CANDIDATE NAME	
CLASS		ADMISSION NUMBER	

2019 Preliminary Exams
Pre-University 3

MATHEMATICS **9758/01**

Paper 1 **3 September 2019**

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your admission number, name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Qn No.	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	*	Total
Score													
Max Score	6	6	6	8	8	9	10	10	12	12	13		100

This document consists of **24** printed pages.

[Turn over

1 The nth term of a sequence $u_1,\ u_2,\ u_3,\ \ldots$ is given by $u_n = n^2 + \dfrac{1}{n!}$.

(i) Show that $u_n - u_{n-1} = 2n - 1 + \dfrac{1-n}{n!}$. [2]

(ii) Hence find $\displaystyle\sum_{r=2}^{2n}\left(2r-1+\frac{1-r}{r!}\right)$. [3]

(iii) State, with a reason, if the series in part **(ii)** converges. [1]

2 It is given that

$$\mathrm{f}(x)=\begin{cases}3-x & \text{for } 0<x\le 2,\\ \dfrac{1}{6}\left(x^2+2\right) & \text{for } 2<x\le 4,\end{cases}$$

and that $\mathrm{f}(x)=\mathrm{f}(x+4)$ for all real values of x.

(i) Sketch the graph of $y=\mathrm{f}(x)$ for $-3\le x\le 10$. [3]

(ii) Find the volume of revolution when the region bounded by the graph of $y=\mathrm{f}(x)$, the lines $x=-1$, $x=2$ and the x-axis is rotated completely about the x-axis. [3]

3 **(i)** Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cos x^2\right)$. [2]

(ii) Hence find $\displaystyle\int x^3\sin x^2\ \mathrm{d}x$. [4]

4 **(a)** The curve C has equation $y=\mathrm{f}(x)$.

(i) Given that $\mathrm{f}(x)=\dfrac{x^2+3x+4}{x+1}$, sketch the curve C, showing the equations of the asymptotes and the coordinates of any turning points and any points of intersection with the axes. [3]

(ii) Hence, state the range of values of x where $\mathrm{f}'(x)>0$. [2]

(b) The curve with equation $y=\mathrm{g}(x)$ is transformed by a stretch with scale factor 2 parallel to the x-axis, followed by a translation of 1 unit in the negative x-direction and followed by a translation of 1 unit in the positive y-direction. The resulting curve has equation $y=\dfrac{x^2+3x+4}{x+1}$. Find $\mathrm{g}(x)$. [3]

5 **(i)** On the same axes, sketch the graphs of $y=\dfrac{1}{|x-a|}$ and $y=-b(x-a)$, where a and b are positive constants and $ab>\dfrac{1}{a}$, stating clearly any axial intercepts and equations of any asymptotes. [3]

(ii) Given that the solution to the inequality $\dfrac{1}{|x-a|}>-b(x-a)$ is $\dfrac{1}{2}<x<1$ or $x>1$, find the values of a and b. [4]

(iii) Using the values of a and b found in part **(ii)**, write down the solution to the inequality $\dfrac{1}{x-a}>-b(x-a)$. [1]

6 **(i)** Given that $\mathrm{f}(x)=\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}$ where a is a constant, find $\mathrm{f}(0)$, $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$ in terms of a. Hence write down the first two non-zero terms in the Maclaurin series for $\mathrm{f}(x)$. Give the coefficients in terms of e. [5]

(ii) The first two non-zero terms in the Maclaurin series for $\mathrm{f}(x)$ are equal to the first two non-zero terms in the series expansion of $\dfrac{1}{\sqrt{b+x^2}}$, where b is a constant. By using appropriate expansions from the List of Formulae (MF26), find the possible values of a and b in terms of e. [4]

7 **(a)** Find the exact value of $\displaystyle\int_0^1 \frac{x}{2-x^2}\,\mathrm{d}x$. [3]

(b) The expression $\dfrac{x^2}{(2-x)^2}$ can be written in the form $A+\dfrac{B}{2-x}+\dfrac{C}{(2-x)^2}$.

(i) Find the values of A, B and C. [3]

(ii) Show that $\displaystyle\int_0^1 \frac{x^2}{(2-x)^2}\,\mathrm{d}x = p+q\ln 2$, where p and q are constants to be found. [4]

8 **(a)** Referred to the origin O, the point Q has position vector $\mathbf{q}$ such that

$$\mathbf{q} = 2\mathbf{i} - \frac{3}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}.$$

(i) Find the acute angle between $\mathbf{q}$ and the y-axis. [2]

It is given that a vector $\mathbf{m}$ is perpendicular to the xy-plane and its magnitude is 1.

(ii) With reference to the xy-plane, explain the geometrical meaning of $|\mathbf{q} \cdot \mathbf{m}|$ and state its value. [2]

(b) Referred to the origin O, the point R has position vector $\mathbf{r}$ given by $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where λ is a positive constant and $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors. It is known that $\mathbf{c}$ is a non-zero vector that is not parallel to $\mathbf{a}$ or $\mathbf{b}$. Given that $\mathbf{c} \times \mathbf{a} = \lambda\mathbf{b} \times \mathbf{c}$, show that $\mathbf{r}$ is parallel to $\mathbf{c}$. [2]

It is also given that $\mathbf{a}$ is a unit vector that is perpendicular to $\mathbf{b}$ and $|\mathbf{b}| = 2$.

By considering $\mathbf{r} \cdot \mathbf{r}$, show that $|\mathbf{c}| = k\sqrt{4\lambda^2 + 1}$, where k is a non-zero constant. [4]

9 The function f is defined by $\mathrm{f} : x \mapsto 1 + 2\mathrm{e}^{-x^2}$, $x \in \mathbb{R}$.

(i) Show that f does not have an inverse. [2]

(ii) The domain of f is further restricted to $x \leq k$, state the largest value of k for which the function f^{-1} exists. [1]

In the rest of the question, the domain of f is $x \in \mathbb{R}$, $x \leq k$, with the value of k found in part **(ii)**.

(iii) Find $\mathrm{f}^{-1}(x)$. [3]

The function g has an inverse such that the range of g^{-1} is given by $(1, 3]$.

(iv) Explain clearly why the composite function gf exists. [2]

It is given that the composite function gf is defined by $\mathrm{gf}(x) = x$.

(v) State the domain and range of gf. [2]

(vi) By considering $\mathrm{gf}(-2)$, find the exact value of $\mathrm{g}^{-1}(-2)$. [2]

10 As a raindrop falls due to gravity, its mass decreases with time due to evaporation. The rate of change of the mass of a raindrop, m grams, with respect to time, t seconds, is a constant c.

(i) **(a)** Write down a differential equation relating m, t and c. State, with a reason, whether c is a positive or negative constant. [2]

(b) Initially the mass of a raindrop is 0.05 grams and, after a further 60 s, the mass of the raindrop is 0.004 grams. Find m in terms of t. [3]

In recent years, scientists are looking for alternative sources of sustainable energy to meet our energy needs. One approach aims to extract kinetic energy from rain to harvest energy. In order to test for the viability of this approach, scientists need to find the maximum kinetic energy that can be harvested from a falling raindrop.

It is known that the kinetic energy K, in millijoules, of an object falling from rest is given by

$$K = \frac{mg^2t^2}{2},$$

where m grams is the mass of the object at time t seconds and g is a positive constant known as the gravitational acceleration.

In the rest of the question, use your answer in part **(i)(b)** as the mass of a raindrop at time t.

(ii) **(a)** Show by differentiation that the maximum value of the kinetic energy of a raindrop falling from rest is pg^2, where p is a constant to be found . Give your answer correct to 2 decimal places. [5]

(b) It is given that $g = 10$. The kinetic energy of a raindrop falling from rest is 1 millijoules when it hits the surface of the ground. Given that it takes more than one minute for a raindrop to fall from rest to the surface of the ground, find the time taken for a raindrop to hit the surface of the ground. [2]

11 In this question, the distance is measured in metres and time is in seconds.

A radio-controlled airplane takes off from ground level and is assumed to be travelling at a steady speed in a straight line. The position vector of the airplane t seconds after it takes off is given by $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where $\mathbf{a}$ refers to the position vector of the point where it departs and $\mathbf{b}$ is known as its velocity vector.

(i) Given that the airplane reaches the point P with coordinates $(-9,\ 4,\ 6)$ after 6 seconds and its velocity vector is $-2\mathbf{i}+\mathbf{j}+\mathbf{k}$, find the coordinates of the point where it departs. [1]

Paul stands at the point C with coordinates $(-7,\ 5,\ 2)$ to observe the airplane.

(ii) Find the shortest distance from Paul's position to the flight path of the airplane. [3]

While the airplane is in the air, a drone is seen flying at a steady speed in a straight line with equation

$$\frac{2x-1}{-6}=\frac{y+7}{4}=\frac{z-10}{k}.$$

(iii) Show that the equation of the flight path of the drone can be written as $\mathbf{r}=\mathbf{p}+\lambda\mathbf{q}$, where λ is a non-negative constant and $\mathbf{p}$ and $\mathbf{q}$ are vectors to be determined, leaving your answer in terms of k. [1]

(iv) Given that the flight paths of the radio-controlled airplane and the drone intersect, find k. [3]

At P, the airplane suddenly changes its speed and direction. The position vector of the airplane s seconds after it leaves P is given by

$$\mathbf{r}=\begin{pmatrix}-9\\4\\6\end{pmatrix}+s\begin{pmatrix}3\\2\\-1\end{pmatrix},\ \text{where } s\in\mathbb{R},\ s\geq 0.$$

It travels at a steady speed in a straight line towards an inclined slope, which is assumed to be a plane with equation

$$x-y+7z=2.$$

(v) Determine if the new flight path is perpendicular to the inclined slope. [2]

(vi) The airplane eventually collides with the slope. Find the coordinates of the point of collision. [3]

End of Paper

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CANDIDATE NAME

CLASS

ADMISSION NUMBER

2019 Preliminary Exams
Pre-University 3

MATHEMATICS **9758/02**

Paper 2 **18 September 2019**

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

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This document consists of **25** printed pages.

[Turn over

Section A: Pure Mathematics [40 marks]

1 It is known that the nth term of a sequence is given by

$$p_n = 3^{n-1} + a,$$

where a is a constant.

Find $\sum_{r=3}^{n} p_r$. [3]

2 **(i)** An architect places 25 rectangular wooden planks in a row as a design for part of the facade of a building. The lengths of the first 18 planks form an arithmetic progression and the first plank has length 4 m. Given that the sum of the first three planks is 11.46 m, find the length of the 18th plank. [2]

(ii) For the last 7 wooden planks, each plank has a length that is $\frac{5}{4}$ of the length of the previous plank. Find the length of the 25th plank. [2]

(iii) The even-numbered planks (2nd plank, 4th plank, 6th plank and so on) are painted blue. Find the total length of the planks that are painted blue. [3]

3 **(a)** Find the general solution for the following differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \mathrm{e}^{-5x+3} + \sin x.$$ [3]

(b) By using the substitution $z = x + \frac{\mathrm{d}y}{\mathrm{d}x}$, show that the following differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - x + 1 = 0$$

can be reduced to

$$\frac{\mathrm{d}z}{\mathrm{d}x} = z.$$ [2]

Hence, given that when $x = 0$, $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$, find y in terms of x. [4]

4 The curve C has parametric equations

$$x = t^3 - t^2, \quad y = t^2 + 2t - 3,$$

where $t \geq -1$.

