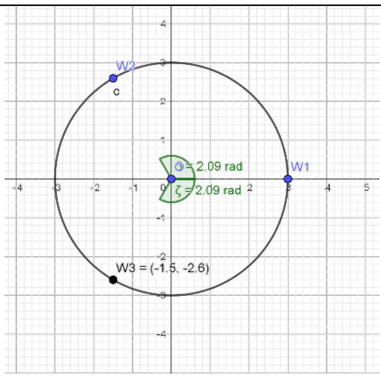
Section A: Pure Mathematics

Question 1

No.	Suggested Solution	Remarks for Student
(i)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\left(\frac{1}{3}y-15\right)^{\frac{1}{3}}$ $3\int\frac{1}{3}\left(\frac{1}{3}y-15\right)^{-\frac{1}{3}}\ \mathrm{d}y=\int\mathrm{d}x$ $\frac{9}{2}\left(\frac{1}{3}y-15\right)^{\frac{2}{3}}=x+C$ Curve passes through $(0,69)\Rightarrow C=\frac{9}{2}\left(8^{\frac{2}{3}}\right)=18$ $\frac{9}{2}\left(\frac{1}{3}y-15\right)^{\frac{2}{3}}=x+18$ $\left(\frac{1}{3}y-15\right)^{\frac{2}{3}}=\frac{2}{9}x+4$ $\frac{1}{3}y-15=\left(\frac{2}{9}x+4\right)^{\frac{3}{2}}$ $\mathrm{f}(x)=y=3\left(\frac{2}{9}x+4\right)^{\frac{3}{2}}+45$	
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\left(\frac{1}{3}y-15\right)^{\frac{1}{3}}=4$ $y=237$ $x=\frac{9}{2}\left(\frac{1}{3}(237)-15\right)^{\frac{2}{3}}-18=54$ Coordinates are (54, 237)	

Question 2

No.	Suggested Solution	Remarks for Student
(a)	$4x^4-20x^3+sx^2-56x+t=0$ Since the coefficients are real, $2+3\text{i}$ is also a root. $\left(x-(2-3\text{i})\right)\left(x-(2+3\text{i})\right)=x^2-4x+13$ $4x^4-20x^3+sx^2-56x+t=\left(x^2-4x+13\right)\left(4x^2+ax+b\right)$ comparing coefficients: $-20=a-16\Rightarrow a=-4$ $-56=13a-4b\Rightarrow b=1$ $t=13b=13$ $s=b-4a+52=69$ $4x^2-4x+1=0\Rightarrow(2x-1)^2=0$ Thus, the other roots are $2+3\text{i}$ and $\frac{1}{2}$.	
(b)	$w^3=27$	
(i)	$w^3-27=(w-3)\left(w^2+cw+d\right)$ Comparing constant terms, $-27=-3d\Rightarrow d=9$ Comparing coefficients of w, $0=-3c+d\Rightarrow c=3$ $w^2+3w+9=0$ $w=\frac{-3\pm\sqrt{9-36}}{2}=-\frac{3}{2}-\text{i}\frac{3\sqrt{3}}{2}$ or $-\frac{3}{2}+\text{i}\frac{3\sqrt{3}}{2}$	
(ii)	$w_1=3=3\text{e}^{\text{i}0}$ $w_2=-\frac{3}{2}+\text{i}\frac{3\sqrt{3}}{2}=3\text{e}^{\text{i}\frac{2\pi}{3}}$ $w_3=-\frac{3}{2}-\text{i}\frac{3\sqrt{3}}{2}=3\text{e}^{-\text{i}\frac{2\pi}{3}}$	

		
(iii)	Sum of roots = 0 Product of roots = 27	

Question 3

No.	Suggested Solution	Remarks for Student
(i)	$\overrightarrow{AD} = \overrightarrow{BC}$ $\overrightarrow{OD} - \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$ $\overrightarrow{OD} = \begin{pmatrix} -10 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix}$ D is (–5, –4, 3)	
(ii)	$\overrightarrow{BC} = \begin{pmatrix} -10 \\ 0 \\ 2 \end{pmatrix}$ $\overrightarrow{BE} = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 10 \end{pmatrix}$ $\overrightarrow{BC} \times \overrightarrow{BE} = \begin{pmatrix} -10 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} -5 \\ -4 \\ 10 \end{pmatrix} = \begin{pmatrix} 8 \\ 90 \\ 40 \end{pmatrix}$ $BCE : \underset{\sim}{r} . \begin{pmatrix} 8 \\ 90 \\ 40 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} . \begin{pmatrix} 8 \\ 90 \\ 40 \end{pmatrix} = 400$ $8x + 90y + 40z = 400 \Rightarrow 4x + 45y + 20z = 200$	

