\right)$

$= 52$ m

A moving particle P starts with a velocity of 7 m/s from a point O and moves in a straight line so that its acceleration after t seconds is given by $a = (20 - 6t)$ m/s². Find

(i) the value of t when the speed is at maximum,

(ii) the total distance travelled by the particle during the fourth second.

For maximum velocity,

$20 - 6t = 0$

$t = 3\frac{1}{3}$

$\frac{d^2v}{dt^2} = -6$

At $t = 3\frac{1}{3}$, velocity is a maximum.

$v = \int 20 - 6t \, dt$

$v = 20t - 3t^2 + c$

At $t = 0, v = 7$

$\therefore c = 7$

$v = 20t - 3t^2 + 7$

$s = \int 20t - 3t^2 + 7 \, dt$

$s = 10t^2 - t^3 + 7t + c$

At $t = 0, s = 0$

$\therefore c = 0$

$s = 10t^2 - t^3 + 7t$

At $t = 3$,

$s = 10(3)^2 - (3)^3 + 7(3)$

$s = 84$ m

At $t = 4$,

$s = 10(4)^2 - (4)^3 + 7(4)$

$s = 124$ m

Total distance travelled $= 124 - 84$

$= 40$ m

The velocity, v ms⁻¹, of a particle travelling in a straight line at time t seconds after leaving a fixed point O, is given by

$$V = 2t^2 + (1 - 3k)t + 8k - 1,$$

where k is a constant. The velocity is a minimum at $t = 5$.

(i) Show that $k = 7$.

(ii) Show that the particle will never return to O with time.

(iii) Find the duration when its velocity is less than 13 ms⁻¹.

(iv) Find the distance travelled by the particle during the third second

(i) $\frac{dv}{dt} = 4t + (1 - 3k)$

When vel is a minimum $\frac{dv}{dt} = 0$

$4(5) + (1 - 3k) = 0$

$3k = 21$

$k = 7$(shown)

(ii) When $k = 7$, $v = 2t^2 - 20t + 55$

Discriminant $= (-20)^2 - 4(2)(55)$

$= 400 - 440$

$= -40$

< 0

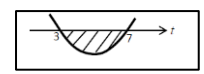
$\Rightarrow$ there is no real values of t such that vel $= 0$, also vel > 0 hence particle will never return to O with time.

(iii) $2t^2 - 20t + 55 < 13$

$2t^2 - 20t + 42 < 0$

$t^2 - 10t + 21 < 0$

$(t - 7)(t - 3) < 0$



$\therefore 3 < t < 7$

Duration $= 7 - 3 = 4$ s

(iv) $s = \int_2^3 (2t^2 - 20t + 55)dt$

$= \left[\frac{2t^3}{3} - 10t^2 + 55t\right]_2^3$

$= [18 - 90 + 165] - \left[\frac{16}{3} - 40 + 110\right]$

$= 17\frac{2}{3}$ m or 17.7 m (3sf)

Kinematics

Acceleration

A particle moving in a straight line passes a fixed point O with a velocity 6 ms^{-1}.
The acceleration of the particle, a ms^{-2}, is given by $a = 2t - 5$, where t seconds is the time after passing O. Find

(i) the values of t when the particle is instantaneously at rest,
(ii) the displacement of the particle from O at t = 3,
(iii) the total distance travelled by the particle in the first 3 seconds of its motion.

$a = 2t - 5$
Let $v = \int 2t - 5\, dt$
$= t^2 - 5t + c$
When $t = 0, v = 6$.
$\therefore c = 6$
$v = t^2 - 5t + 6$
When $v = 0$, $t^2 - 5t + 6 = 0$.
$(t - 2)(t - 3) = 0$
$t = 2$ or $t = 3$
Particle is instantaneously at rest when t = 2 and t = 3

