VICTORIA JUNIOR COLLEGE

JC 1 PROMOTIONAL EXAMINATION 2023

H2 Further Mathematics 9649/01

Paper 1 3 hours

Additional Materials: Answer Paper
Graph Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

1 **Do not use a calculator in answering this question.**

Let w be a non-real cube root of unity.

(a) Prove that $w^2 + w = -1$. [2]

(b) Hence, or otherwise, show that $(1+2w+3w^2)(1+2w^2+3w) = 3$. [3]

2 The set of points (x, y) which are equidistant from the line $x = d$ and the point $(0, 2d)$, where d is a positive constant, is denoted by P.

Using this geometrical definition, show that $(y-2d)^2 = d(d-2x)$. [2]

Determine the volume of the solid formed when the finite region bounded by P and the y-axis is rotated completely about the x-axis. Leave your answer in exact form in terms of π and d. [5]

~~**3** **(a)** Show that~~ $\cancel{\dfrac{2k}{2k+1} < \dfrac{2k+2}{2k+3}}$ ~~for~~ $\cancel{k > 0}$. ~~[2]~~

~~**(b)** The sequence~~ $\cancel{u_1, u_2, u_3, \ldots}$ ~~is such that~~ $\cancel{u_1 = \dfrac{2}{3}}$ ~~and~~ $\cancel{u_n = \dfrac{2n}{2n+1}u_{n-1}}$.

~~Prove by induction that~~ $\cancel{u_n \le \left(\dfrac{2n}{2n+1}\right)^n}$ ~~for all positive integers~~ ~~n.~~ ~~[5]~~

4 Let $f(x) = x^k - a$, where k is a positive integer and $a > 0$. The number $a^{1/k}$ is a root of the equation $f(x) = 0$.

(a) Show that for any initial estimate x_1 $(x_1 > 0)$ of $a^{1/k}$, the Newton-Raphson method gives the iteration formula $x_{n+1} = \dfrac{1}{k}\left[(k-1)x_n + \dfrac{a}{x_n^{\,k-1}}\right]$. [2]

The iteration formula established in part **(a)** is used to approximate $\sqrt[3]{25}$.

(b) Without using a calculator, explain why $x_1 = 3$ is reasonable. [1]

(c) Determine the value of $\sqrt[3]{25}$, giving your answer correct to 3 decimal places. You need not verify the correctness of your answer. [2]

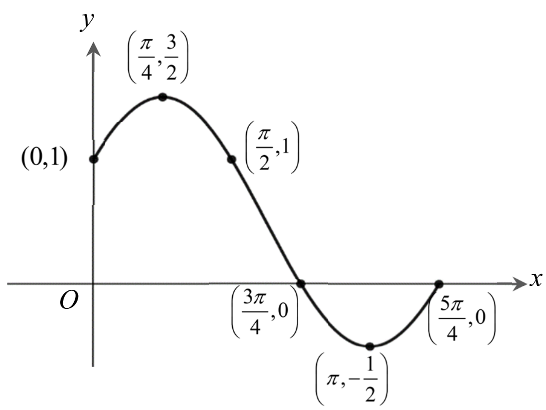
(d) By sketching suitable graphs, illustrate clearly how the iterates obtained in part **(c)** converge to the value of $\sqrt[3]{25}$. [3]

5 The sequence $\{v_n\}$ is given by $v_1 = 2$, $v_2 = 0$ and $v_{n+2} - 2v_{n+1} + 2v_n = 4$ for $n \ge 1$. The sequence $\{w_n\}$ satisfies a second order homogeneous recurrence relation and $v_n + w_n = k$, where k is a constant.

(a) Find the value of k and obtain an expression for v_n as a function of n. [6]

(b) Hence show that v_{4n+2} can be expressed as $k\left[1 + f(n)\right]$, where the two possible forms of $f(n)$ are to be determined. [2]

6 (a) The diagram shows the curve with equation $y = \mathrm{f}(x)$ for $0 \le x \le \frac{5\pi}{4}$. Some points on the curve are given. The stationary points are $\left(\frac{\pi}{4}, \frac{3}{2}\right)$ and $\left(\pi, -\frac{1}{2}\right)$.



Use Simpson's rule with 2 strips to estimate $\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} y^2 \, \mathrm{d}x$. [3]

(b) The polar curve C has equation $r = \mathrm{f}(\theta)$, $0 \le \theta \le \frac{5\pi}{4}$, where the function f is as defined in part **(a)**. For examples, when θ are 0 and $\frac{\pi}{4}$, the values of r are $\mathrm{f}(0) = 1$ and $\mathrm{f}\left(\frac{\pi}{4}\right) = \frac{3}{2}$ respectively.