(iii)	$\overrightarrow{BC}=\begin{pmatrix}-10\\0\\2\end{pmatrix}$ $\overrightarrow{BA}=\begin{pmatrix}5\\-4\\1\end{pmatrix}-\begin{pmatrix}5\\4\\0\end{pmatrix}=\begin{pmatrix}0\\-8\\1\end{pmatrix}$ $\overrightarrow{BC}\times\overrightarrow{BA}=\begin{pmatrix}-10\\0\\2\end{pmatrix}\times\begin{pmatrix}0\\-8\\1\end{pmatrix}=\begin{pmatrix}16\\10\\80\end{pmatrix}=2\begin{pmatrix}8\\5\\40\end{pmatrix}$ $\cos\theta=\dfrac{\begin{pmatrix}8\\5\\40\end{pmatrix}.\begin{pmatrix}4\\45\\20\end{pmatrix}}{\sqrt{8^2+5^2+40^2}\sqrt{4^2+45^2+20^2}}$ $\theta=58.630^\circ=58.6^\circ$	
(iv)	Let M be the mid-point of AD $\overrightarrow{OM}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OD}\right)=\frac{1}{2}\left[\begin{pmatrix}5\\-4\\1\end{pmatrix}+\begin{pmatrix}-5\\-4\\3\end{pmatrix}\right]=\begin{pmatrix}0\\-4\\2\end{pmatrix}$ Required distance $=\left\lvert\overrightarrow{ME}.\hat{\underset{\sim}{n}}\right\rvert$ $=\dfrac{\left[\begin{pmatrix}0\\0\\10\end{pmatrix}-\begin{pmatrix}0\\-4\\2\end{pmatrix}\right].\begin{pmatrix}4\\45\\20\end{pmatrix}}{\sqrt{2441}}$ $=\dfrac{340}{\sqrt{2441}}\approx 6.88$	

Question 4

No.	Suggested Solution	Remarks for Student
(i)	$\ln(\cos 2x) = \ln\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\dots\right)$ $= \ln\left(1+\left(-2x^2+\frac{2x^4}{3}-\frac{4x^6}{45}+\dots\right)\right)$ $=\left(-2x^2+\frac{2x^4}{3}-\frac{4x^6}{45}+\dots\right)-\frac{1}{2}\left(-2x^2+\frac{2x^4}{3}-\frac{4x^6}{45}+\dots\right)^2$ $+\frac{1}{3}\left(-2x^2+\frac{2x^4}{3}-\frac{4x^6}{45}+\dots\right)^3$ $\approx -2x^2+\frac{2x^4}{3}-\frac{4x^6}{45}-\frac{1}{2}\left(4x^4-\frac{8x^6}{3}\right)-\frac{8x^6}{3}$ $= -2x^2-\frac{4x^4}{3}-\frac{64x^6}{45}$ Not valid for $x=\frac{\pi}{4}$ since $\ln\left(\cos 2\left(\frac{\pi}{4}\right)\right)=\ln 0$ which is undefined	
(ii)	$\int \frac{\ln(\cos 2x)}{x^2}\,dx \approx \int \frac{-2x^2-\frac{4x^4}{3}-\frac{64x^6}{45}}{x^2}\,dx$ $=\int\left(-2-\frac{4x^2}{3}-\frac{64x^4}{45}\right)dx$ $=-2x-\frac{4x^3}{9}-\frac{64x^5}{225}+C$ $\int_0^{0.5}\frac{\ln(\cos 2x)}{x^2}\,dx \approx \left[-2x-\frac{4x^3}{9}-\frac{64x^5}{225}\right]_0^{0.5} = -1.0644$ (4 d.p.)	
(iii)	$\int_0^{0.5}\frac{\ln(\cos 2x)}{x^2}\,dx \approx -1.0670$ (4 d.p.)	To remind us that answers from GC is an approximation.