Let $s = \int t^2 - 5t + 6\, dt$
$= \frac{t^3}{3} - \frac{5t^2}{2} + 6t + c_1$
When $t = 0, s = 0, \therefore c_1 = 0$.
$s = \frac{t^3}{3} - \frac{5t^2}{2} + 6t$
When $t = 3$, $s = \frac{3^3}{3} - \frac{5\times3^2}{2} + 6 \times 3$
$= 9 - \frac{45}{2} + 18$
$= 4\frac{1}{2}$
Displacement of particle from O at t = 3 is $4\frac{1}{2}$ m
When $t = 2$, $s = \frac{2^3}{3} - \frac{5\times2^2}{2} + 6 \times 2$
$= \frac{8}{3} - 10 + 12$
$= 4\frac{2}{3}$
Distance travelled in the first 3 seconds $= 4\frac{2}{3} + 4\frac{2}{3} - 4\frac{1}{2}$ m
$= 4\frac{5}{6}$ m

A particle traveling in a straight line passes through a fixed point O with a speed of −10 m/s. The acceleration, a m/s^2, of the particle, t s after passing through O, is given by $a = \frac{24}{(2t+1)^2}$. The particle comes to instantaneous rest at the point P.

(i) Find the time when the particle reaches P.
(ii) Calculate the distance travelled by the particle in the first 3 sec.
(iii) Show that the particle is again at O at some instant during the ninth second after first passing through O.

$v = \frac{24(2t+1)^{-1}}{2(-1)} + c$
$= \frac{12}{2t+1} + c$
When $t = 0, v = -10$ m/s
$\therefore c = 2$
$\therefore v = 2 - \frac{12}{2t+1}$
At P, $v = 0 \Rightarrow 2 - \frac{12}{2t+1} = 0$
$\Rightarrow t = 2.5s$

$s = 2t - 12\frac{\ln(2t+1)}{2} + c_1$.
$= 2t - 6\ln(2t + 1) + c_1$
When $t = 0, s = 0, \therefore c_1 = 0$
$\therefore s = 2t - 6\ln(2t + 1)$
$t = 0, s = 0$
$t = 2.5, s = 2(2.5) - 6\ln 6 = -5.750\,5$
$t = 3, s = 2(3) - 6\ln 7 = -5.675\,4$
Distance travelled $= 5.750\,5 + (5.750\,5 - 5.675\,4)$
$= 5.83$ m (3 sf)

[9^{th} second means from t = 8 s to t = 9 s]
When $t = 8$, $s = 2(8) - 6\ln 17 = -0.999\,28$ m
When $t = 9$, $s = 2(9) - 6\ln 19 = +0.333\,36$ m
$\therefore s = 0$ for $8 < t < 9$
i.e The particle is again at O during the 9^{th} sec.

Important Concepts

Validation

Kinematics

2 Particles

Two particles A and B, leave a point O at the same time and travel in the same direction along the same straight line.
Particle A starts with a velocity of 9 m/s and moves with a constant acceleration of 2 m/s^2.
Particle B starts from the rest and moves with an acceleration of a m/s^2, where a = 1 + $\frac{t}{3}$ and t seconds is the time travel since leaving O. Find

(a) Expression for the velocity of each particle in terms of t,
(b) Expression for the displacement of each particle in terms of t,
(c) The distance from O at which particle B collides with A,
(d) The speed of each particle at the point of collision.

(a) For Particle A,
$v_A = \int 2\,dt$
$v_A = 2t + c$
When $t = 0, v_A = 9$
$c = y$
$v_A = 2t + 9$

For particle B,
$v_B = \int \left(1 + \frac{t}{3}\right) dt$
$v_B = t + \frac{1}{6}t^2 + c$
When $t = 0$, $v_B = 0$,
$c = 0$

(b) For particle A,
$s_A = \int (2t + 9) dt$
$s_A = t^2 + 9t + c$
When $t = 0, s_A = 0$
$c = 0$
$s_A = t^2 + 9t$

For particle B,
$s_B = \int \left(t + \frac{1}{6}t^2\right) dt$
$s_B = \frac{1}{2}t^2 + \frac{1}{18}t^3 + c$
When $t = 0$, $s_B = 0$,
$c = 0$

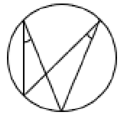
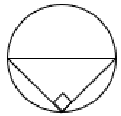
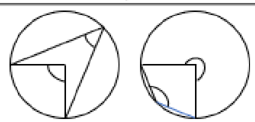
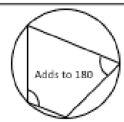
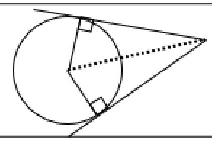
(c) When particle B collides with particle A,
$s_A = s_B$
$t^2 + 9t = \frac{1}{2}t^2 + \frac{1}{18}t^3$
$18t^2 + 162t = 9t^2 + t^3$
$t^3 - 9t^2 - 162t = 0$
$t = (t^2 - 9t - 162) = 0$
$t(t - 18)(t + 9) = 0$
$t = 0$ (N.A.) or $t = 18$ or $t = -9$(N.A.)