(i) Sketch the curve C, stating clearly any axial intercepts and the coordinates of any end-points. [2]

The point P on the curve C has parameter $t = 2$.

(ii) Without using a calculator, find the equation of tangent to C at P. [3]

(iii) Without using a calculator, find the area of the region bounded by C, the tangent to C at P and the y-axis. [4]

5 **(a)** One of the roots of the equation $z^4 - 2z^3 + az^2 - 8z + 40 = 0$ is bi, where a and b are positive real constants.

(i) Find the values of a and b and hence find the other roots of the equation. [4]

(ii) Deduce the roots of the equation $w^4 + 2w^3 + aw^2 + 8w + 40 = 0$. [2]

(b) **(i)** The complex number $-6 + \left(2\sqrt{3}\right)\mathrm{i}$ is denoted by w. Without using a calculator, find an exact expression of w^n in modulus-argument form. [3]

(ii) Hence find the two smallest positive integers n such that $w^n w^*$ is purely imaginary. [3]

Section B: Probability and Statistics [60 marks]

6 The government of Country X uses a particular method to create a unique Identification Number (IN) for each of its citizens. The IN consists of 4 digits from 0 to 9 followed by one of the ten letters A – J. The digits in the IN can be repeated.

(i) Find the number of different Identification Numbers that can be created. [1]

Suppose the letter at the end of the IN is determined by the following steps:

Step 1: Find the sum of all the digits in the IN.

Step 2: Divide the sum by 10 and note the remainder.

Step 3: Use the table below to determine the letter that corresponds to the remainder.

Remainder	0	1	2	3	4	5	6	7	8	9
Letter	A	B	C	D	E	F	G	H	I	J

(ii) It is known that all citizens who are born in the year 1990 must have the digit '9' appearing twice in their IN and the sum of all the digits in their IN is at least 20. Eric, who is born in the year 1990, has 'I' as the last letter of his IN. Find the number of INs that could possibly belong to Eric. [4]

7 In a carnival, a player begins a game by rolling a fair 12-sided die which consists of 3 red faces, 7 blue faces and 2 white faces. When the die thrown comes to rest, the colour of the uppermost face of the die is noted. If the colour of the uppermost face is red, a ball is picked from box A that contains 3 red and 4 blue balls. If the colour of the uppermost face is blue, a ball is picked from box B that contains 2 red, 3 blue and 3 green balls. Otherwise, the game ends. A mystery gift is only given when the colour of the uppermost face of the die is the same as the colour of the ball picked.

It is assumed that Timothy plays the game only once.

(i) Find the probability that the colour of the uppermost face of the die is blue and a red ball is picked. [1]

(ii) Find the probability the mystery gift is given to Timothy. [2]

(iii) Given that Timothy did not win the mystery gift, find the probability that a red ball is picked. [3]

One of the rules of the game is changed such that if the colour of the uppermost face is white when the die thrown comes to rest, the player gets to roll the die again. The other rules of the game still hold.

Alicia plays the game once. Find the probability that she obtains the mystery gift in the end. [2]

8 **(a)** Sketch a scatter diagram that might be expected when x and y are related approximately as given in each of the cases (A), (B) and (C) below. In each case your diagram should include 6 points, approximately equally spaced with respect to x, with all x- and y-values positive. The letters a, b, c, d, e and f represent constants.

(A) $y = a + bx^2$, where a and b are positive.

(B) $y = c + d\ln x$, where c is positive and d is negative.

(C) $y = e + \dfrac{f}{x}$, where e is positive and f is negative. [3]

(b) The following table shows the population of a certain country, P, in millions, at various times, t years after the year 1990.

t	4	10	12	14	15	18	20	24
P	3.65	3.89	3.95	4.02	4.15	4.30	4.45	4.95

(i) Draw the scatter diagram for these values, labelling the axes. Give a reason why a linear model may not be appropriate. [2]

(ii) Explain which of the three cases in part **(a)** is the most appropriate for modelling these values, and calculate the product moment correlation coefficient for this case. [2]

(iii) A statistician wants to predict the population of the country in the year 2020. Use the case that you identified in part **(b)(ii)** to find the equation of a suitable regression line, and use your equation to find the required prediction. [3]

(iv) Comment on the reliability of the statistician's prediction. [1]

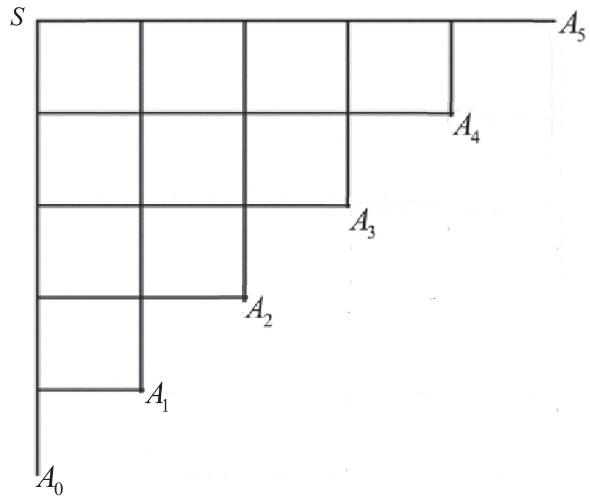
9 A chemist is conducting experiments to analyse the amount of active ingredient A in a particular type of health supplement tablets. Each tablet is said to contain an average amount of 50 mg of active ingredient A. A random sample of 40 tablets is taken and the amount of active ingredient A per tablet is recorded. The sample mean is 50.6 mg and the sample variance is 2.15 mg^2 respectively.

(i) Explain the meaning of 'a random sample' in the context of the question. [1]

(ii) Test, at the 1% level of significance, whether the mean amount of active ingredient A in a tablet has changed. You should state your hypotheses and define any symbols you use. [6]

In a revised formula of the same type of health supplement tablets, the amount of active ingredient A now follows a normal distribution with population variance 1.5 mg^2. The chemist wishes to test his claim that the mean amount of active ingredient A per tablet is more than 50 mg. A second random sample of n such tablets is analysed and its mean is found to be 50.4 mg. Find the set of values that n can take such that the chemist's claim is valid at 2.5% level of significance. [4]

10



A particle moves one step each time either to the right or downwards through a network of connected paths as shown above. The particle starts at S, and, at each junction, randomly moves one step to the right with probability p, or one step downwards with probability q, where $q = 1 - p$. The steps taken at each junction are independent. The particle finishes its journey at one of the 6 points labelled A_i, where $i = 0,\ 1,\ 2,\ 3,\ 4,\ 5$ (see diagram). Let $\{\ X = i\ \}$ be the event that the particle arrives at point A_i.

(i) Show that $\mathrm{P}(X = 2) = 10p^2q^3$. [2]

(ii) After experimenting, it is found that the particle will end up at point A_2 most of the time. By considering the mode of X or otherwise, show that $\frac{1}{3} < p < \frac{1}{2}$. [4]

The above setup is a part of a two-stage computer game.

- If the particle lands on A_0, the game ends immediately and the player will not win any points.
- If the particle lands on A_i, where $i = 2$ or 4, then the player gains 2 points.
- If the particle lands on A_i, where $i = 1$, 3 or 5, then the player proceeds on to the next stage, where there is a probability of 0.4 of winning the stage. If he wins the stage, he gains 5 points; otherwise he gains 3 points.

Let Y be the number of points gained by the player when one game is played.

(iii) If $p = 0.4$, determine the probability distribution of Y. [4]

(iv) Hence find the expectation and variance of Y. [2]

11 The two most popular chocolates sold by the *Dolce* chocolatier are the dark truffles and salted caramel ganaches and their masses have independent normal distributions. The masses, in grams, of dark truffles have the distribution $\mathrm{N}(17, 1.3^2)$.

(i) Find the probability that the total mass of 4 randomly chosen dark truffles is more than 70 g. [2]

(ii) The dark truffles are randomly packed into boxes of 4. In a batch of 20 boxes, find the probability that there are more than 3 boxes of dark truffles that have a mass more than 70 g. State an assumption you made in your calculations. [4]

The masses of salted caramel ganaches are normally distributed such that the proportion of them having a mass less than 12 grams is the same as the proportion of them having a mass greater than 15 grams. It is also given that 97% of the salted caramel ganaches weigh at most 15 grams.

(iii) Find the mean and variance of the masses of salted caramel ganaches. [3]

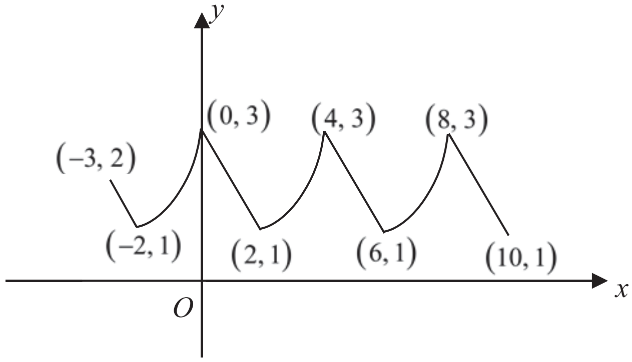
Dark truffles are sold at \$0.34 per gram and the salted caramel ganaches are sold at \$0.28 per gram.

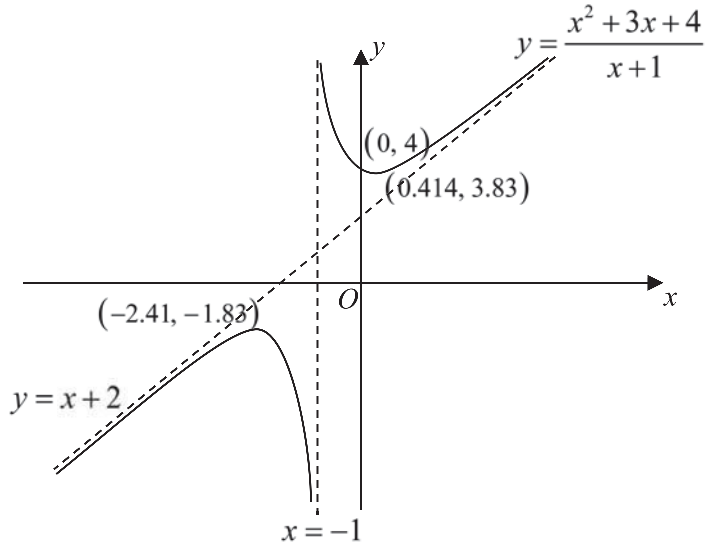
(iv) Find the probability that the total cost of 6 randomly chosen salted caramel ganaches is less than the total cost of 4 randomly chosen dark truffles. State the distribution you use and its parameters. [4]

End of Paper

Solution to Paper 9758/01

Qn	Solution
1(i)	$u_n = n^2 + \frac{1}{n!}$ $\text{LHS} = u_n - u_{n-1}$ $= n^2 + \frac{1}{n!} - \left((n-1)^2 + \frac{1}{(n-1)!} \right)$ $= n^2 + \frac{1}{n!} - n^2 + 2n - 1 - \frac{1}{(n-1)!}$ $= 2n - 1 + \frac{1}{n!} - \frac{n}{(n-1)! \times n}$ $= 2n - 1 + \frac{1-n}{n!}$ $= \text{RHS}$
1(ii)	$\sum_{r=2}^{2n} 2r - 1 + \frac{1-r}{r!}$ $= \sum_{r=2}^{2n} u_r - u_{r-1}$ $= \quad u_2 - u_1$ $+ u_3 - u_2$ $+ u_4 - u_3$ $+ \ldots$ $+ u_{2n-2} - u_{2n-3}$ $+ u_{2n-1} - u_{2n-2}$ $+ u_{2n} - u_{2n-1}$ $= u_{2n} - u_1$ $= (2n)^2 + \frac{1}{(2n)!} - (1+1)$ $= 4n^2 - 2 + \frac{1}{(2n)!}$
1(iii)	As $n \to \infty$, $4n^2 - 2 \to \infty$, $\frac{1}{(2n)!} \to 0$. Hence $\sum_{r=2}^{2n} 2r - 1 + \frac{1-r}{r!}$ does not converge. It diverges.