Section B: Statistics

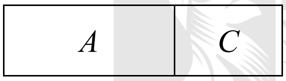
Question 5

No.	Suggested Solution	Remarks for Student
(i)	As the manager does not know how the MTTF is distributed, he needs to have a random sample of size large enough so that he could apply central limit theorem on the sample mean on MTTF. In general, 30 is considered large. The fans have to chosen randomly, example he could label the *N* number (assume very large population) of fans manufactured in the day from 1 to *N* and generate *n* (sample size of at least 30) distinct numbers from {1, 2, … , *N*) using random function of a calculator. The fans labeled according to the numbers generated will be the sample.	
(ii)	Null hypothesis, $H_0 : \mu = 65000$ Alternative hypothesis, $H_1 : \mu < 65000$ where μ is the population MTTF.	
(iii)	Perform an one-tailed test at 5% significance level. Under H_0, $\bar{X} \sim N\left(65000, \frac{s^2}{43}\right)$ approximately by Central Limit Theorem since $n = 43$ is large No reason to reject H_0 (that is do not reject H_0), p-value > 0.05 $P\left(\bar{X} < 64230\right) > 0.05$ $P\left(Z < \frac{64230 - 65000}{s/\sqrt{43}}\right) > 0.05$ $P\left(Z < \frac{-770\sqrt{43}}{s}\right) > 0.05$ $s > 3069.7127$ $s^2 > 9423136.061$ that is, $s^2 \geq 9420000$ (3 s.f.)	

Question 6

No.	Suggested Solution	Remarks for Student
(i)	Note that for any path allowing the bug to move from S to D, the bug has to take 5 left forks and 3 right forks. The required probability is ${}^8C_5 p^5 q^3 = 56p^5q^3$.	
(ii)	This is a binomial distribution in disguise. Let X = no. of left forks out of 8. (the rest will be right forks) $X \sim \mathrm{B}(8, p)$ We want ${}^8C_5 p^5 q^3 = \mathrm{P}(X=5)$ to be the largest, so $\mathrm{P}(X=0) \le \mathrm{P}(X=1) \ldots \le \mathrm{P}(X=4) < \mathrm{P}(X=5) > \mathrm{P}(X=6) \ge \mathrm{P}(X=7) \ge \mathrm{P}(X=8)$ $\mathrm{P}(X=4) < \mathrm{P}(X=5)$ *and* $\mathrm{P}(X=5) > \mathrm{P}(X=6)$ ${}^8C_4 p^4 q^4 < {}^8C_5 p^5 q^3$ ${}^8C_5 p^5 q^3 > {}^8C_6 p^6 q^2$ $70q < 56p$ $56q > 28p$ $70(1-p) < 56p$ $56(1-p) > 28p$ $p > \frac{5}{9}$ $p < \frac{2}{3}$ Thus, $\frac{5}{9} < p < \frac{2}{3}$	
(iii)	$0.9^8 = 0.430$ (3 s.f.)	

Question 7

No.	Suggested Solution	Remarks for Student
	$\mathrm{P}(A)=a,\mathrm{P}(B)=b,\mathrm{P}(C)=c$	
(i)	$\mathrm{P}(A)=a,\mathrm{P}(B)=b,\mathrm{P}(C)=c$ $\mathrm{P}(A'\cap B')=1-\mathrm{P}(A\cup B)$ $=1-\mathrm{P}(A)-\mathrm{P}(B)+\mathrm{P}(A\cap B)$ $=1-a-b+ab$ since A and B are independent, so $\mathrm{P}(A\cap B)=\mathrm{P}(A)\mathrm{P}(B)$ $=1-a-b(1-a)$ $=(1-a)(1-b)$ $=\mathrm{P}(A')\mathrm{P}(B')$ Thus, A' and B' are independent.	
(ii)	$\mathrm{P}(A'\cap C')=1-\mathrm{P}(A\cup C)$ $=1-\mathrm{P}(A)-\mathrm{P}(C)+\mathrm{P}(A\cap C)$ $=1-a-c$ since A and C are mutually exclusive, so $\mathrm{P}(A\cap C)=0$ Below is a possible venn diagram where $A\cup C=\Omega$ with $A'=C$ and $C'=A$ $A'\cap C'=C\cap A=\phi$  Note that if A' and C' are mutually exclusive , $\mathrm{P}(A'\cap C')=0$ $1-a-c=0$ $\mathrm{P}(A)+\mathrm{P}(C)=1$ Note also $C\cap A=\phi$ $\Rightarrow C\subseteq A'$ (1) $A'\cap C'=\phi$ $\Rightarrow A'\subseteq(C')'=C$ (2) $(1),(2)$ $A'=C$	
	$\mathrm{P}(A)=\frac{2}{5},\mathrm{P}(B\cap C)=\frac{1}{5},\mathrm{P}(A'\cap B'\cap C')=\frac{1}{10}$ $\mathrm{P}(A'\cap C')=\frac{3}{5}-c$	

