Differentiation – Practice Questions 1a Answers

1. Differentiate the following with respect to x:

(a) $2x^2 + 5x + 7$

$$\frac{d}{dx} 2x^2 + 5x + 7 = 4x + 5$$

(b) $\frac{7}{2}x^4 + \frac{5}{3x^4} + x$

$$\frac{d}{dx}\left(\frac{7}{2}x^4 + \frac{5}{3x^4} + x\right) = \frac{d}{dx}\left(\frac{7}{2}x^4 + \frac{5}{3}x^{-4} + x\right)$$

$$= 4\left(\frac{7}{2}x^3\right) + (-4)\left(\frac{5}{3}x^{-5}\right) + 1$$

$$= 14x^3 - \frac{20}{3}x^{-5} + 1$$

$$= 14x^3 - \frac{20}{3x^5} + 1$$

(c) $4x^2 + \frac{2}{3\sqrt{x}} + 7$

$$\frac{d}{dx}\left(4x^2 + \frac{2}{3\sqrt{x}} + 7\right) = \frac{d}{dx}\left(4x^2 + \frac{2}{3}x^{-\frac{1}{2}} + 7\right)$$

$$= 8x + \left(-\frac{1}{2}\right)\left(\frac{2}{3}x^{-\frac{3}{2}}\right)$$

$$= 8x - \frac{1}{3x^{\frac{3}{2}}}$$

$$= 8x - \frac{1}{3\sqrt{x^3}}$$

(d) $5ax^2 + 3bx^3 + 5$

$$\frac{d}{dx}(5ax^2 + 3bx^3 + 5) = 2(5ax) + 3(3bx^2)$$

$$= 10ax + 9bx^2$$

(e) $\frac{3x^2 - 4x^3}{x^3}$

$$\frac{d}{dx}\left(\frac{3x^2 - 4x^3}{x^3}\right) = \frac{d}{dx}\left(\frac{3}{x} - 4\right)$$

$$= \frac{d}{dx}(3x^{-1} - 4)$$

$$= -\frac{3}{x^2}$$

(f) $\frac{3\sqrt{x}-4}{\sqrt{x}}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3\sqrt{x}-4}{\sqrt{x}}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(3 - \frac{4}{\sqrt{x}}\right)$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(3 - 4x^{-\frac{1}{2}}\right)$$
$$= -\frac{1}{2}\left(-4x^{-\frac{3}{2}}\right)$$
$$= 2x^{-\frac{3}{2}}$$
$$= \frac{2}{\sqrt{x^3}}$$

(g) $\sqrt{x} + \frac{1}{\sqrt{x}}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)$$
$$= \frac{1}{2}x^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$$
$$= \frac{1}{2}x^{-\frac{3}{2}}(x-1) \qquad \text{or} \qquad = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$$
$$= \frac{x-1}{2\sqrt{x^3}}$$

2. At x-axis, $y = 0$

$$0 = \frac{5x-4}{x^2}$$
$$5x - 4 = 0$$
$$x = \frac{4}{5}$$
$$y = \frac{5x-4}{x^2} = \frac{5}{x} - \frac{4}{x^2} = 5x^{-1} - 4x^{-2}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -5x^{-2} + 8x^{-3} = -\frac{5}{x^2} + \frac{8}{x^3}$$

At $x = \frac{4}{5}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{5}{\left(\frac{4}{5}\right)^2} + \frac{8}{\left(\frac{4}{5}\right)^3} = 7\frac{13}{16} = \frac{125}{16}$

3. $y = \frac{a}{x^2} + \frac{b}{x} = ax^{-2} + bx^{-1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2ax^{-3} - bx^{-2} = -\frac{2a}{x^3} - \frac{b}{x^2}$$

At $(-1, 5)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 4 = -\frac{2a}{(-1)^3} - \frac{b}{(-1)^2}$

$$4 = -\frac{2a}{-1} - \frac{b}{1}$$

$4 = 2a - b$ ----- (1)