Qn	Solution
2(i)	
2(ii)	$\text{Volume} = \pi \int_0^2 (3-x)^2 \, dx + \pi \int_3^4 \left(\frac{x^2+2}{6}\right)^2 dx$ $= 45.5$
3(i)	$\frac{d}{dx} \cos x^2 = -2x \sin x^2$
3(ii)	$\int x^3 \sin x^2 \, dx$ $= -\frac{1}{2} \int \left(-2x \sin x^2\right)\left(x^2\right) dx$ $= -\frac{1}{2}\left[x^2 \cos x^2 - \int (2x)\left(\cos x^2\right) dx\right]$ $= -\frac{x^2}{2} \cos x^2 + \int x\left(\cos x^2\right) dx$ $= -\frac{x^2}{2} \cos x^2 + \frac{1}{2} \int 2x\left(\cos x^2\right) dx$ $= -\frac{x^2}{2} \cos x^2 + \frac{1}{2} \sin x^2 + c$ $u = x^2 \quad v' = -2x \sin x^2$ $u' = 2x \quad v = \cos x^2$ Note: $\frac{d}{dx} \sin x^2 = 2x \cos x^2$

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>4a(i)</td><td>$y=\dfrac{x^2+3x+4}{x+1}$
$=x+2+\dfrac{2}{x+1}$
Vertical asymptote: $x=-1$, Oblique asymptote: $y=x+2$
</td></tr>
<tr><td>4a(ii)</td><td>$x<-2.41$ or $x>0.414$</td></tr>
<tr><td>4(b)</td><td>Method 1a: Applying transformations backward
We are given the transformations in the order of A, B and C, and the resulting curve $y=\dfrac{x^2+3x+4}{x+1}$. In order to find the original curve $y=\mathrm{g}(x)$, we use our resulting curve, apply the C', B' then A', where A', B' and C' are the "opposites" of A, B and C respectively.
Original Curve $\underset{A'}{\overset{A}{\rightleftarrows}}$ ♠ $\underset{B'}{\overset{B}{\rightleftarrows}}$ ♥ $\underset{C'}{\overset{C}{\rightleftarrows}}$ Resulting Curve,
where ♠ and ♥ represent intermediate curves in the entire process.

Hence, we have
C': Translate 1 unit in negative y-direction (Replace y by $y+1$);
B': Translate 1 unit in positive x-direction (Replace x by $x-1$);
A': Stretch with scale factor $\dfrac{1}{2}$ parallel to x-axis (Replace x by $2x$).</td></tr>
</table>

Qn	Solution

$$y=\frac{x^2+3x+4}{x+1}\xrightarrow[C']{\text{Replace } y \text{ by } y+1} y+1=\frac{x^2+3x+4}{x+1} \quad \left(\text{i.e. } y=\frac{x^2+2x+3}{x+1}\right)$$

$$y=\frac{x^2+2x+3}{x+1}\xrightarrow[B']{\text{Replace } x \text{ by } x-1} y=\frac{(x-1)^2+2(x-1)+3}{(x-1)+1}$$

$$=\frac{x^2-2x+1+2x-2+3}{x}$$

$$=\frac{x^2+2}{x}$$

$$=x+\frac{2}{x}$$

$$y=x+\frac{2}{x}\xrightarrow[A']{\text{Replace } x \text{ by } 2x} y=2x+\frac{2}{2x}$$

$$=2x+\frac{1}{x}=\mathrm{g}(x)$$

Method 1b: Applying transformations backward

Note that:

Hence, we have

C': Translate 1 unit in negative y-direction (Replace y by $y+1$);

B': Translate 1 unit in positive x-direction (Replace x by $x-1$);

A': Stretch with scale factor $\frac{1}{2}$ parallel to x-axis (Replace x by $2x$).

Note that $y=\frac{x^2+3x+4}{x+1}=x+2+\frac{2}{x+1}$

$$y=x+2+\frac{2}{x+1}\xrightarrow[C']{\text{Replace } y \text{ by } y+1} y+1=x+2+\frac{2}{x+1} \quad \left(\text{i.e. } y=x+1+\frac{2}{x+1}\right)$$

$$y=x+1+\frac{2}{x+1}\xrightarrow[B']{\text{Replace } x \text{ by } x-1} y=(x-1)+1+\frac{2}{(x-1)+1}$$

$$=x+\frac{2}{x}$$

$$y=x+\frac{2}{x}\xrightarrow[A']{\text{Replace } x \text{ by } 2x} y=2x+\frac{2}{2x}$$

$$=2x+\frac{1}{x}=\mathrm{g}(x)$$

- **Method 2:** Start from the original curve $y=\mathrm{g}(x)$

Original Curve $y=\mathrm{g}(x)\xrightarrow{A}\spadesuit\xrightarrow{B}\heartsuit\xrightarrow{C}$ Resulting Curve $y=\frac{x^2+3x+4}{x+1}$

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td></td><td>

A : Stretch with scale factor 2 parallel to x-axis (Replace x by $\frac{1}{2}x$);

B : Translate 1 unit in negative x-direction (Replace x by $x+1$);
C : Translate 1 unit in positive y-direction (Replace y by $y-1$).

$$y=\mathrm{g}(x)\xrightarrow[\text{by } \frac{1}{2}x]{\text{Replace } x} y=\mathrm{g}\left(\frac{x}{2}\right)\xrightarrow[\text{by } x+1]{\text{Replace } x} y=\mathrm{g}\left(\frac{x+1}{2}\right)\xrightarrow[\text{by } y\text{-}1]{\text{Replace } y} y-1=\mathrm{g}\left(\frac{x+1}{2}\right)$$

$$y=\mathrm{g}\left(\frac{x+1}{2}\right)+1$$

Let $\mathrm{g}\left(\frac{x+1}{2}\right)+1=\frac{x^2+3x+4}{x+1}$

$\mathrm{g}\left(\frac{x+1}{2}\right)+1=x+2+\frac{2}{x+1}$

$\mathrm{g}\left(\frac{x+1}{2}\right)=x+1+\frac{2}{x+1}$

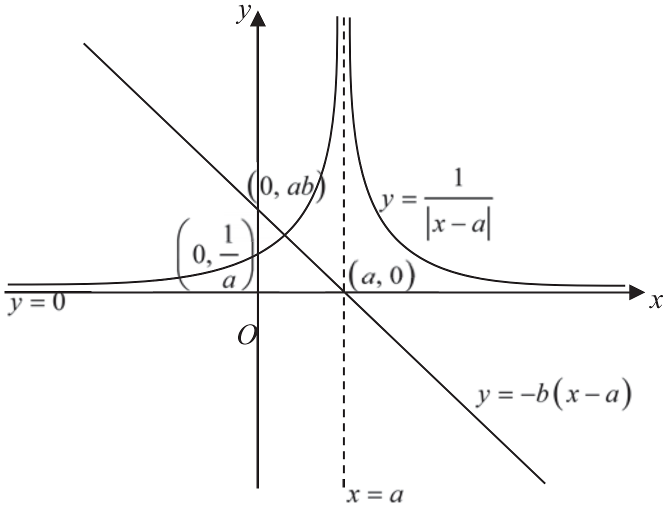
Note that we need to progress till we reach $\mathrm{g}(x)$.

To do that, we need to replace x by $2x-1$ in $\mathrm{g}\left(\frac{x+1}{2}\right)=x+1+\frac{2}{x+1}$.

$\mathrm{g}\left(\frac{(2x-1)+1}{2}\right)=(2x-1)+1+\frac{2}{(2x-1)+1}$

$\therefore \mathrm{g}(x)=2x+\frac{1}{x}$

</td></tr>
<tr><td>5(i)</td><td>



</td></tr>
<tr><td>5(ii)</td><td>

<u>Method 1</u>:

From graph in (i), one of the solution to $\frac{1}{|x-a|}>-b(x-a)$ is $x>a$.

By observation, since $x>1$, $a=1$

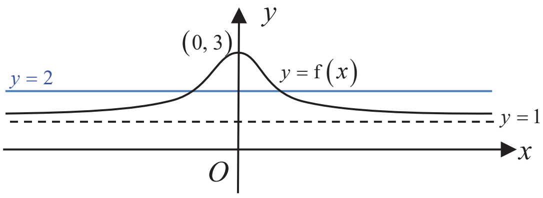
</td></tr>
</table>

Qn	Solution
	When $x=\frac{1}{2}, y=\frac{1}{\left\|\frac{1}{2}-1\right\|}=2$ Subst $x=\frac{1}{2}$ and $y=2$ into $y=-b(x-1)$ $2=-b\left(\frac{1}{2}-1\right)$ $b=4$ ----- **Method 2**: From graph in **(i)**, one of the solution to $\frac{1}{\|x-a\|}>-b(x-a)$ is $x>a$. By observation, since $x>1$, $a=1$ To find the *x*-coordinate of the point of intersection: $\frac{1}{-(x-1)}=-b(x-1)$ $(x-1)^2=\frac{1}{b}$ $x-1=\pm\frac{1}{\sqrt{b}}$ When $x=\frac{1}{2}$, $\frac{1}{2}-1=\pm\frac{1}{\sqrt{b}}$ $-\frac{1}{2}=-\frac{1}{\sqrt{b}}$ or $-\frac{1}{2}=\frac{1}{\sqrt{b}}$ (rej. since $\sqrt{b}>0$) $\sqrt{b}=2$ $b=4$
5(iii)	$x>1$
6(i)	**Method 1**: $\mathrm{f}(x)=\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}$ $\mathrm{f}'(x)=a\cos\left(ax+\frac{\pi}{2}\right)\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}$ $\mathrm{f}''(x)=a^2\cos^2\left(ax+\frac{\pi}{2}\right)\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}+a\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}\left[-a\sin\left(ax+\frac{\pi}{2}\right)\right]$ $\mathrm{f}''(x)=a^2\cos^2\left(ax+\frac{\pi}{2}\right)\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}-a^2\sin\left(ax+\frac{\pi}{2}\right)\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}$ ----