(iii)

$$P(A' \cap B' \cap C') = \frac{1}{10}$$

$$\Rightarrow 1 - P(A \cup B \cup C) = \frac{1}{10}$$

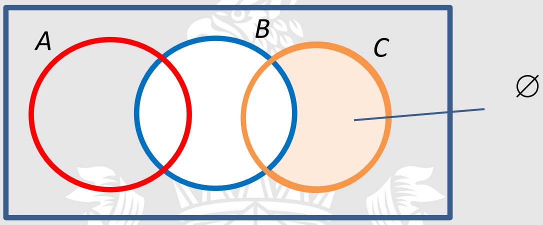
$$\Rightarrow P(A \cup B \cup C) = \frac{9}{10} \cdots (1)$$

$P(A \cap B) = ab$ ($\because A, B$ are independent)

$$= \frac{2}{5}b \quad \cdots (2)$$

<u>Method 1</u>

Maximum value of $P(A \cap B)$ occurs when b is maximum and c is minimum, ie when $C \subseteq B$.



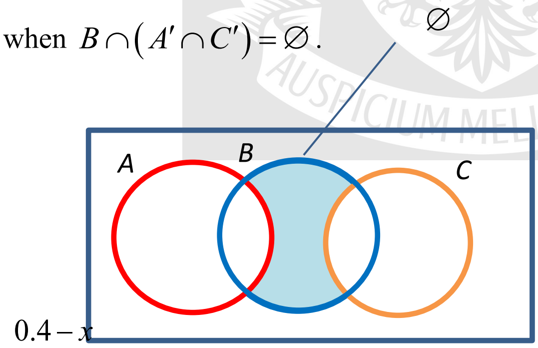
$$P(A \cup B \cup C) = P(A) + P(B) - P(A \cap B) = \frac{2}{5} + b - \frac{2}{5}b = \frac{9}{10}$$

$$\Rightarrow b = \frac{1}{2}\left(\frac{5}{3}\right) = \frac{5}{6}$$

So maximum value of $P(A \cap B) = \frac{2}{5}\left(\frac{5}{6}\right) = \frac{1}{3}$

Minimum value of $P(A \cap B)$ occurs when b is minimum and c is maximum,

ie when $B \cap (A' \cap C') = \varnothing$.

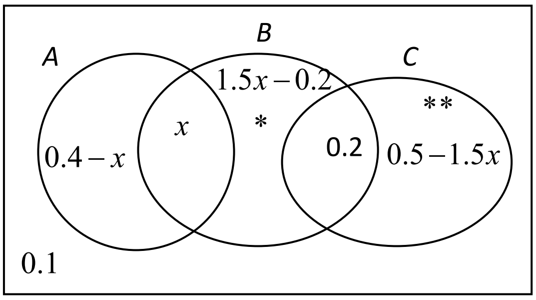


So $P(B) = P(A \cap B) + P(B \cap C) \Rightarrow b = \frac{2}{5}b + \frac{1}{5} \Rightarrow b = \frac{1}{3}$

Minimum value of $P(A \cap B) = \frac{2}{5}\left(\frac{1}{3}\right) = \frac{2}{15}$

Method 2

Let $\mathrm{P}\left(A \cap B\right) = x$. From (2), $b = \frac{5}{2}x$.



(*): $\frac{5}{2}x - x - 0.2$

(**): $0.9 - [0.4 + \frac{5}{2}x - x\,]$

Since all the probabilities are nonnegative, we have

$x \geq 0$, $0.4 - x \geq 0$, $1.5x - 0.2 \geq 0$ and $0.5 - 1.5x \geq 0$.

Thus we have $x \geq 0$, $x \leq 0.4$, $x \geq \frac{2}{15}$ and $x \leq \frac{1}{3}$.

Combining all together we have $\frac{2}{15} \leq x \leq \frac{1}{3}$.

