National Junior College
2016 – 2017 H2 Mathematics
Differentiation and its Applications **Lecture Questions**

Part 1. Key Questions to answer:

- ❑ How do you differentiate the following functions?
  - ❑ *polynomial functions*
  - ❑ *trigonometric functions*
  - ❑ *exponential functions*
  - ❑ *logarithmic functions*
  - ❑ *constant multiples, sums and differences of any combination of the above functions*
- ❑ How do you use implicit differentiation, parametric differentiation and logarithmic differentiation?

Prerequisite knowledge: Differentiation by product rule, quotient rule and chain rule.
Lecture Readings: Section 1

Question 1.1

Differentiate the following with respect to x and simplify your answers.

(a) $\ln\left(\frac{x}{1+x^2}\right)$,

(b) $\log_2(x+1)$.

Solution:

Question 1.2 (Implicit Differentiation)

The equation of a curve is given by $4x^3+3x^2y=y^3-2$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

Solution:

Question 1.3 (Logarithmic Differentiation)

Differentiate

(a) $3^{\tan x}$

(b) $x^{\sin x}$

with respect to x, leaving your answer in terms of x.

Solution:

(a) Method 1: $\frac{\mathrm{d}}{\mathrm{d}x}3^{\tan x}=\left(\sec^2 x\right)\left(\ln 3\right)3^{\tan x}$

Method 2: Let $y=3^{\tan x}$

Taking "ln" on both sides, we get $\ln y=\left(\tan x\right)\ln 3$

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\left(\ln 3\right)\sec^2 x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=y\left(\ln 3\right)\sec^2 x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\left(\sec^2 x\right)\left(\ln 3\right)3^{\tan x}$$

(b) Let $y=x^{\sin x}$.

Taking "ln" on both sides, we get $\ln y=(\sin x)\ln x$.

Differentiate implicitly w.r.t. x :

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=(\sin x)\left(\frac{1}{x}\right)+(\ln x)\cos x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=y\left(\frac{1}{x}\sin x+(\ln x)\cos x\right)$$

$$=x^{\sin x}\left(\frac{1}{x}\sin x+(\ln x)\cos x\right).$$

Question 1.4 (Differentiating Inverse Trigonometric Functions)

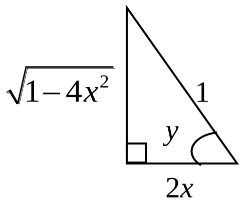
The equation of a curve is given by $y=\cos^{-1}(2x)$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of *x*.

Solution:

$$y=\cos^{-1}(2x) \Rightarrow \cos y = 2x$$

$$-\sin y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=2$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2}{\sin y}=\frac{-2}{\sqrt{1-4x^2}}$$



Question 1.5 (Parametric Differentiation)

The curve C is defined parametrically by the equations

$$x=t+\ln t,\quad y=t+1,\ \text{where } t>0.$$

Using parametric differentiation, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of *t*.

Solution:

$$x=t+\ln t,\quad y=t+1$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=1+\frac{1}{t},\quad \frac{\mathrm{d}y}{\mathrm{d}t}=1$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{1}{\frac{t+1}{t}}=\frac{t}{t+1}$$

Learning Point(s):

(a) Implicit differentiation to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ is useful if *y* cannot be expressed in terms of the independent variable *x* explicitly.

(b) Logarithmic differentiation is useful for expressions of the form u^v, where both *u* and *v* are **non-constant** expressions of *x* and/or *y*.

(c) Referring to Question 1.4, alternatively, we can convert the parametric equation into the Cartesian form and differentiate implicitly w.r.t. *x*.

(d) Note that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\neq\frac{\frac{\mathrm{d}^2y}{\mathrm{d}t^2}}{\frac{\mathrm{d}^2x}{\mathrm{d}t^2}}$.