Qn	Solution
	Method 2: Let $y=\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}$ $\Rightarrow \ln y=\sin\left(ax+\frac{\pi}{2}\right)$ Differentiate w.r.t. x: $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=a\cos\left(ax+\frac{\pi}{2}\right)$ Differentiate w.r.t. x: $\frac{1}{y}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(-\frac{1}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-a^2\sin\left(ax+\frac{\pi}{2}\right)$ --- **Method 3:** $\mathrm{f}(x)=\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}=\mathrm{e}^{\sin ax\cos\frac{\pi}{2}+\cos ax\sin\frac{\pi}{2}}=\mathrm{e}^{\cos ax}$ $\mathrm{f}'(x)=-(\sin ax)\mathrm{e}^{\cos ax}$ $\mathrm{f}''(x)=\mathrm{e}^{\cos ax}(-a\cos ax)+(\sin^2 ax)\mathrm{e}^{\cos ax}$ --- Hence $\mathrm{f}(0)=\mathrm{e}^{\sin\left(\frac{\pi}{2}\right)}=\mathrm{e}$ $\mathrm{f}'(0)=0$ $\mathrm{f}''(0)=-a^2\mathrm{e}$ By Maclaurin Series, $\mathrm{f}(x)=\mathrm{f}(0)+x\,\mathrm{f}'(0)+\frac{x^2}{2!}\mathrm{f}''(0)+\ldots$ $\mathrm{f}(x)=\mathrm{e}-\frac{a^2\mathrm{e}}{2}x^2+\ldots$ Or it can be presented as $\mathrm{f}(x)\approx\mathrm{e}-\frac{a^2\mathrm{e}}{2}x^2$
6(ii)	$\frac{1}{\sqrt{b+x^2}}=\left(b+x^2\right)^{-\frac{1}{2}}$ $=b^{-\frac{1}{2}}\left(1+\frac{x^2}{b}\right)^{-\frac{1}{2}}$ $=\frac{1}{\sqrt{b}}\left(1+\left(-\frac{1}{2}\right)\left(\frac{x^2}{b}\right)+\ldots\right)$ $=\frac{1}{\sqrt{b}}-\frac{x^2}{2b\sqrt{b}}+\ldots$

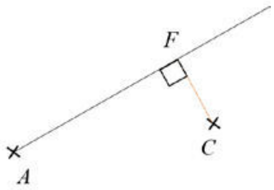
Qn	Solution
	Comparing the constant term, $\frac{1}{\sqrt{b}}=\mathrm{e} \quad \Rightarrow \quad b=\frac{1}{\mathrm{e}^2}$ Comparing coefficient of x^2, $-\frac{1}{2b\sqrt{b}}=\frac{-a^2\mathrm{e}}{2}$ $a^2\mathrm{e}=\frac{1}{\left(\frac{1}{\mathrm{e}^2}\right)\left(\frac{1}{\mathrm{e}}\right)}$ $a^2=\mathrm{e}^2$ $a=\pm\mathrm{e}$
7(a)	$\int_0^1 \frac{x}{2-x^2}\,\mathrm{d}x=-\frac{1}{2}\int_0^1 \frac{-2x}{2-x^2}\,\mathrm{d}x$ $=-\frac{1}{2}\left[\ln\left\|2-x^2\right\|\right]_0^1$ $=-\frac{1}{2}(\ln 1-\ln 2)$ $=-\frac{1}{2}\ln\frac{1}{2} \quad \text{or} \quad \frac{1}{2}\ln 2$
7(b)i)	$x^2=A(2-x)^2+B(2-x)+C$ $=Ax^2-4Ax+4A+x^2+2B-Bx+C$ $=Ax^2+(-4A-B)x+4A+2B+C$ By comparing coefficients, $A=1$, $-4(1)-B=0\Rightarrow B=-4,$ $4(1)+2(-4)+C=0\Rightarrow C=4$
7(b)ii)	$\int_0^1 \frac{x^2}{(2-x)^2}\,\mathrm{d}x=\int_0^1 1-\frac{4}{2-x}+\frac{4}{(2-x)^2}\,\mathrm{d}x$ $=\left[x+4\ln\left\|2-x\right\|+\frac{4}{2-x}\right]_0^1$ $=1+4\ln(1)+\frac{4}{2-1}-4\ln 2-2$ $=3-4\ln 2$ $p=3,\ q=-4$

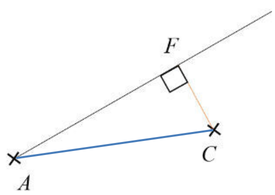
Qn	Solution
8(a)i)	$\overrightarrow{OQ}=\begin{pmatrix}2\\-\frac{3}{2}\\-\frac{1}{2}\end{pmatrix}$ Acute angle $=\cos^{-1}\dfrac{\left\lvert\begin{pmatrix}2\\-1.5\\-0.5\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}\right\rvert}{\sqrt{6.5}\sqrt{1}}$ $=\cos^{-1}\left(\dfrac{1.5}{\sqrt{6.5}}\right)$ $=54.0^{\circ}$ (or 0.942 rad)
8(a)ii)	xy-plane: $\mathbf{r}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=0$ Since $\mathbf{m}$ is perpendicular to xy-plane, $\mathbf{m}\,/\!/\begin{pmatrix}0\\0\\1\end{pmatrix}$. Since $\lvert\mathbf{m}\rvert=1$, $\mathbf{m}=\begin{pmatrix}0\\0\\1\end{pmatrix}$. $\lvert\mathbf{q}\cdot\mathbf{m}\rvert$ refers to the perpendicular distance from Q to xy-plane. $\lvert\mathbf{q}\cdot\mathbf{m}\rvert=\left\lvert\begin{pmatrix}2\\-\frac{3}{2}\\-\frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert=\frac{1}{2}$
8 (b)	$\mathbf{c}\times\mathbf{a}=\lambda\mathbf{b}\times\mathbf{c}$ $\mathbf{c}\times\mathbf{a}-\lambda\mathbf{b}\times\mathbf{c}=\mathbf{0}$ $\mathbf{c}\times\mathbf{a}+\mathbf{c}\times\lambda\mathbf{b}=\mathbf{0}$ $\mathbf{c}\times(\mathbf{a}+\lambda\mathbf{b})=\mathbf{0}$ $\mathbf{c}\times\mathbf{r}=\mathbf{0}$ $\therefore\ \mathbf{r}\,/\!/\,\mathbf{c}$ (shown) Since $\mathbf{c}\,/\!/\,\mathbf{r}$, $\mu\mathbf{c}=\mathbf{r}$.

Qn	Solution
	$\mathbf{r} \bullet \mathbf{r} = \mu\mathbf{c} \bullet \mu\mathbf{c}$ $\mathbf{r} \bullet \mathbf{r} = \mu^2 \lvert\mathbf{c}\rvert^2$ Since a is a unit vector, $\lvert\mathbf{a}\rvert = 1$. $\mathbf{r} \bullet \mathbf{r} = (\mathbf{a}+\lambda\mathbf{b}) \bullet (\mathbf{a}+\lambda\mathbf{b})$ $= \mathbf{a} \bullet \mathbf{a} + \mathbf{a} \bullet \lambda\mathbf{b} + \lambda\mathbf{b} \bullet \mathbf{a} + \lambda\mathbf{b} \bullet \lambda\mathbf{b}$ $= \lvert\mathbf{a}\rvert^2 + 2\lambda(\mathbf{a} \bullet \mathbf{b}) + \lambda^2 \lvert\mathbf{b}\rvert^2$ $= 1 + 4\lambda^2$ $\mu^2 \lvert\mathbf{c}\rvert^2 = 1 + 4\lambda^2$ $\lvert\mathbf{c}\rvert^2 = \frac{1}{\mu^2}(1+4\lambda^2)$ $\lvert\mathbf{c}\rvert = \sqrt{\frac{1}{\mu^2}(1+4\lambda^2)}$ (rej. $-\sqrt{\frac{1}{\mu^2}(1+4\lambda^2)}$ since $\lvert\mathbf{c}\rvert > 0$) $\lvert\mathbf{c}\rvert = \sqrt{\frac{1}{\mu^2}}\left(\sqrt{(1+4\lambda^2)}\right)$ $\lvert\mathbf{c}\rvert = k\sqrt{(1+4\lambda^2)}$ where $k = \sqrt{\frac{1}{\mu^2}}$ (or $k = \frac{1}{\lvert\mu\rvert}$)
9(i)	**<u>Method 1: Horizontal Line Test</u>** $\mathrm{f} : x \mapsto 1 + 2\mathrm{e}^{-x^2}$, $x \in \mathbb{R}$  Since there exists **a/the horizontal line** $y = 2$ that intersects the graph of $y = \mathrm{f}(x)$ **more than once**, f is **not one-one**, f does not have an inverse. **<u>Method 2: Counterexample</u>** Since $\mathrm{f}(-1) = \mathrm{f}(1) = 1 + \frac{2}{\mathrm{e}}$, f is **not one-one**, f does not have an inverse.
9(ii)	Largest value of k is 0.
9(iii)	Let $y = 1 + 2\mathrm{e}^{-x^2}$ $y - 1 = 2\mathrm{e}^{-x^2}$ $\mathrm{e}^{-x^2} = \frac{y-1}{2}$ $-x^2 = \ln\left(\frac{y-1}{2}\right)$ $x^2 = -\ln\left(\frac{y-1}{2}\right)$ $x = \pm\sqrt{-\ln\left(\frac{y-1}{2}\right)}$

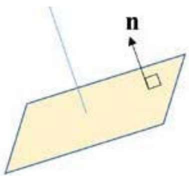
Qn	Solution
	Since $x \le 0$ (restricted domain of f), $x = -\sqrt{-\ln\left(\frac{y-1}{2}\right)}$ $\mathrm{f}^{-1}(x) = -\sqrt{-\ln\left(\frac{x-1}{2}\right)}$ OR $-\sqrt{\ln\left(\frac{2}{x-1}\right)}$
9(iv)	Range of $\mathrm{f} = (1,\ 3]$ Domain of g = Range of $\mathrm{g}^{-1} = (1,\ 3]$ Since Range of f $\subseteq$ Domain of g, gf exists.
9(v)	Domain of gf = Domain of f = $(-\infty,\ 0]$ Range of gf = $(-\infty,\ 0]$
9(vi)	Given that $\mathrm{gf}(x) = x$, $\mathrm{gf}(-2) = -2$ $\mathrm{g}^{-1}\mathrm{gf}(-2) = \mathrm{g}^{-1}(-2)$ $\mathrm{g}^{-1}(-2) = \mathrm{f}(-2)$ $= 1 + 2\mathrm{e}^{-4}$
10(i) a)	**Method 1:** $\frac{\mathrm{d}m}{\mathrm{d}t} = c$ where c is a negative constant because the mass of the raindrop is decreasing with time due to evaporation. **Method 2:** $\frac{\mathrm{d}m}{\mathrm{d}t} = -c$ where c is a positive constant because the mass of the raindrop is decreasing with time due to evaporation.
10(i) b)	$\frac{\mathrm{d}m}{\mathrm{d}t} = c$ $m = ct + D$ When $t = 0$, $m = 0.05$, $0.05 = c(0) + D \Rightarrow D = 0.05$ When $t = 60$, $m = 0.004$, $0.004 = c(60) + 0.05$ $c = -0.00076667$ (5s.f.) $c = -0.000767$ (3s.f.) $\therefore m = -0.000767t + 0.05$
10(ii) a)	$K = \frac{mg^2t^2}{2}$ From (i)(b), $m = -0.00076667t + 0.05$,