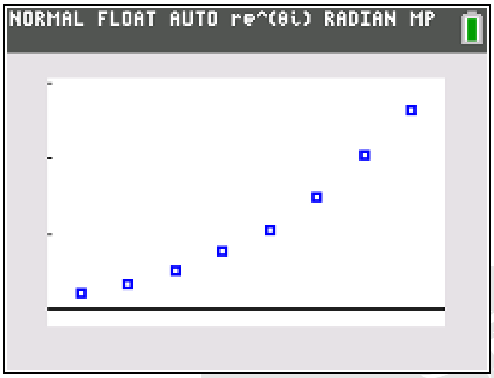
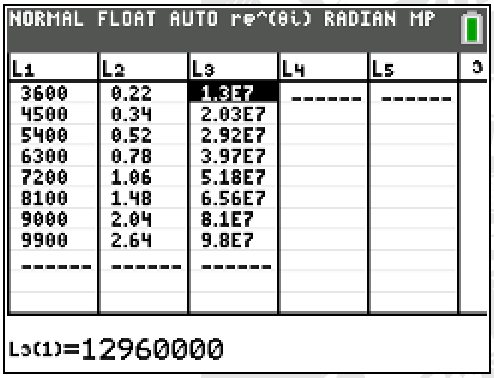
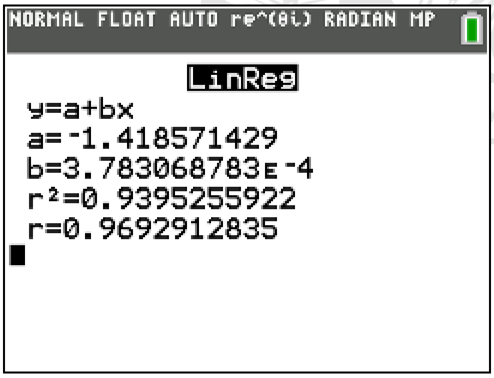
Minimum and maximum value of $\mathrm{P}(A \cap B)$ is $\frac{2}{15}$ and $\frac{1}{3}$ respectively.

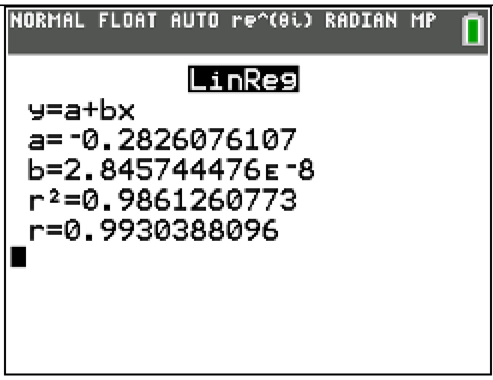
Question 8

No.	Suggested Solution	Remarks for Student
	$3,3,4,4,4,\underbrace{5,5,...,5}_{n \text{ terms}}$	
(i)	S = sum of numbers on the 2 balls taken <table><tr><td>s</td><td>$\mathrm{P}(S=s)$</td></tr><tr><td>6</td><td>$\frac{{}^2C_2}{{}^{n+5}C_2}$ $=\frac{2}{(n+5)(n+4)}$</td></tr><tr><td>7</td><td>$\frac{{}^2C_1\,{}^3C_1}{{}^{n+5}C_2}$ $=\frac{12}{(n+5)(n+4)}$</td></tr><tr><td>8</td><td>$\frac{{}^3C_2+{}^2C_1\,{}^nC_1}{{}^{n+5}C_2}$ $=\frac{6+4n}{(n+5)(n+4)}$</td></tr><tr><td>9</td><td>$\frac{{}^3C_1\,{}^nC_1}{{}^{n+5}C_2}$ $=\frac{6n}{(n+5)(n+4)}$</td></tr><tr><td>10</td><td>$\frac{{}^nC_2}{{}^{n+5}C_2}$ $=\frac{n(n-1)}{(n+5)(n+4)}$</td></tr></table>	
(ii)	$\mathrm{P}(S=10)=0$ This is an impossible event as there is only one ball with number 5 and it is impossible to get a sum of 10 with two 3s and/or 4s.	