At $(-1, 5)$, $5 = \frac{a}{(-1)^2} + \frac{b}{-1}$

$5 = a - b$

$a = 5 + b$ ----- (2)

$Sub\ (2)\ into\ (1)$

$4 = 2(5 + b) - b$

$4 = 10 + 2b - b$

$4 - 10 = b$

$b = -6$

$a = 5 - 6 = -1$

4. **CHAIN RULE**

(a) $(2x + 5)^7$

$$\frac{d}{dx}[(2x + 5)^7] = 7(2x + 5)^6(2)$$
$$= 14(2x + 5)^6$$

(b) $3(x + 4)^5$

$$\frac{d}{dx}[3(x + 4)^5] = 3(5)(x + 4)^4(1)$$
$$= 15(x + 4)^4$$

(c) $\frac{2}{3}\left(\frac{x}{6} - 1\right)^4$

$$\frac{d}{dx}\left[\frac{2}{3}\left(\frac{x}{6} - 1\right)^4\right] = \frac{2}{3}(4)\left(\frac{x}{6} - 1\right)^3\left(\frac{1}{6}\right)$$
$$= \frac{4}{9}\left(\frac{x}{6} - 1\right)^3$$

(d) $\frac{1}{3x+2}$

$$\frac{d}{dx}\left(\frac{1}{3x+2}\right) = \frac{d}{dx}[(3x + 2)^{-1}]$$
$$= -(3x + 2)^{-2}(3)$$
$$= -\frac{3}{(3x+2)^2}$$

(e) $\frac{12}{2+3x^2}$

$$\frac{d}{dx}\left(\frac{12}{2+3x^2}\right) = 12\frac{d}{dx}(2 + 3x^2)^{-1}$$

$= 12(-1)(2+3x^2)^{-2}(6x)$

$= -\frac{72x}{(2+3x^2)^2}$

(f) $\sqrt{5x^2+6}$

$\frac{D}{dx}\left(\sqrt{5x^2+6}\right) = \frac{d}{dx}(5x^2+6)^{\frac{1}{2}}$

$= \frac{1}{2}(5x^2+6)^{-\frac{1}{2}}(10x)$

$= \frac{5x}{\sqrt{5x^2+6}}$

5. $y = (3x^2-5x+3)^3$

$\frac{dy}{dx} = 3(3x^2-5x+3)^2(6x-5)$

At $x = 1, \frac{dy}{dx} = 3(3(1)^2-5(1)+3)^2(6(1)-5)$

$= 3(3-5+3)^2(6-5)$

$= 3(1)^2(1)$

$= 3$

6. $y = 2 + \frac{12}{(3x-4)^2} = 2 + 12(3x-4)^{-2}$

$\frac{dy}{dx} = 12(-2)(3x-4)^{-3}(3) = -\frac{72}{(3x-4)^3}$.

At $(2,5), \frac{dy}{dx} = -\frac{72}{(3(2)-4)^3} = -\frac{72}{8} = -9$

Thus $y - 5 = -9(x-2)$

$y = -9x + 18 + 5$

$y = -9x + 23$

7. $y = \frac{a}{2+bx} = a(2+bx)^{-1}$

$\frac{dy}{dx} = a(-1)(2+bx)^{-2}(b) = -\frac{ab}{(2+bx)^2}$

At $x = 1, \frac{dy}{dx} = \frac{3}{5} = -\frac{ab}{(2+b)^2}$

$3(2+b)^2 = -5ab$ ------ (1)

At $x = 1, y = 1, 1 = \frac{a}{2+b}$

$a = 2 + b$ ------ (2)