Qn	Solution
	$K=\dfrac{\left(-0.00076667t+0.05\right)g^2t^2}{2}$ $=-0.00038333g^2t^3+0.025g^2t^2$ At stationary points, $\dfrac{\mathrm{d}K}{\mathrm{d}t}=-0.00115g^2t^2+0.05g^2t=0$ $t\left(-0.00115g^2t+0.05g^2\right)=0$ $t=0$ (rejected since $K=0$ when $t=0$) or $t=43.478$ $\dfrac{\mathrm{d}^2K}{\mathrm{d}t^2}=-0.0023g^2t+0.05g^2$ When $t=43.478$, $\dfrac{\mathrm{d}^2K}{\mathrm{d}t^2}=-0.0500g^2$ <0 (since $g^2>0$) $\therefore t=43.478$ gives maximum K. When $t=43.478$, $K=15.75g^2$, where $p=15.75$ (2d.p.)
10(ii) b)	$K=-0.00038333(10)^2t^3+0.025(10)^2t^2$ $=-0.038333t^3+2.5t^2$ At surface of ground, $1=-0.038333t^3+2.5t^2$ Using GC, $t=-0.629,\ 0.636$ or 65.2 Since $t>60$, $\therefore t=65.2$ (3 s.f.) Required time taken = 65.2 s. (3 s.f.)
11(i)	Let A be the point where it departs from the ground. $\mathbf{r}=\overrightarrow{OA}+t\begin{pmatrix}-2\\1\\1\end{pmatrix}$ $\begin{pmatrix}-9\\4\\6\end{pmatrix}=\overrightarrow{OA}+6\begin{pmatrix}-2\\1\\1\end{pmatrix}$ $\overrightarrow{OA}=\begin{pmatrix}-9\\4\\6\end{pmatrix}-6\begin{pmatrix}-2\\1\\1\end{pmatrix}=\begin{pmatrix}3\\-2\\0\end{pmatrix}$ Coordinates: $A\left(3,\ -2,\ 0\right)$

Qn	Solution
11(ii)	 Shortest distance from C to the flight path (line) refers to the perpendicular distance from C to the flight path. **Method 1 (via foot of perpendicular)** Let F be the foot of perpendicular from C to the flight path $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ for $t \in \mathbb{R}, t \geq 0$ $\overrightarrow{OF} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ for some $t \in \mathbb{R}, t \geq 0$ $\overrightarrow{CF} = \overrightarrow{OF} - \overrightarrow{OC} = \begin{pmatrix} 3-2t \\ -2+t \\ t \end{pmatrix} - \begin{pmatrix} -7 \\ 5 \\ 2 \end{pmatrix}$ $= \begin{pmatrix} 10-2t \\ -7+t \\ t-2 \end{pmatrix}$ Since $\overrightarrow{CF} \perp$ flight path, $\begin{pmatrix} 10-2t \\ -7+t \\ t-2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = 0$ $-20+4t-7+t+t-2=0$ $t = \frac{29}{6}$ (or 4.8333) $\overrightarrow{CF} = \begin{pmatrix} 10-2\left(\frac{29}{6}\right) \\ -7+\left(\frac{29}{6}\right) \\ \left(\frac{29}{6}\right)-2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{13}{6} \\ \frac{17}{6} \end{pmatrix}$ $\left\lvert\overrightarrow{CF}\right\rvert = \sqrt{\left(\frac{1}{3}\right)^2 + \left(-\frac{13}{6}\right)^2 + \left(\frac{17}{6}\right)^2} = \sqrt{\frac{77}{6}}$ $\left\lvert\overrightarrow{CF}\right\rvert = 3.58236 = 3.58$ m (3 sf) ----

Qn	Solution
	<u>Method 2 (Vector product)</u> $\overrightarrow{OC}=\begin{pmatrix}-7\\5\\2\end{pmatrix}$ $\overrightarrow{AC}=\begin{pmatrix}-7\\5\\2\end{pmatrix}-\begin{pmatrix}3\\-2\\0\end{pmatrix}=\begin{pmatrix}-10\\7\\2\end{pmatrix}$ $\text{Perpendicular distance}=\dfrac{\left\lvert\overrightarrow{AC}\times\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\rvert}$ $=\dfrac{\left\lvert\begin{pmatrix}-10\\7\\2\end{pmatrix}\times\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\rvert}{\sqrt{4+1+1}}$ $=\dfrac{\left\lvert\begin{pmatrix}5\\6\\4\end{pmatrix}\right\rvert}{\sqrt{6}}$ $=\dfrac{\sqrt{77}}{\sqrt{6}}$ $=3.58236=3.58\text{ m}\ \ (3\text{ sf})$ **<u>Method 3 (Pythagoras Theorem)</u>** $\overrightarrow{OC}=\begin{pmatrix}-7\\5\\2\end{pmatrix}$  $\overrightarrow{AC}=\begin{pmatrix}-7\\5\\2\end{pmatrix}-\begin{pmatrix}3\\-2\\0\end{pmatrix}=\begin{pmatrix}-10\\7\\2\end{pmatrix}$

Qn	Solution
	$\left\lvert\overrightarrow{AF}\right\rvert = \dfrac{\left\lvert \overrightarrow{AC} \bullet \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\rvert}$ $= \dfrac{\left\lvert \begin{pmatrix} -10 \\ 7 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\rvert}{\sqrt{4+1+1}}$ $= \dfrac{29}{\sqrt{6}}$ Perpendicular distance $= \sqrt{\left\lvert\overrightarrow{AC}\right\rvert^2 - \left\lvert\overrightarrow{AF}\right\rvert^2}$ $= \sqrt{\left((-10)^2 + 7^2 + 2^2\right) - \left(\dfrac{29}{\sqrt{6}}\right)^2}$ $= 3.58236 = 3.58 \text{ m} \ \ (3 \text{ sf})$
11(iii)	Let $\lambda = \dfrac{2x-1}{-6} = \dfrac{y+7}{4} = \dfrac{z-10}{k}$ $\dfrac{2x-1}{-6} = \lambda \ \Rightarrow \ x = \dfrac{1}{2} - 3\lambda$ $\dfrac{y+7}{4} = \lambda \ \Rightarrow \ y = -7 + 4\lambda$ $\dfrac{z-10}{k} = \lambda \ \Rightarrow \ z = 10 + k\lambda$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -7 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ k \end{pmatrix}$ $\mathbf{r} = \begin{pmatrix} \frac{1}{2} \\ -7 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ k \end{pmatrix}$, where $\lambda \in \mathbb{R}, \lambda \geq 0$
11(iv)	Given that both flight paths (lines) intersect,

Qn	Solution
	$\begin{pmatrix} \frac{1}{2} \\ -7 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ k \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} \frac{1}{2} - 3\lambda \\ -7 + 4\lambda \\ 10 + k\lambda \end{pmatrix} = \begin{pmatrix} 3 - 2t \\ -2 + t \\ t \end{pmatrix}$ $\frac{1}{2} - 3\lambda = 3 - 2t \quad \Rightarrow \quad -3\lambda + 2t = \frac{5}{2}$ ---(1) $-7 + 4\lambda = -2 + t \quad \Rightarrow \quad 4\lambda - t = 5$ ---(2) $10 + k\lambda = t$ Using GC to solve simultaneously, $\lambda = 2.5, \quad t = 5$ $10 + k(2.5) = 5$ $k = -2$
11(v)	Equation of line: $\mathbf{r} = \begin{pmatrix} -9 \\ 4 \\ 6 \end{pmatrix} + s \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ Equation of plane: $\mathbf{r} \bullet \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = 2$  If the new flight path is perpendicular to the slope, it will be parallel to the normal vector of the slope. **Method 1:** If **a** and **b** are parallel, then $\mathbf{a} = k\,\mathbf{b}$ for all $k \in \mathbb{R}$. Since $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$ for all $k \in \mathbb{R}$, $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ is not parallel to $\begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$. Thus the new flight path and the inclined slope are not perpendicular to each other.

Qn	Solution
	Method 2: If **a** and **b** are parallel, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. **Note:** **0** means a zero vector, not a constant 0. If the new flight path (line) is perpendicular to the slope (plane), then this means the direction vector of the line and the normal vector of the plane is parallel. This is to then show that the vector (cross) product of the direction vector of the line and the normal vector of the plane is **0**. $\begin{pmatrix}3\\2\\-1\end{pmatrix} \times \begin{pmatrix}1\\-1\\7\end{pmatrix} = \begin{pmatrix}13\\-22\\-5\end{pmatrix} \neq \begin{pmatrix}0\\0\\0\end{pmatrix}$. This shows that $\begin{pmatrix}3\\2\\-1\end{pmatrix}$ is not parallel to $\begin{pmatrix}1\\-1\\7\end{pmatrix}$. Thus the new flight path and the inclined slope are not perpendicular to each other. **Method 3:** If **a** and **b** are parallel, then the angle between **a** and **b** is 0°. If the new flight path (line) is perpendicular to the slope (plane), then this means the direction vector of the line and the normal vector of the plane is parallel. This is to then show that the angle between the line and the plane is 0°. $\sin\theta = \dfrac{\left\lvert \begin{pmatrix}3\\2\\-1\end{pmatrix} \bullet \begin{pmatrix}1\\-1\\7\end{pmatrix} \right\rvert}{\sqrt{14}\sqrt{51}} \Rightarrow \theta = 13.0^{\circ} \neq 0^{\circ}$ This shows that $\begin{pmatrix}3\\2\\-1\end{pmatrix}$ is not parallel to $\begin{pmatrix}1\\-1\\7\end{pmatrix}$. Thus the new flight path and the inclined slope are not perpendicular to each other.
11(vi)	Equation of line: $\mathbf{r} = \begin{pmatrix}-9\\4\\6\end{pmatrix} + s\begin{pmatrix}3\\2\\-1\end{pmatrix}$ Equation of plane: $\mathbf{r} \bullet \begin{pmatrix}1\\-1\\7\end{pmatrix} = 2$

Qn	Solution
	$\begin{pmatrix} -9+3s \\ 4+2s \\ 6-s \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = 2$ $-9+3s-4-2s+42-7s=2$ $6s=27$ $s=4.5$ $\mathbf{r} = \begin{pmatrix} -9 \\ 4 \\ 6 \end{pmatrix} + 4.5 \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{9}{2} \\ 13 \\ \frac{3}{2} \end{pmatrix}$ Coordinates: $\left(\frac{9}{2},\ 13,\ \frac{3}{2}\right)$