Distance from $0 = (18)^2 + 9(18) = 486$ m
Speed of particle A = $2(18) + 9 = 45$ m/s
Speed of particle B = $(18) + \frac{1}{6}(18)^2 = 72$ m/s

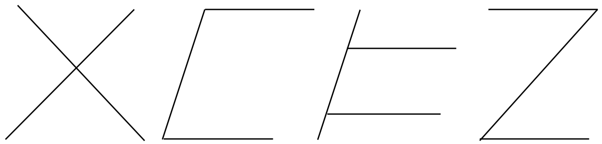
Important Concepts

Validation

Circle Properties

Angles in the same segment are equal.	
Angle in a semicircle is a right angle, 90.	
Angle at the centre is twice the angles at the circumference.	
Angles in opposite segments of a circle are supplementary. (opposite angles in a cyclic quadrilateral are supplementary.)	
A tangent to a circle is perpendicular to the radius at the point of contact. Tangent from external point are equal.	

Angle Properties



<u>V</u>ertically Opposite Angles

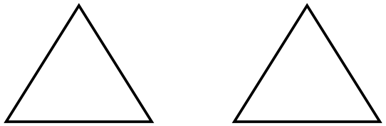
<u>I</u>nterior Angles

<u>C</u>orresponding Angles

<u>A</u>lternate Angles

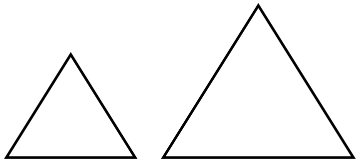
Congruence and Similarity

<u>Congruent Triangles</u>



You can prove by SSS, SAS, AAS, RHS
You **<u>CANNOT</u>** prove using AAA or ASS!

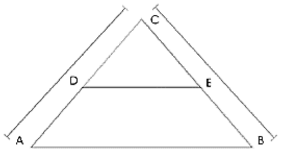
<u>Similar Triangles</u>



All 3 Corresponding Angles are same (AA)

All 3 Corresponding Sides have the same **<u>RATIO</u>.**

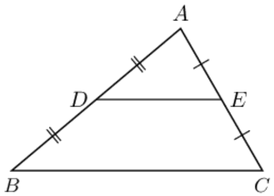
2 of the Corresponding Sides have the same **<u>RATIO</u>**
& included Angles are the same.



Triangle CDE is similar to CAB.

$\frac{DC}{AC} = \frac{EC}{BC}$ Diagram is confusing, Look the name of triangle

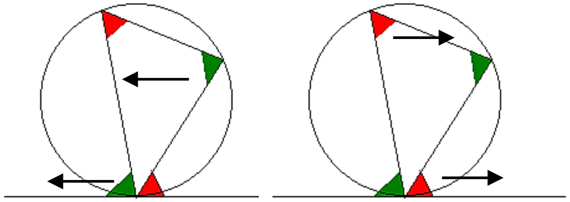
Midpoint Theorem



D is a midpoint of AB
E is a midpoint of AC
DE is parallel to BC

This means that ADE is similar to ABC.
Therefore, $DE = \frac{1}{2}BC$

Alternate Segment Theorem

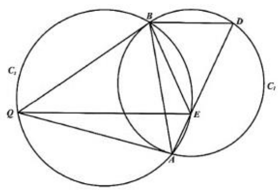


PLANE GEOMETRY

Alternate Segment Theorem

The diagram shows two intersecting circles, C_1 and C_2. C_1 passes through the vertices of the triangle ABD. The tangents to C_1 at A and B intersect at the point Q on C_2. A line is drawn from Q to intersect the line AD at E on C_2.

Prove that
(i) QE bisects angle AEB,
(ii) $EB = ED$,
(iii) BD is parallel to QE.