Part 2. Key Questions to answer:

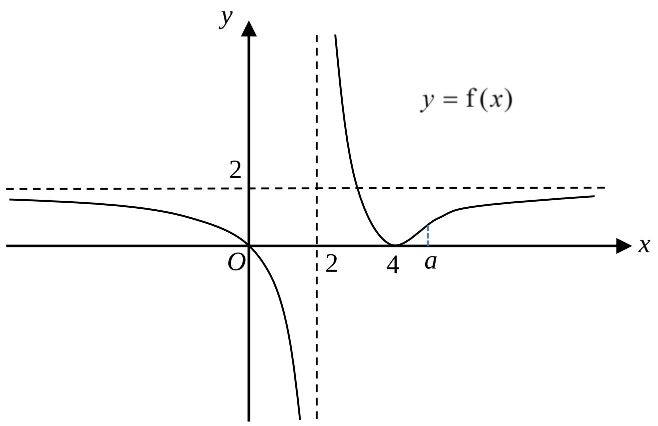
- ❑ What are increasing and decreasing functions?
- ❑ How do you determine the concavity of a curve?
- ❑ How do you sketch the graph of the derivative function?

Prerequisite knowledge: Curve Sketching, Functions
Lecture Readings: Section 2

Question 2.1

Write down the largest set of values of x for which the graph of $y = \mathrm{f}(x)$ is

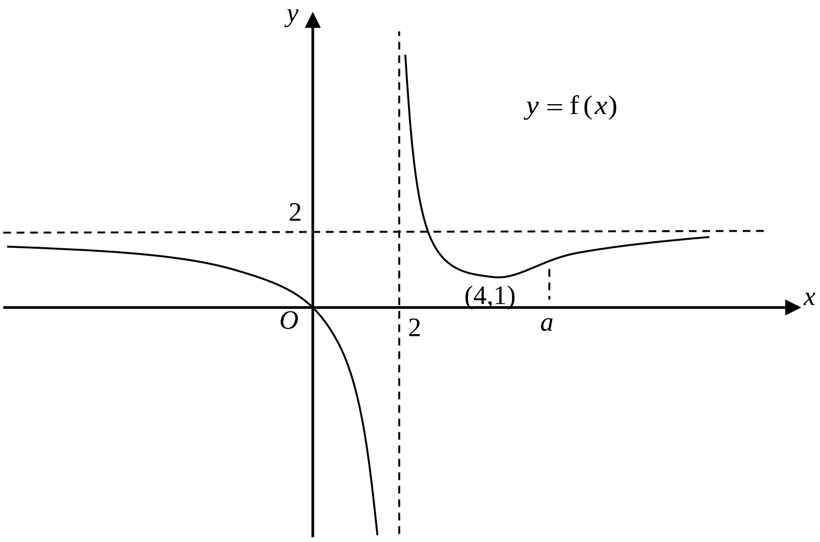
(i) increasing, (ii) decreasing, (iii) concave upwards, (iv) concave downwards.



Solution:

Question 2.2 (Sketching the derivative function)

Sketch the graph of $y = \mathrm{f}'(x)$.



Solution:

Learning Point(s):

(a) The stationary point of $y = \mathrm{f}(x)$ becomes the *x*-intercept of $y = \mathrm{f}'(x)$.
(b) The point of inflexion of $y = \mathrm{f}(x)$ becomes the stationary point of $y = \mathrm{f}'(x)$.
(c) The vertical asymptote of $y = \mathrm{f}(x)$ remains unchanged in $y = \mathrm{f}'(x)$.
(d) The horizontal asymptote of $y = \mathrm{f}(x)$ becomes the horizontal asymptote $y = 0$ in $y = \mathrm{f}'(x)$.
(e) If $\mathrm{f}'(x) > 0$, then the graph of $y = \mathrm{f}'(x)$ lies above the *x*-axis. If $\mathrm{f}'(x) < 0$, then the graph of $y = \mathrm{f}'(x)$ lies below the *x*-axis.

Part 3. Key Questions to answer:

- ❑ How do you find the gradient of the tangent (and normal) of a curve at a point via implicit and parametric differentiation, and hence the equation of the tangent (and normal)?
- ❑ How do you solve problems that require you to maximise or minimise a variable?
- ❑ How do you solve problems that involve connected rates of change?

Prerequisite knowledge: Curve Sketching, Tangents and Normals, Differentiation Rules
Lecture Readings: Sections 3 and 4

Question 3.1 (Tangent/Normal)

Refer to the equation of the curve given in Question 1.2.
Find the equations of the tangents to the curve that are parallel to the *y*-axis.

Solution:

Learning Points:

(a) The gradient of the tangent to the curve $y = \mathrm{f}(x)$ is given by $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(b) What should we do if were asked to find the equations of the tangents that are parallel to the *x*-axis instead?