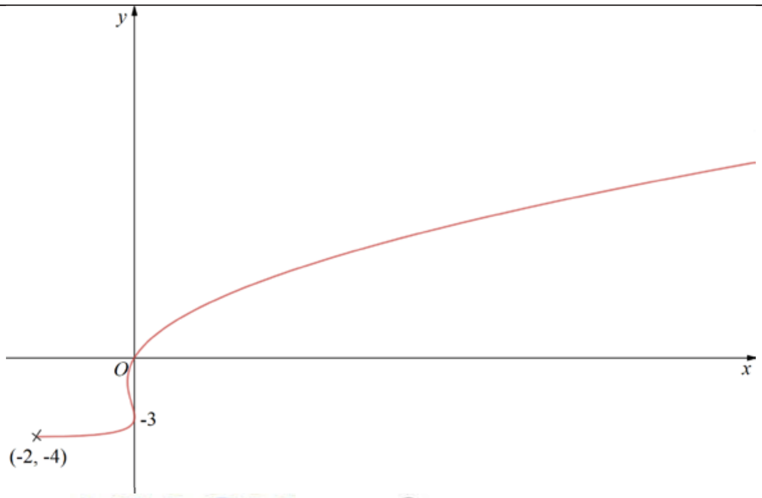
Solution to Paper 9758/02

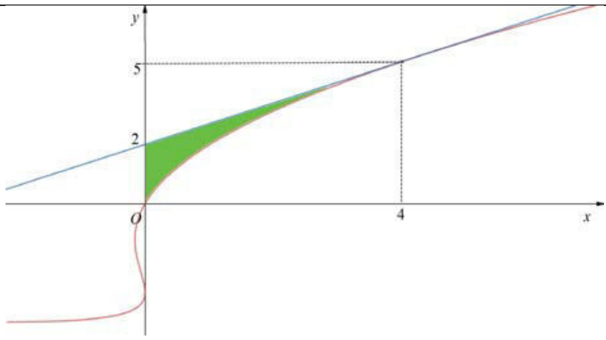
Section A: Pure Mathematics

Qn	Solution
1	$p_n = 3^{n-1} + a$ **<u>Method 1</u>** $\sum_{r=3}^{n} p_r = \sum_{r=3}^{n}\left(3^{r-1}+a\right) = \sum_{r=3}^{n}\left(3^{r-1}\right)+\sum_{r=3}^{n}a$ $= \left[3^2+3^3+3^4+\ldots+3^{n-1}\right]+(n-3+1)a$ $= \dfrac{3^2\left(3^{n-2}-1\right)}{3-1}+(n-3+1)a$ $= \dfrac{9}{2}\left(3^{n-2}-1\right)+(n-2)a$ **<u>Method 2</u>** $\sum_{r=3}^{n} p_r = \sum_{r=3}^{n}\left(3^{r-1}+a\right)$ $= \dfrac{1}{3}\sum_{r=3}^{n}\left(3^r\right)+\sum_{r=3}^{n}a$ $= \dfrac{1}{3}\left(3^3+3^4+3^5+\ldots+3^n\right)+(n-3+1)a$ $= \dfrac{1}{3}\left(\dfrac{3^3\left(3^{n-2}-1\right)}{3-1}\right)+(n-3+1)a$ $= \dfrac{9}{2}\left(3^{n-2}-1\right)+(n-2)a$

Qn	Solution
2(i)	Let T_n be the length of the nth plank Given that it is an AP: Let $a = 4$, $S_3 = 11.46$. Let d be the common difference. $S_3 = 11.46 = \frac{3}{2}(2(4)+2d)$ $d = -0.18$ $T_{18} = 4 + 17(-0.18)$ $= 0.94$ Hence the length of the 18th plank is 0.94 m.
(ii)	The length of the remaining **7** planks follows a GP: first term of GP: $T_{19} = 0.94\left(\frac{5}{4}\right)$ and $r = \frac{5}{4}$. Length of last plank $= T_{25}$ $= ar^{n-1}$ $= T_{19}\left(\frac{5}{4}\right)^{7-1}$ $= 0.94\left(\frac{5}{4}\right)\left(\frac{5}{4}\right)^{6}$ $= 0.94\left(\frac{5}{4}\right)^{7}$ $= 4.4822$ $= 4.48$ m (3 sf)
(iii)	**<u>Method 1</u>** Total length of blue planks $= \underbrace{T_2 + T_4 + T_6 + \dots + T_{18}}_{\text{Sum of AP with } a=T_2,\ d=-0.36,\ l=T_{18}} + \underbrace{T_{20} + T_{22} + T_{24}}_{\text{Sum of GP with } a=T_{20},\ r=\left(\frac{5}{4}\right)^2}$ $= \frac{9}{2}\left[(4-0.18)+0.94\right] + 0.94\left(\frac{5}{4}\right)^2\left(\frac{\left(\left(\frac{5}{4}\right)^2\right)^3 - 1}{\left(\frac{5}{4}\right)^2 - 1}\right)$ $= 21.42 + 7.34948$ $= 28.8$ m (3 sf) **<u>Method 2</u>** Total length of blue planks

Qn	Solution
	$=\frac{9}{2}\left[2(4-0.18)+(9-1)(2(-0.18))\right]$ $+0.94\left(\frac{5}{4}\right)^2\left(\frac{\left(\left(\frac{5}{4}\right)^2\right)^3-1}{\left(\frac{5}{4}\right)^2-1}\right)$ $=21.42+7.34948$ $=28.8$ m (3 sf)
3(a)	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\mathrm{e}^{-5x+3}+\sin x$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\int\left(\mathrm{e}^{-5x+3}+\sin x\right)\,\mathrm{d}x=-\frac{1}{5}\mathrm{e}^{-5x+3}-\cos x+c$ $y=\int\left(-\frac{1}{5}\mathrm{e}^{-5x+3}-\cos x+c\right)\mathrm{d}x$ $y=\frac{1}{25}\mathrm{e}^{-5x+3}-\sin x+cx+d$ where $c,\ d$ are arbitrary constants.
3(b)	$z=x+\frac{\mathrm{d}y}{\mathrm{d}x}\Rightarrow\frac{\mathrm{d}z}{\mathrm{d}x}=1+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ Hence, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{\mathrm{d}y}{\mathrm{d}x}-x+1=0\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+1=\left(\frac{\mathrm{d}y}{\mathrm{d}x}+x\right)$ Thus, replacing accordingly, $\frac{\mathrm{d}z}{\mathrm{d}x}=z$ (Shown) $\frac{\mathrm{d}z}{\mathrm{d}x}=z$ $\Rightarrow\int\frac{1}{z}\,\mathrm{d}z=\int 1\,\mathrm{d}x$ $\ln\lvert z\rvert=x+c$ **Method 1:** $\lvert z\rvert=\mathrm{e}^{x+c}$ $z=\pm\mathrm{e}^{x+c}$ $z=A\mathrm{e}^{x}$ where $A=\pm\mathrm{e}^{c}$ Since $z=x+\frac{\mathrm{d}y}{\mathrm{d}x}$, when $x=0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=1\Rightarrow z=1$ $z=A\mathrm{e}^{x}\Rightarrow 1=A\mathrm{e}^{0}\Rightarrow A=1$ $z=\mathrm{e}^{x}$

Qn	Solution
	Thus, $z = \mathrm{e}^x \Rightarrow x + \dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x - x$ $\Rightarrow y = \mathrm{e}^x - \dfrac{1}{2}x^2 + d$ When $x = 0$, $y = 1 \Rightarrow 1 = 1 + d \Rightarrow d = 0$ Hence, $y = \mathrm{e}^x - \dfrac{1}{2}x^2$ **<u>Method 2:</u>** $\ln\lvert z\rvert = x + c$ $\Rightarrow \ln 1 = 0 + c \Rightarrow c = 0$ Thus, $z = \pm\mathrm{e}^x \Rightarrow x + \dfrac{\mathrm{d}y}{\mathrm{d}x} = \pm\mathrm{e}^x$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x - x$ or $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^x - x$ (rej since $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right\rvert_{x=0} \neq 1$) $\Rightarrow y = \mathrm{e}^x - \dfrac{1}{2}x^2 + d$ When $x = 0$, $y = 1$ $\Rightarrow 1 = 1 + d \Rightarrow d = 0$ Hence, $y = \mathrm{e}^x - \dfrac{1}{2}x^2$
4(i)	
(ii)	$\dfrac{\mathrm{d}x}{\mathrm{d}t} = 3t^2 - 2t,\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 2t + 2$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2t+2}{3t^2 - 2t}$ At $t = 2$, $x = 4,\ y = 5,\ \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3}{4}$

Qn	Solution
	Equation of tangent: $y-5=\frac{3}{4}(x-4)$ $y=\frac{3}{4}x+2$
(iii)	 **<u>Method 1: Integrate wrt *y*</u>** When $y=0 \Rightarrow t^2+2t-3=0$ $\Rightarrow t=-3$ (rej) or $t=1$ When $y=5 \Rightarrow t=2$ (from part (ii)) Area $=\int_0^5 x\ \mathrm{d}y-\frac{1}{2}(4)(5-2)$ $=\int_1^2 (t^3-t^2)(2t+2)\ \mathrm{d}t-6$ $=\int_1^2 2t^4-2t^2\ \mathrm{d}t-6$ $=\left[\frac{2t^5}{5}-\frac{2t^3}{3}\right]_1^2-6$ $=\frac{26}{15}$ **<u>Method 2: Integrate wrt *x*</u>** When $x=0 \Rightarrow t^3-t^2=0$ $\Rightarrow t=0$ or $t=1$ $\Rightarrow y=-3$ (not this point) or $y=0$ When $x=4 \Rightarrow t=2$ (from part (ii)) Area $=\frac{1}{2}(2+5)(4)-\int_0^4 y\ \mathrm{d}x$ $=14-\int_1^2 (t^2+2t-3)(3t^2-2t)\ \mathrm{d}t$ $=14-\int_1^2 3t^4+4t^3-13t^2+6t\ \mathrm{d}t$ $=14-\left[\frac{3t^5}{5}+\frac{4t^4}{4}-\frac{13t^3}{3}+\frac{6t^2}{2}\right]_1^2$ $=\frac{26}{15}$

Qn	Solution
5(a) (i)	**<u>Method 1</u>** $z^4 - 2z^3 + az^2 - 8z + 40 = 0$ Since bi is a root, $(b\text{i})^4 - 2(b\text{i})^3 + a(b\text{i})^2 - 8(b\text{i}) + 40 = 0$ $b^4 + 2b^3\text{i} - ab^2 - 8b\text{i} + 40 = 0$ By comparing real and imaginary parts, $2b^3 - 8b = 0$ $2b(b^2 - 4) = 0$ Since $b \neq 0$, $b = -2$ (rej) or $b = 2$ When $b = 2$, $b^4 - ab^2 + 40 = 0$ $16 - 4a + 40 = 0$ $a = 14$ $z^4 - 2z^3 + 14z^2 - 8z + 40 = 0$ Using GC, Other roots are $z = 1 - 3\text{i},\ 1 + 3\text{i},\ -2\text{i}$. **<u>Method 2</u>** $z^4 - 2z^3 + az^2 - 8z + 40 = 0$ Since all coefficients are real, $-b$i is also a root. Quadratic factor: $(z - b\text{i})(z + b\text{i}) = z^2 + b^2$ $z^4 - 2z^3 + az^2 - 8z + 40 = (z^2 + b^2)(z^2 + cz + d)$ $z^4 - 2z^3 + az^2 - 8z + 40 = z^4 + cz^3 + (d + b^2)z^2 + cb^2z + b^2d$ By comparison of: Coefficient of z^3 : $c = -2$ Coefficient of z : $-8 = (-2)b^2 \Rightarrow b = 2$ (since $b > 0$) Constant: $40 = 4d \Rightarrow d = 10$ Coefficient of z^2 : $a = 10 + 4 = 14$ $z^4 - 2z^3 + 14z^2 - 8z + 40 = 0$ Using GC, Other roots are $z = 1 - 3\text{i},\ 1 + 3\text{i},\ -2\text{i}$ **<u>Method 3</u>** $z^4 - 2z^3 + az^2 - 8z + 40 = 0$ Since bi is a root, $(b\text{i})^4 - 2(b\text{i})^3 + a(b\text{i})^2 - 8(b\text{i}) + 40 = 0$ $b^4 + 2b^3\text{i} - ab^2 - 8b\text{i} + 40 = 0$