Answer:

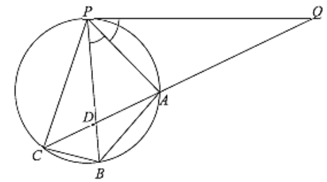
(i) Let $\angle QEA = x^\circ$

$\angle QBA = \angle QEA$ (angles in same segment in C_2)
$= x^\circ$
$QB = QA$ (tangents to C_1 from external point Q)

$\angle QAB = \angle QBA$ (base angles of isosceles triangle)
$= x^\circ$
$\therefore \angle QEB = \angle QEA$
Hence, *QE* bisects angle *AEB*

(ii) $\angle QBA = x^\circ$(from (i))
$\angle ADB$= $\angle QBA$ (angles in alternate segment in C_1) either
$= x^\circ$
$\angle AEB = 2x^\circ$(from (i))
$\angle DBE = \angle AEB$- $\angle ADB$ (exterior angle of triangle *BDE*)
$= 2x^\circ - x^\circ$
$= x^\circ$
$\therefore \angle ADB = \angle EDB = \angle EBD = x^\circ$(base angles of isosceles triangle BDE)
Hence $EB = ED$
(iii) From (i) $\angle EBD = \angle QEB = x$
$\therefore \angle EBD$ and $\angle QEB$ are alternate angles of parallel lines.
(alternate angles are equal)
BD is parallel to *QE*

The diagram shows a point P on a circle and PQ is a tangent to the circle. Points A, B and C lie on the circle such that PA bisects angle QPB and QAC is a straight line. The lines QC and PB intersect at D.



(i) Prove that $AP = AB$.
(ii) Prove that CD bisects angle PCB.
(iii) Prove that triangles CDP and CBA are similar.

Answer:
(i) $\angle ABP = \angle APQ$ (alt. segment theorem)
Since *PA* bisects $\angle QPB$,
$\angle$APQ $= \angle APB$
$\therefore \angle ABP = \angle APB$ (base $\angle s$ of isosceles triangle *APB*)
Hence, $AP = AB$.

(ii) $\angle ACB = \angle APB$ ($\angle s$ in the same segment)
$\angle ACP = \angle ABP$ ($\angle s$ in the same segment)
$= \angle APB$ (shown)
$\angle ACB = \angle ACP$
Hence, *CD* bisects $\angle PCB$

(iii) $\angle ACB = \angle ACP$ (from ii)
$\angle CPD = \angle CAB$ ($\angle s$ in the same segment)

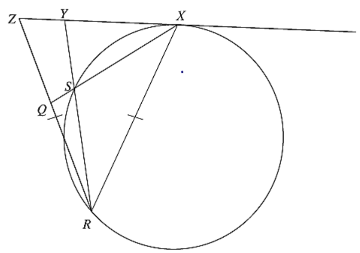
Hence, ΔCDX and ΔCBA are similar.

Important Concepts

PLANE GEOMETRY

Circles Application

In the figure, XYZ is a straight line that is tangent to the circle at X. XQ bisects $\angle RXZ$ and cuts the circle at S. RS produced meets XZ at Y and $ZR = XR$.



Prove that

(a) $SR = SX$,
(b) a circle can be drawn passing through Z, Y, S and Q.

Answer:

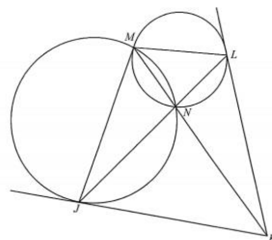
(a) $\angle ZXQ = \angle SRX$ (Alternate Segment Theorem)
$\angle ZXQ = \angle QXR$ (XQ is the angle bisector of $\angle RXZ$)
$\angle QXR = \angle SRX$
By base angles of isosceles triangles, $SR = SX$

(b) Let $\angle QXR$ be x
$\angle RSX = 180° - 2x$ (Isosceles Triangle)
$\angle YSQ = 180° - 2x$ (Vertically Opposite Angles)
$\angle RZX = \angle ZXR = 2x$ (Base angles of Isosceles Triangle)
$\angle RZX + \angle YSQ = 180° - 2x + 2x = 180°$

Since opposite angles are supplementary in cyclic quadrilaterals, a circle that passes through Z, Y, S and Q can be drawn.