Question 3.2 (Tangent/Normal)

Refer to the equation of the curve given in Question 1.4.

(i) Find the equation of the normal to the curve at $t = 1$.

(ii) Determine if the normal meets the curve again.

Solution:

(i) Equation of curve: $x = t + \ln t,\ \ y = t + 1$, where $t > 0$.

As shown in Question 1.4, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{t}{t+1}$.

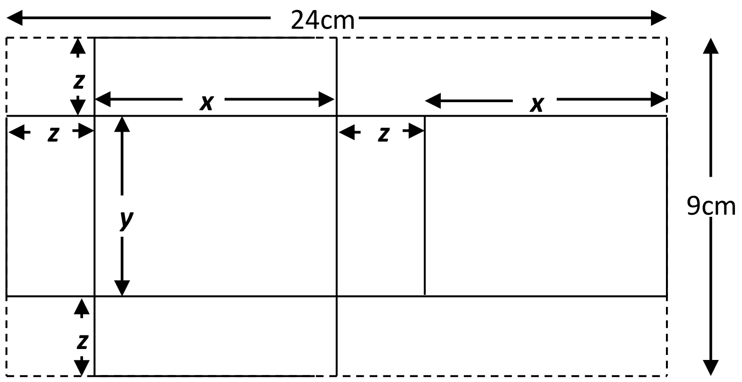
Thus at t = 1, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$ and hence, gradient of the normal is −2.

Also, at t = 1, $x = 1,\ \ y = 2$.

Equation of the normal is $y - 2 = -2(x-1) \Rightarrow y = -2x + 4$.

Question 3.3 (Max/Min Problem)

The figure below shows a 24cm by 9cm cardboard sheet. It can be folded into a box with length x, width y and height z.



(i) Show that the volume of the box, $V\,\text{cm}^3$, can be expressed as $V = -2x^3 + 39x^2 - 180x$.
(ii) Find, using differentiation, the largest volume of the box.

Solution:

(i) $2x + 2z = 24$, $\quad y + 2z = 9$

Solving the above simultaneous equation, $z = 12 - x$, $y = 2x - 15$.

$V = xyz = x(2x-15)(12-x)$

$= -2x^3 + 39x^2 - 180x$ (Shown)

(ii) $\dfrac{\mathrm{d}V}{\mathrm{d}x} = 0$

$-6x^2 + 78x - 180 = 0$

$x^2 - 13x + 30 = 0$

$(x-10)(x-3) = 0$

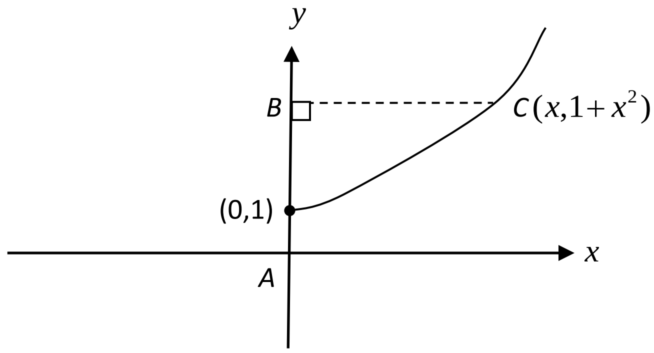
$x = 3$ or $x = 10$

$\dfrac{\mathrm{d}^2V}{\mathrm{d}x^2} = -12x + 78$

When $x = 3$, $\dfrac{\mathrm{d}^2V}{\mathrm{d}x^2} = 42 > 0$. When $x = 10$, $\dfrac{\mathrm{d}^2V}{\mathrm{d}x^2} = -42 < 0$.

Hence, when $x = 10$ cm, maximum volume = (10)(5)(2) = 100 cm^3.

Question 3.4 (Connected rates of change)



A variable right-angled triangle *ABC* has its right angle at point *B*. *B* starts at (0,1) at time $t = 0$ and moves upward along the *y*-axis at a constant rate of 2 units per second. Point *A* is fixed at the origin and point *C* moves on the curve $y = 1 + x^2$, $x > 0$. Find, when $t = 2$ seconds,

(i) the rate of change of *x*,

(ii) the rate at which the area of the triangle *ABC* is increasing.

Learning Point: Chain rule is used to find the rate of change of a variable.