Qn	Solution
	By comparing real and imaginary parts, $2b^3-8b=0$ $2b(b^2-4)=0$ Since $b>0$, $b=-2$ (rejected) or $b=2$ When $b=2$, $b^4-ab^2+40=0$ $16-4a+40=0$ $a=14$ $z^4-2z^3+14z^2-8z+40=0$ Since all coefficients are real, –2i is also a root. Quadratic factor: $(z-2\mathrm{i})(z+2\mathrm{i})=z^2+4$ $z^4-2z^3+az^2-8z+40=(z^2+4)(z^2-2z+10)=0$ $z^2-2z+10=0$ $z=\dfrac{2\pm\sqrt{4-4(10)}}{2}=\dfrac{2\pm\sqrt{36\mathrm{i}^2}}{2}=1\pm3\mathrm{i}$ Hence, other roots are $z=1-3\mathrm{i},\ 1+3\mathrm{i},\ -2\mathrm{i}$.
5(a) (ii)	$w^4+2w^3+aw^2+8w+40=0$ Let $z=-w$ For $z^4-2z^3+14z^2-8z+40=0$ $\Rightarrow\ (-w)^4-2(-w)^3+14(-w)^2-8(-w)+40=0$ $w^4+2w^3+14w^2+8w+40=0$ $-w=1-3\mathrm{i},\ 1+3\mathrm{i},\ -2\mathrm{i}$ or $2\mathrm{i}$ $w=-1+3\mathrm{i},\ -1-3\mathrm{i},\ 2\mathrm{i}$ or $-2\mathrm{i}$
5(b) (i)	$w=-6+(2\sqrt{3})\mathrm{i}$ $\lvert w\rvert=\sqrt{(-6)^2+(2\sqrt{3})^2}=\sqrt{48}$ (or $4\sqrt{3}$) $\arg(w)=\pi-\tan^{-1}\left(\dfrac{2\sqrt{3}}{6}\right)=\dfrac{5\pi}{6}$ $\lvert w^n\rvert=\lvert w\rvert^n=(\sqrt{48})^n=48^{\frac{n}{2}}$ (or $(4\sqrt{3})^n$) $\arg(w^n)=n\arg(w)=\dfrac{5n\pi}{6}$ $w^n=48^{\frac{n}{2}}\mathrm{e}^{\mathrm{i}\left(\frac{5n\pi}{6}\right)}=(4\sqrt{3})^n\mathrm{e}^{\mathrm{i}\left(\frac{5n\pi}{6}\right)}$ Or $w^n=48^{\frac{n}{2}}\left[\cos\left(\dfrac{5n\pi}{6}\right)+\mathrm{i}\sin\left(\dfrac{5n\pi}{6}\right)\right]$

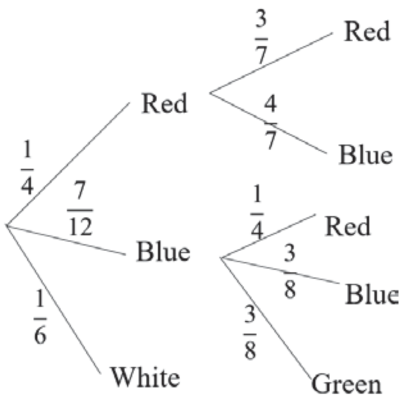
Qn	Solution
5(b) (ii)	$\arg\left(w^n w^*\right)=\dfrac{5n\pi}{6}-\dfrac{5\pi}{6}=\dfrac{5(n-1)\pi}{6}$ For $w^n w^*$ to be purely imaginary $\arg\left(w^n w^*\right)=\dfrac{\pi}{2},\ \dfrac{\pi}{2}\pm\pi,\ \dfrac{\pi}{2}\pm 2\pi,\ldots$ $\arg\left(w^n w^*\right)=\dfrac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$ $\arg\left(w^n w^*\right)=\dfrac{(2k+1)\pi}{2},\ k\in\mathbb{Z}$ $\dfrac{5(n-1)\pi}{6}=\dfrac{(2k+1)\pi}{2}$ $n-1=\dfrac{2k+1}{2}\left(\dfrac{6}{5}\right)$ $n=1+\dfrac{3(2k+1)}{5},\ k\in\mathbb{Z}$ $n=4,\ 10$ (when $k=2$ and $k=7$)

Section B: Probability and Statistics

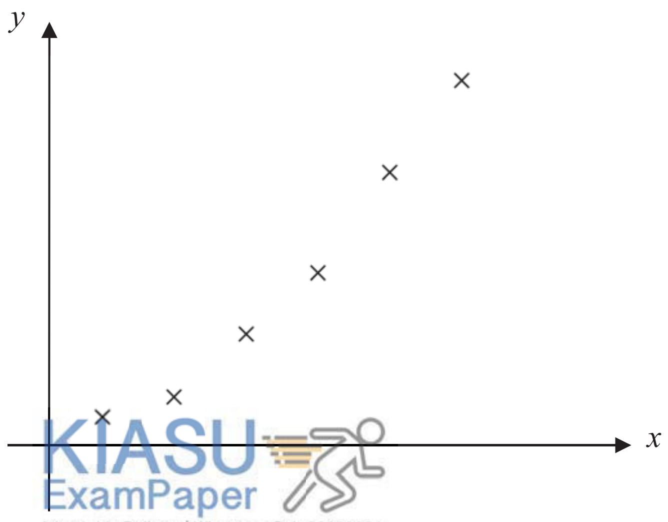
<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>6(i)</td><td>Total no. of possible IN $= 10^4 \times 10 = 100\ 000$ ways</td></tr>
<tr><td>6(ii)</td><td>Since the last letter of his IN is I, i.e. the remainder is 8, then the possible sum is 28 (since the sum must be at least 20)

The possible sets of digits are $\{9,9,a,b\}$, where $a+b=10$.
The possible values of $\{a,b\}$ are $\{2,8\}$, $\{3,7\}$, $\{4,6\}$ and $\{5,5\}$.

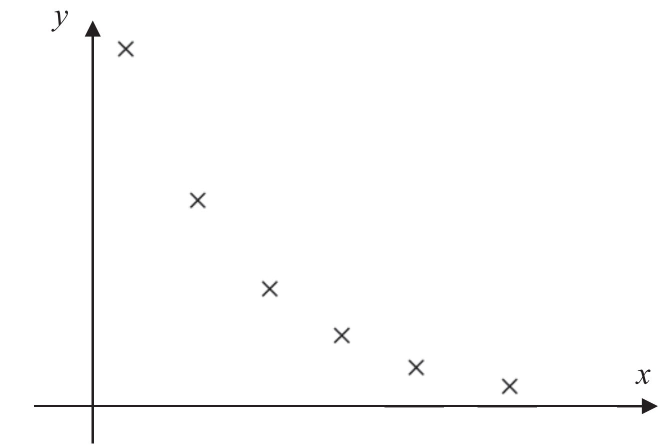
Hence, no. of ways $= 3\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!2!}\right) =$ 42 ways

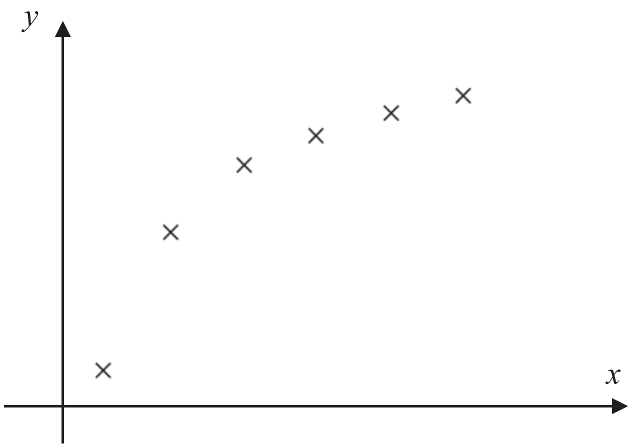
Hence, there are a total of 42 ways.</td></tr>
<tr><td>7(i)</td><td>

P(blue face shown and ball is red)
$= \frac{7}{12}\left(\frac{1}{4}\right)$
$= \frac{7}{48} = 0.146$ (3 sf)</td></tr>
<tr><td>7(ii)</td><td>P(mystery gift given) =P(all red) + P(all blue)
$= \frac{1}{4}\left(\frac{3}{7}\right)+\left(\frac{7}{12}\right)\left(\frac{3}{8}\right)$
$= \frac{73}{224}$
$= 0.32589 = 0.326$ (3 sf)</td></tr>
</table>

Qn	Solution
7(iii)	$\mathrm{P}(\text{one red ball picked} \mid \text{did not win mystery gift})$ $= \dfrac{\mathrm{P}(\text{one red ball picked and did not win mystery gift})}{1-\dfrac{73}{224}}$ $= \dfrac{7}{48} \div \left(1-\dfrac{73}{224}\right)$ $= \dfrac{98}{453}$ $= 0.216$ (3sf)
7 Last part	P(mystery gift given) = P(gift given in the first roll) + P(white face in 1st roll and gift given in the second roll) + P(white face in 1st 2 rolls and gift given in the third roll) $+\ldots$ $= \underbrace{\dfrac{73}{224} + \left(\dfrac{1}{6}\right)\left(\dfrac{73}{224}\right) + \left(\dfrac{1}{6}\right)^2\left(\dfrac{73}{224}\right) + \left(\dfrac{1}{6}\right)^3\left(\dfrac{73}{224}\right) + \ldots}_{\text{Sum of infinity with } a=\frac{73}{224} \text{ and } r=\frac{1}{6}}$ $= \dfrac{73}{224}\left[\dfrac{1}{1-\dfrac{1}{6}}\right]$ $= \dfrac{219}{560}$ $= 0.391$ (3 sf)
8 (a)	

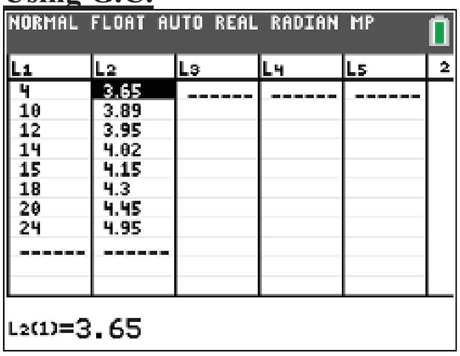
Qn	Solution



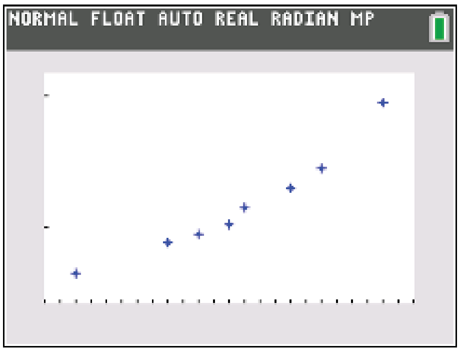


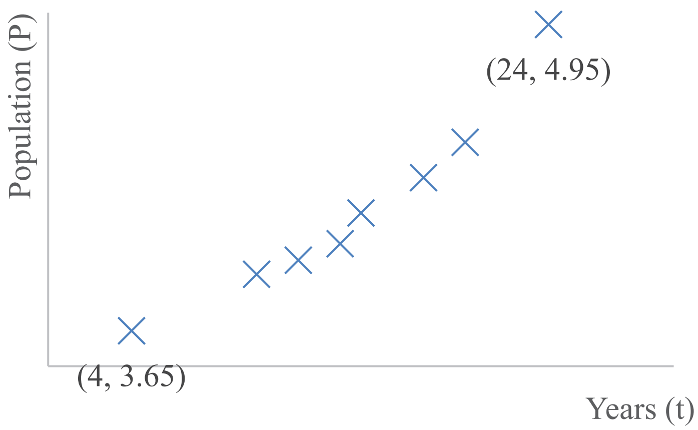
8 (b) (i)