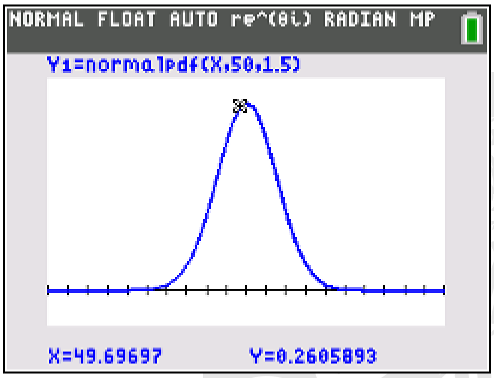
(iii)	$\begin{aligned} \mathrm{E}(S) &= \sum s\mathrm{P}(S=s) \\ &= \frac{12+84+48+32n+54n+10n(n-1)}{(n+5)(n+4)} \\ &= \frac{10n^2+76n+144}{(n+5)(n+4)} \\ &= \frac{(10n+36)(n+4)}{(n+5)(n+4)} \\ &= \frac{10n+36}{n+5} \end{aligned}$	
(iv)	$\begin{aligned} \mathrm{E}(S^2) &= \sum s^2\mathrm{P}(S=s) \\ &= \frac{72+588+384+256n+486n+100n(n-1)}{(n+5)(n+4)} \\ &= \frac{100n^2+642n+1044}{(n+5)(n+4)} \\ \mathrm{var}(S) &= \frac{100n^2+642n+1044}{(n+5)(n+4)} - \left(\frac{10n+36}{n+5}\right)^2 \\ &= \frac{(100n^2+642n+1044)(n+5)-(100n^2+720n+1296)(n+4)}{(n+5)^2(n+4)} \\ &= \frac{100n^3+642n^2+1044n+500n^2+3210n+5220-(100n^3+720n^2+1296n+400n^2+2880n+5184)}{(n+5)^2(n+4)} \\ &= \frac{22n^2+78n+36}{(n+5)^2(n+4)} \end{aligned}$	Idea is simple but you need to be careful in algebraic manipulation.

Question 9

No.	Suggested Solution	Remarks for Student
(i)	 The scatter diagram suggests that there is a non-linear (looks quadratic) spread of the points. It may not be well-modeled by $P = aR + b$	
(ii)	  For model, $P = aR + b$ $r = 0.96929$	

	 For model, $P = aR^2 + b$ $r = 0.99304$ Since the product moment correlation coefficient, r, for model $P = aR^2 + b$ is nearer to 1 compared to model $P = aR + b$, we confirmed the relationship by drawing the scatter diagram. Thus, $P = 0.000000028457R^2 - 0.28261$	
(iii)	$\sqrt{41557055} = 6446.476$ Reliable as 1. 0.9 is within the given range. 2. We have justified in (ii) the model is good, using the value of r and the scatter diagram.	
(iv)	0.027294. Not reliable as 3300 is not within the given range.	
(v)	$P = 0.000000028457R^2 - 0.28261$ where R is in rev/min Given R in rev/s, then we need to multiply R by 60 $P = 0.000000028457(60R)^2 - 0.28261$ $= 0.00010245R^2 - 0.28261$	

Question 10

No.	Suggested Solution	Remarks for Student
	Let X denote mass of a light bulb in grams. $X \sim \mathrm{N}\left(50, 1.5^2\right)$	
(i)		Use pdf instead of cdf when sketching the graph
(ii)	$\mathrm{P}(X < 50.4) = 0.60514 \approx 0.605$	
	Let E denote mass of an empty box in grams. $E \sim \mathrm{N}\left(75, 2^2\right)$	Assume (logically) in grams same as bulb
(iii)	Let $W = E_1 + E_2 + E_3 + E_4$ $W \sim \mathrm{N}\left(300, 4^2\right)$ $\mathrm{P}(W > 297) = 0.77337 \approx 0.773$	
(iv)	$X + E \sim \mathrm{N}\left(125, 2.5^2\right)$ $\mathrm{P}(124.9 < X + E < 125.7) = 0.12621 \approx 0.126$	
	Let Y denote mass of the padding of a bulb in grams. $Y = 0.3X \sim \mathrm{N}\left(15, 0.45^2\right)$	Assume (logically) in grams same as bulb
(v)	$X + Y + E \sim \mathrm{N}(140, 6.4525)$ $\mathrm{P}(X + Y + E > k) = 0.9$ $k = 136.745 \approx 137$	

(vi)	Let mass of a box in grams, $B = X + Y + E \sim \mathrm{N}(140, 6.4525)$ $T = B_1 + B_2 + B_3 + B_4 \sim \mathrm{N}(560, 25.81)$ $\mathrm{P}(T > 565) = 0.16251 \approx 0.163$	