Two circles intersect at M and N. K is a point on MN produced such that KL and KJ are tangents to the circles at L and J respectively and $KL = KJ$.
Given that LNJ is a straight line, show that

(i) Line KM bisects $\angle LMJ$.
(ii) If MK is perpendicular to LJ, explain why a circle with MK as diameter passes through L and J.



Answer:

(i) $\angle KJN = \angle KLN$ (given)
$\angle KJN = \angle JMN$ (tan-chord thm)
$\angle KLN = \angle LMN$ (tan-chord thm)
$\Rightarrow \angle JMN = \angle LMN$
$\therefore$ Line LON bisects $\angle LMJ$ (shown)

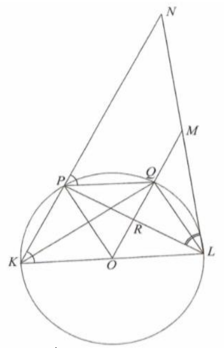
(ii) Given that $\angle MNL = \angle MNJ = 90°$ and
Let $\angle LMN = \angle JMN = \angle KLN = \angle KJL = x$,
$\therefore \angle NLM = 90° - x$ ($\angle s$ sum in ΔNML)
$\angle MLK = 90° - x + x$
$= 90°$
$\therefore \angle NJM = 90° - x$ ($\angle s$ sum in ΔNMJ)
$\angle MJK = 90° - x + x$
$= 90°$
Since $\angle MLK$ and $\angle MJK$ are 90°, they obey $\angle$ in semicircle property.
$\Rightarrow MK$ is a diameter of a circle which passes through L and J. (shown)

Important Concepts

PLANE GEOMETRY

Similar Triangles Method

The diagram shows a circle with centre O, diameter KL. NML is a tangent to the circle at L and M is the midpoint of NL. The lines KN and OM cut the circle at P and Q respectively. The lines PL and OQ intersect at R. The line LQ bisects $\angle RLM$ and $\angle NPQ = \angle NKL$.



(i) Prove that $OKPQ$ is a rhombus.
(ii) Prove that
$KQ \times RQ = LQ \times LR$.

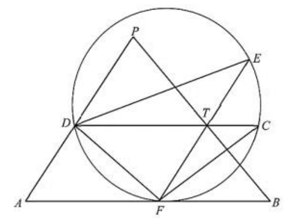
Answer:

(i) O is the centre of the circle
and KL is diameter (given)
$\Rightarrow O$ is the midpoint of KL
M is the midpoint of NL (given)
By Midpoint Theorem,

By Midpoint Theorem, $OM \parallel KN$ and $OM = \frac{1}{2}KN$.
$\Rightarrow OQ \parallel KP$
$\angle NPQ = \angle NKL$ (given)
$\Rightarrow PQ \parallel KO$ (corresponding angles are equal)
As $OQ \parallel KP$ and $PQ \parallel KO$, then $OKPQ$ is a parallelogram.
(2 pairs of opposite and parallel sides)
$OK = OQ$ (radii) of circle
As $OKPQ$ is a parallelogram and adjacent sides, OK and OQ are equal,
thus $OKPQ$ is a rhombus.
(ii) $\angle QKL = \angle NLQ$ (Alternate Segment Theorem)
$\angle RLQ = \angle NLQ$ (LQ bisects $\angle RLM$)
$\therefore \angle QKL = \angle RLQ$
Since $OQ = OL$ (radii of circle)
ΔOQL is an isosceles triangles,
$\angle OLQ = \angle OQL$ (base $\angle s$ of isosceles Δ)
$\Rightarrow \angle KLQ = \angle RQL$
$\therefore \Delta KLQ$ is similar to ΔLQR (AA Similarity)

Hence, $\frac{KQ}{LR} = \frac{LQ}{RQ}$ (ratio of corresponding sides are equal)
$\therefore KQ \times RQ = LQ \times LR$ (shown)

The diagram shows a circle passing through points D, E, C and F, where $FC = FD$. The point D lies on AP such that $AD = DP.DC$ and EF cut PB at T such that $PT = TB$.



(i)Show that AB is a tangent to the circle at point F.
(ii)By showing that triangle DFT and triangle EFD are similar show that $DF^2 - FT^2 = FT \times ET$.