<u>**Using G.C.**</u>

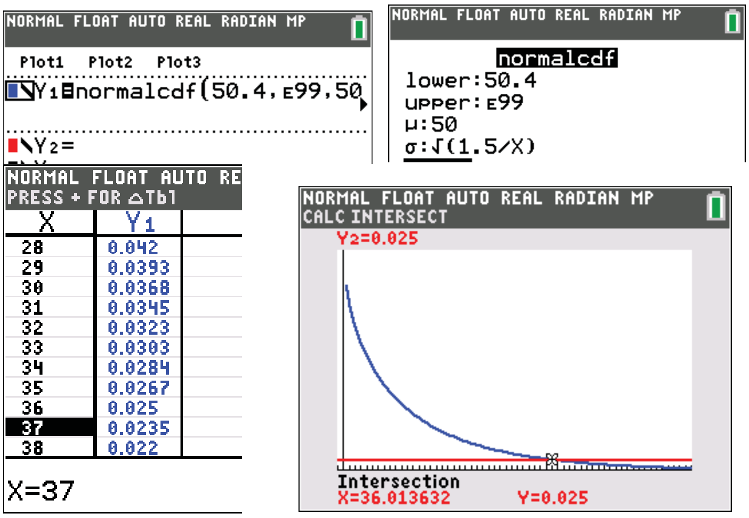


L1	L2	L3	L4	L5
4	3.65	------	------	------
10	3.89			
12	3.95			
14	4.02			
15	4.15			
18	4.3			
20	4.45			
24	4.95			
------	------			



Qn	Solution
	 A linear model may not be appropriate as the points do not lie close to a straight line. (Or the points do not follow a linear trend.)
8 (b)(ii)	**<u>Using G.C.</u>**  Note that from the scatter diagram suggests that as t increases, P increases at an increasing rate. Hence, Case (A) is most appropriate. Using G.C., $r = 0.995$ (3 s.f.)
8 (b) (iii)	Using G.C., $P = 3.6211 + 0.0022167t^2$ $P = 3.62 + 0.00222t^2$ (3 s.f.) In the year 2020, $t = 30$ Hence, when $t = 30$, $P = 3.6211 + 0.0022167(30)^2$ $= 5.6161$ Hence, the predicted population is approximately 5.62 million (3 s.f.)
8 (b) (iv)	The statistician's prediction is **not be reliable** as $t = 30$ is **not within data range**.

Qn	Solution
9(i)	A random sample means that each of the tablets were **selected independently** from each other *and* **each** tablet has an **equal chance** of being **selected**.
9(ii)	Let X be the amount in grams of active ingredient A in a particular type of health supplement tablets. Define μ: Let μ be the **population mean amount/mass of active ingredient** A in a particular type of health supplement tablets. Given that sample mean $\bar{x} = 50.6$, sample variance $= 2.15$ **Unbiased estimate** of population variance $=\frac{40}{39}(2.15) = 2.2051$ $H_0 : \mu = 50$ $H_1 : \mu \neq 50$ Under H_0, since ***n* = 40 is large, by Central Limit Theorem**, $\bar{X} \sim N\left(50, \frac{2.2051}{40}\right)$ approximately. Use z-test at $\alpha = 0.01$ Using GC, p-value $= 0.010605 = 0.0106$ (3 sf) Since p-value $> \alpha$, do not reject H_0. There is **insufficient evidence** at 1% level of significance to conclude that the mean amount of active ingredient A has changed (or is not 50 mg).
9 last part	Let Y be the amount in grams of active ingredient A in the revised formula $H_0 : \mu = 50$ $H_1 : \mu > 50$ Under H_0, $\bar{Y} \sim N\left(50, \frac{1.5}{n}\right)$ Use z-test at $\alpha = 0.025$. <u>**Method 1**</u> Test-statistic distribution: $Z = \frac{\bar{Y} - 50}{\sqrt{\frac{1.5}{n}}} \sim N(0, 1)$

Qn	Solution
	Corresponding test-statistic: $z=\dfrac{50.4-50}{\sqrt{\dfrac{1.5}{n}}}=\dfrac{0.4\sqrt{n}}{\sqrt{1.5}}$ Critical region: $z \geq 1.95996$ Since the chemist claim is valid, H0 is rejected. Test-statistic falls within the critical region. $\dfrac{0.4\sqrt{n}}{\sqrt{1.5}} \geq 1.95996$ $n \geq 36.0135$ $n \geq 37$ $\{n \in \mathbb{Z} : n \geq 37\}$ **<u>Method 2 (Using GC, Table/Graph method)</u>** For H0 to be rejected, p - value ≤ 0.025. p-value = $\mathrm{P}\left(\overline{Y} > 50.4\right) \leq 0.025$  $n \geq 37$ $\{n \in \mathbb{Z} : n \geq 37\}$
10 (i)	$\mathrm{P}(X=2)=\mathrm{P}(\text{point } A_2)$ $=\mathrm{P}(\text{2 steps to the right and 3 steps downwards})$ $={}^5C_2\,p^2q^3$ $=10p^2q^3$ (Shown)
10 (ii)	Note that $\mathrm{P}(X=3)=\mathrm{P}(\text{point } A_3)$ $={}^5C_3\,p^3q^2=10p^3q^2$ Since the particle will end up at point A_2 most of the time, then mode occurs at $X=2$.

Qn	Solution
	$\therefore \mathrm{P}(X=3) < \mathrm{P}(X=2)$ $10p^3q^2 < 10p^2q^3$ $\frac{q}{p} > 1 \Rightarrow 1-p > p$ $2p < 1 \Rightarrow p < 0.5$ In the same way, $\mathrm{P}(X=2) > \mathrm{P}(X=1)$ $10p^2q^3 > 5pq^4$ $2p > q$ $2p > 1-p$ $3p > 1 \Rightarrow p > \frac{1}{3}$ Combining both inequalities, $\frac{1}{3} < p < \frac{1}{2}$ (Shown)
10 (iii)	<table><tr><td>Y</td><td>0</td><td>2</td><td>3</td><td>5</td></tr><tr><td>$\mathrm{P}(Y=y)$</td><td>0.07776</td><td>0.4224</td><td>0.299904</td><td>0.199936</td></tr></table> $\mathrm{P}(Y=0) = \mathrm{P}(X=0) = {}^5C_0(0.4)^0(0.6)^5 = 0.07776$ $\mathrm{P}(Y=2) = \mathrm{P}(X=i,\text{ where } i \text{ is even})$ $= \mathrm{P}(X=2) + \mathrm{P}(X=4)$ $= 10(0.4)^2(0.6)^3 + {}^5C_4(0.4)^4(0.6)^1 = 0.4224$ $\mathrm{P}(Y=3) = \mathrm{P}(X=i,\text{ where } i \text{ is odd and lose the game})$ $= 0.6(\mathrm{P}(X=1) + \mathrm{P}(X=3) + \mathrm{P}(X=5))$ $= 0.6\left[{}^5C_1(0.4)^1(0.6)^4 + {}^5C_3(0.4)^3(0.6)^2 + {}^5C_5(0.4)^5(0.6)^0\right]$ $= 0.299904$ $\mathrm{P}(Y=5) = \mathrm{P}(X=i,\text{ where } i \text{ is odd and win the game})$ $= 0.4(\mathrm{P}(X=1) + \mathrm{P}(X=3) + \mathrm{P}(X=5))$ $= 0.4\left[{}^5C_1(0.4)^1(0.6)^4 + {}^5C_3(0.4)^3(0.6)^2 + {}^5C_5(0.4)^5(0.6)^0\right]$ $= 0.199936$
10 (iv)	Using G.C. $\mathrm{E}(Y) = 2.744192 = 2.74$ (3 sf) $\mathrm{Var}(Y) = 1.3626^2 = 1.8567 = 1.86$ (3 sf)

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>11(i)</td><td>Let X be the mass, in g, of a randomly chosen dark truffle.
$X \sim \mathrm{N}\left(17,\ 1.3^2\right)$

Let $T = X_1 + X_2 + X_3 + X_4$

$\mathrm{E}(T) = 4(17) = 68$
$\mathrm{Var}(T) = 4\left(1.3^2\right) = 6.76$
$T \sim \mathrm{N}(68,\ 6.76)$

$\mathrm{P}(T > 70) = 0.220878 = 0.221$ (3 sf)</td></tr>
<tr><td>11(ii)</td><td>Let Y be the number of boxes that <u>weigh more than 70 g</u>, out of 20 boxes.

$Y \sim \mathrm{B}(20,\ 0.220878)$

$\mathrm{P}(Y > 3) = 1 - \mathrm{P}(Y \le 3)$
$= 0.67448$
$= 0.674$ (3 sf)

Assumption: The mass of the empty box is negligible.

(Other possible answer: The event that a mass of a box of 4 dark truffles has mass more than 70g is independent of other boxes.)</td></tr>
<tr><td>11(iii)</td><td>Let W be the mass, in g, of a randomly chosen salted caramel ganache.
$W \sim \mathrm{N}\left(\mu,\ \sigma^2\right)$

Given $\mathrm{P}(W < 12) = \mathrm{P}(W > 15)$ and $\mathrm{P}(W \le 15) = 0.97$
<u>Method 1</u>
By symmetry,
$\mu = \dfrac{12+15}{2} = 13.5$
$\mathrm{P}(W \le 15) = 0.97$
$\mathrm{P}\left(Z \le \dfrac{15-13.5}{\sigma}\right) = 0.97$
$\dfrac{1.5}{\sigma} = 1.88079$
$\sigma = 0.797537$
$\sigma^2 = 0.636065$
$\sigma^2 = 0.636$ (3 sf)</td></tr>
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Qn	Solution
	<u>Method 2</u> $\mathrm{P}(W < 12) = 0.03$ $\mathrm{P}\left(Z < \dfrac{12-\mu}{\sigma}\right) = 0.03$ $\dfrac{12-\mu}{\sigma} = -1.88079$ $\mu - 1.88079\sigma = 12$ ---(1) $\mathrm{P}(W \le 15) = 0.97$ $\mathrm{P}\left(Z < \dfrac{15-\mu}{\sigma}\right) = 0.03$ $\dfrac{15-\mu}{\sigma} = 1.88079$ $\mu + 1.88079\sigma = 15$ ---(2) Solving equation (1) and (2), $\mu = 13.5, \quad \sigma = 0.797537$ $\sigma^2 = 0.636065$ $\sigma^2 = 0.636$ (3 sf)
11(iv)	$W \sim \mathrm{N}(13.5,\ 0.636065)$ Find $\mathrm{P}\left(0.28(W_1 + W_2 + \ldots + W_6) < 0.34T\right)$. Let $S = 0.28(W_1 + W_2 + \ldots + W_6) - 0.34T$ $\mathrm{E}(S) = 0.28(6)(13.5) - 0.34(68) = -0.44$ $\mathrm{Var}(S) = 0.28^2(6)(0.636065) + 0.34^2(6.76) = 1.08065$ $S \sim \mathrm{N}(-0.44,\ 1.08065)$ $\mathrm{P}(S < 0) = 0.66394 = 0.664$ (3 sf)