$Sub\ (2)\ into\ (1)$

$3(2 + b)^2 = -5b(2 + b)$

$3(4 + 4b + b^2) = -10b - 5b^2$

$12 + 12b + 3b^2 = -10b - 5b^2$

$8b^2 + 22b + 12 = 0$

$(b + 2)(8b + 6) = 0$

$b = -2$ or $b = -\frac{3}{4}$

$a = 2 - 2 = 0$ or $a = 2 - \frac{3}{4} = 1\frac{1}{4}$

Reject a =0 and $b = -2$

Differentiation – Practice Questions 1b Answers

1. Answer

(a) $(3x+2)(2-x^2)$

$$\frac{\mathrm{d}}{\mathrm{d}x}[(3x+2)(2-x^2)] = (3x+2)\frac{\mathrm{d}}{\mathrm{d}x}(2-x^2) + (2-x^2)\frac{\mathrm{d}}{\mathrm{d}x}(3x+2)$$

$$= (3x+2)(-2x) + (2-x^2)(3)$$

$$= -6x^2 - 4x + 6 - 3x^2$$

$$= -9x^2 - 4x + 6$$

(b) $(x+1)^3(x+3)^5$

$$\frac{\mathrm{d}}{\mathrm{d}x}[(x+1)^3(x+3)^5] = (x+1)^3\frac{\mathrm{d}}{\mathrm{d}x}(x+3)^5 + (x+3)^5\frac{\mathrm{d}}{\mathrm{d}x}(x+1)^3$$

$$= (x+1)^3(5)(x+3)^4 + (x+3)^5(3)(x+1)^2$$

$$= (x+1)^2(x+3)^4[5(x+1) + (x+3)(3)]$$

$$= (x+1)^2(x+3)^4[5x + 5 + 3x + 9]$$

$$= (x+1)^2(x+3)^4(8x+14)$$

$$= 2(x+1)^2(x+3)^4(4x+7)$$

(c) $(x+5)^3(x-4)^6$

$$\frac{\mathrm{d}}{\mathrm{d}x}[(x+5)^3(x-4)^6] = (x+5)^3\frac{\mathrm{d}}{\mathrm{d}x}(x-4)^6 + (x-4)^6\frac{\mathrm{d}}{\mathrm{d}x}(x+5)^3$$

$$= (x+5)^3(6)(x-4)^5 + (x-4)^6(3)(x+5)^2$$

$$= (x+5)^2(x-4)^5[6(x+5) + 3(x-4)]$$

$$= (x+5)^2(x-4)^5[6x + 30 + 3x - 12]$$

$$= (x+5)^2(x-4)^5(9x+18)$$

$$= 9(x+5)^2(x-4)^5(x+2)$$

(d) $(3x-1)\sqrt{2x^2+3} = (3x-1)(2x^2+3)^{\frac{1}{2}}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(3x-1)(2x^2+3)^{\frac{1}{2}}\right] = (3x-1)\frac{\mathrm{d}}{\mathrm{d}x}(2x^2+3)^{\frac{1}{2}} + (2x^2+3)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}(3x-1)$$

$$= (3x-1)\left(\frac{1}{2}\right)(2x^2+3)^{-\frac{1}{2}}(4x) + (2x^2+3)^{\frac{1}{2}}(3)$$

$$= (2x^2+3)^{-\frac{1}{2}}[(3x-1)(2x) + (2x^2+3)^1(3)]$$

$$= (2x^2+3)^{-\frac{1}{2}}[6x^2 - 2x + 6x^2 + 9]$$

$$= \frac{[6x^2-2x+6x^2+9]}{(2x^2+3)^{\frac{1}{2}}}$$

$$= \frac{12x^2-2x+9}{\sqrt{2x^2+3}}$$

(e) $(x^3 + x^2)(x-2)^7$

$$\frac{\mathrm{d}}{\mathrm{d}x}[(x^3+x^2)(x-2)^7] = (x^3+x^2)\frac{\mathrm{d}}{\mathrm{d}x}(x-2)^7 + (x-2)^7\frac{\mathrm{d}}{\mathrm{d}x}(x^3+x^2)$$

$$= (x^3+x^2)(7)(x-2)^6 + (x-2)^7(3x^2+2x)$$

$$= (x-2)^6[7(x^3+x^2) + (x-2)(3x^2+2x)]$$

$$= (x-2)^6[7x^3+7x^2+3x^3+2x^2-6x^2-4x]$$

$$= (x-2)^6[10x^3+3x^2-4x]$$

$$= x(x-2)^6(10x^2+3x-4)$$

$$= x(x-2)^6(2x-1)(5x+4)$$

2. $y = (x+3)^4(x-5)^7$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x+3)^4(7)(x-5)^6 + (x-5)^7(4)(x+3)^3$$