Answer:

(i) DT is parallel to Ab (Midpoint Theorem)
$\angle AFD = \angle TDF$ (alt angles)
$= \angle FED$
Since $\angle AFD$ and $\angle FED$ satisfies the alternate segment theorem, AB is a tangent at F.

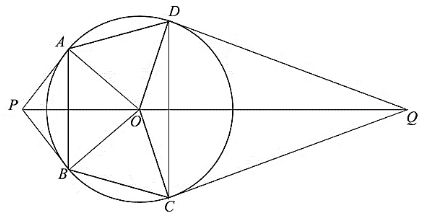
(ii) $\angle DFE$ is common.
$\angle TDF = \angle DCF$ (base angles of an isos triangle)
$\angle DCF = \angle DEF$ (angles in the same segment)
$\therefore DFT$ and EFD are similar triangles (AA)
$\frac{DF}{EF} = \frac{FT}{FD}$
$DF^2 = FT \times EF$
$DF^2 = FT \times (ET + TF)$
$DF^2 = FT^2 + FT \times ET$
$DF^2 - FT^2 = FT \times ET$

Important Concepts

PLANE GEOMETRY

Deduction Questions

In the diagram, A, B, C and D are points on the circle centre O. AP and BP are tangents to the circle at A and B respectively. Q and CQ are tangents to the circle at D and C respectively. POQ is a straight line.
(i)Prove that angle $COD = 2 \times$ angle CDQ.



(ii)Make a similar deduction about angle AOB.
(iii)Prove that $2 \times$ angle $OAD =$ angle $CDQ +$ angle BAP

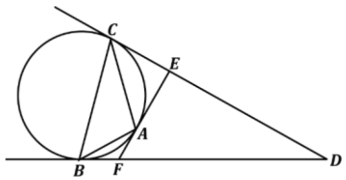
Answer:

(i) Let $\angle CDQ = a$
$\angle ODQ = 90°(tan \perp rad)$
$\therefore \angle ODC = 90° - a$
$\therefore \angle COD = 180° - 2(90° - a)$ $(\angle sum, \Delta COD)$

(ii) $\angle AOB = 2 \times \angle BAP$

(iii) From (i) and (ii),
$2(\angle CDQ + \angle BAP) = \angle COD + \angle AOB$
$\angle CDQ + \angle BAP = \frac{1}{2}(\angle COD + \angle AOB)$
$\angle CDQ + \angle BAP = \angle AOP + \angle DOQ$($\perp$ prop of chord)
$\angle CDQ + \angle BAP = 180° - \angle AOD$
$\angle CDQ + \angle BAP = 2\angle OAD$

The diagram shows a triangle ABC whose vertices lie on the circumference of a circle. The triangle DEF is formed by tangents drawn to the circle at the points A, B and C.



(i) Prove that angle $DEF = 2 \times$ angle ABC.
(ii) Make a similar deduction about angle DFE.
(iii) Prove that $2 \times$ angle $BAC = 180° +$ angle EDF.

Answer:

(i) ∠ABC = ∠ACE (Alternate Segment Theorem)
$AE = CE$ (tangents from ext. points)
$\therefore \Delta ACE$ is an isosceles triangle. Hence, $\angle ACE = \angle EAC$
$\therefore \angle DEF = \angle ACE + \angle EAC$ (ext ∠ of Δ),
$= \angle ACE + \angle ACE$
$= 2\angle ACE$
$= 2 \times \angle ABC$ (proven)
(ii) $\angle DFE = 2 \times \angle ACB$
(iii) $\angle BAC$
$= 180° - \angle ABC - \angle ACB$ (sum ∠ in Δ)
$= 180° - \frac{1}{2}\angle DEF - \frac{1}{2}\angle DFE$
$= 180° - \frac{1}{2}(\angle DEF + \angle DFE)$
$= 180° - \frac{1}{2}(180° - \angle EDF)$ ($\angle sum$ in Δ)
$= 90° + \frac{1}{2}\angle EDF$
$\therefore 2 \times \angle BAC = 2 \times \left(90° + \frac{1}{2}\angle EDF\right)$
$= 180° + \angle EDF$ (proven)

Important Concepts