(i) Sketch C, indicating clearly all key features. [3]

(ii) Using your answer to part **(a)**, estimate the area of the inner loop of C. [2]

~~**7** Point F lies on the x-axis at $(2c, 0)$ with $c > 0$. O is the origin and a variable point G moves on the x-y plane in such a way that the total distance $OG + GF = 2a$, where $a > c$. It is given that $OG = r$ and $\angle FOG = \theta$.~~

~~**(a)** State what type of conic section the locus of G is. [1]~~

~~**(b)** Show that $r = \frac{p}{1 - q\cos\theta}$, giving the constants p and q in terms of a and c. [4]~~

~~**(c)** State what property of the conic section is represented by the value of q. [1]~~

(d) ~~Write down in terms of θ, an integral that represents the arc length of the curve traced out by G as θ varies from 0 to π. State, in terms of π and a, an estimate for the value of this integral for extremely small values of c, giving a reason for your answer. [3]~~

~~**8** A Mathematics club consists of 12 members, of which 7 are men and 5 are women. They need to arrange themselves in 2 rows for a photoshoot. The first row has 5 seats while the second row has 7 seats.~~

~~Find the number of ways in which the members can be arranged if~~

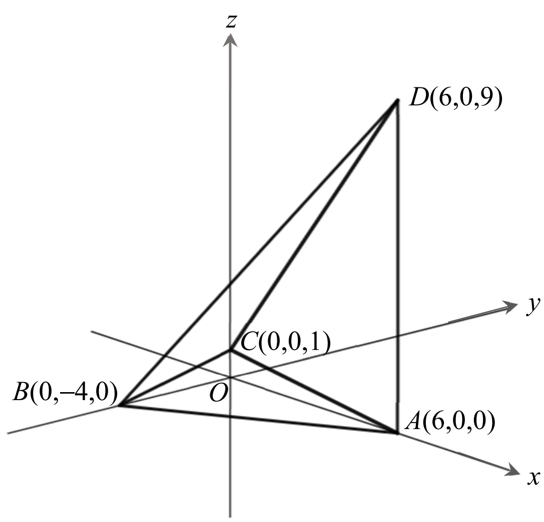
~~**(a)** there is no restriction, [1]~~

~~**(b)** all women are to sit next to each other, [3]~~

~~**(c)** 2 particular women must not sit next to each other. [3]~~

~~After the photoshoot, the members decide to play bridge. Find the total number of ways in which they can be divided into 3 groups of 4 players each. [2]~~

9 A tetrahedron has a plane base ABC. The coordinates of the vertices are $A(6,0,0)$, $B(0,-4,0)$, $C(0,0,1)$ and $D(6,0,9)$ (see diagram).



(a) Find a cartesian equation of plane ABC. [3]

(b) Determine the coordinates of the point of intersection of OD with plane ABC. [3]

(c) Calculate exactly the shortest distance from D to plane ABC. [3]

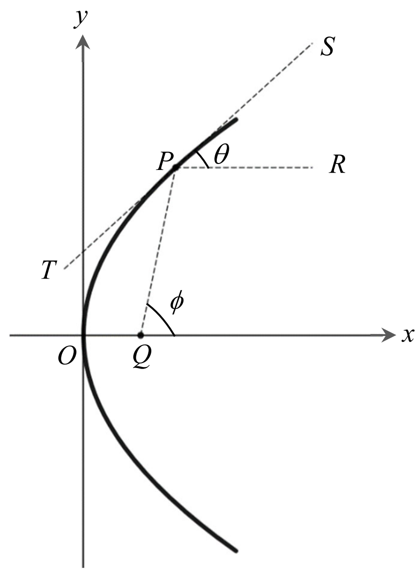
The volume of a tetrahedron is $\frac{1}{3} \times$ base area $\times$ height.

(d) Find the volume of the tetrahedron $ABCD$. [3]

~~**10** The diagram below shows the cross-section of a car headlight whose inner reflective surface is modelled by the parabolic curve~~

$$\cancel{x = at^2, \quad y = 2at, \quad -\sqrt{a} \le t \le \sqrt{a},}$$

~~where a is a positive constant greater than 1.~~



~~$P(at^2, 2at)$ is a point on the curve with parameter t. Q is the point $(a, 0)$. TS is the tangent to the curve at P, and PR is the line through P and parallel to the x-axis. The angle that PS and QP make with the positive x-direction are θ and ϕ respectively.~~

~~**(a)** By considering gradients of TS and QP, show that~~

$$\cancel{\tan\theta = \frac{1}{t} \text{ and } \phi = 2\theta.} \quad [5]$$

~~**(b)** Deduce that angle TPQ is equal to θ.~~ ~~[2]~~

~~**(c)** Describe the significance of this result in relation to the design of the car headlight.~~ ~~[2]~~

~~The reflective surface is formed by rotating the parabolic curve completely about the x-axis.~~

(d) ~~Assuming negligible thickness, calculate in terms of a and π, the area of the reflective surface.~~ ~~[4]~~

11 Drones are used during hot and dry seasons to monitor the temperatures of forested areas. Two drones D_1 and D_2 are being flown over an area of rainforest to detect hotspots. Taking the observation centre as the origin and the surrounding ground as the x-y plane, the positions of D_1 and D_2 are

$$\mathbf{r}_1 = \begin{pmatrix} 0 \\ 0 \\ 30 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 10 \\ 8 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix},$$

at time t seconds after take-off. Both drones take off simultaneously and all distances are measured in metres.

(a) State the height of D_1 above ground. [1]

(b) Show that both drones will not collide and by first expressing the distance between D_1 and D_2 in terms of t, determine the shortest distance between them. [6]

(c) Find the angle of elevation of D_2 from point (10, 8, 0) on the ground after take-off. [3]

Suppose each drone can observe a circular area A m^2, of the ground directly below such that $A = 2h^2$, where h m is the height of the drone above ground.

(d) Show that the area of ground that can be observed by D_1 7 seconds after it takes off overlaps with the area of ground that can be observed by D_2 at that time. [4]