$$= (x+3)^3(x-5)^6[7(x+3) + (4)(x-5)]$$

$$= (x+3)^3(x-5)^6[7x+21+4x-20]$$

$$= (x+3)^3(x-5)^6(11x+1)$$

When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $(x+3)^3(x-5)^6(11x+1) = 0$

Thus $(x+3)^3 = 0$, $(x-5)^6 = 0$, $(11x+1) = 0$

$x = -3, 5, -\frac{1}{11}$

3. $y = (3x+2)(2-x)^{-1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (3x+2)(-1)(2-x)^{-2}(-1) + (2-x)^{-1}(3)$$

$$= (2-x)^{-2}[(3x+2) + (2-x)^1(3)]$$

$$= (2-x)^{-2}[3x+2+6-3x]$$

$$= \frac{8}{(2-x)^2}$$

When $\frac{\mathrm{d}y}{\mathrm{d}x} = 8$,

$$8 = \frac{8}{(2-x)^2}$$

$(2-x)^2 = 1$

$2 - x = \pm 1$

$x = 1, 3$

4. Answer

(a) $\frac{3x+2}{1-4x}$

$$\frac{d}{dx}\left(\frac{3x+2}{1-4x}\right) = \frac{(1-4x)(3)-(3x+2)(-4)}{(1-4x)^2}$$

$$= \frac{3-12x+12x+8}{(1-4x)^2}$$

$$= \frac{11}{(1-4x)^2}$$

(b) $\frac{x^2}{2x-1}$

$$\frac{d}{dx}\left(\frac{x^2}{2x-1}\right) = \frac{(2x-1)(2x)-(x^2)(2)}{(2x-1)^2}$$

$$= \frac{4x^2-2x-2x^2}{(2x-1)^2}$$

$$= \frac{2x^2-2x}{(2x-1)^2}$$

$$= \frac{2x(x-1)}{(2x-1)^2}$$

(c) $\frac{\sqrt{x}}{3+x}$

$$\frac{d}{dx}\left(\frac{\sqrt{x}}{3+x}\right) = \frac{(3+x)\left(\frac{1}{2}\right)\left(x^{-\frac{1}{2}}\right)-x^{\frac{1}{2}}(1)}{(3+x)^2}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}[(3+x)-2x]}{(3+x)^2}$$

$$= \frac{[3+x-2x]}{2\sqrt{x}(3+x)^2}$$

$$= \frac{3-x}{2\sqrt{x}(3+x)^2}$$

(d) $\frac{2x-7}{\sqrt{x+1}}$

$$\frac{d}{dx}\left(\frac{2x-7}{\sqrt{x+1}}\right) = \frac{(x+1)^{\frac{1}{2}}(2)-(2x-7)\left(\frac{1}{2}\right)(x+1)^{-\frac{1}{2}}}{x+1}$$

$$= \frac{\frac{1}{2}(x+1)^{-\frac{1}{2}}[2(2)-(2x-7)]}{x+1}$$

$$= \frac{[4-2x+7]}{2\sqrt{(x+1)^3}}$$

$$= \frac{11-2x}{2\sqrt{(x+1)^3}}$$

8. $y = \frac{x+2}{\sqrt{3x+1}}$

$$\frac{dy}{dx} = \frac{(3x+1)^{\frac{1}{2}}(1)-(x+2)\left(\frac{1}{2}\right)(3x+1)^{-\frac{1}{2}}(3)}{3x+1}$$

$$= \frac{\frac{1}{2}(3x+1)^{-\frac{1}{2}}[2(3x+1)-(x+2)(3)]}{3x+1}$$

$$= \frac{[2(3x+1)-(x+2)(3)]}{2(3x+1)^{\frac{3}{2}}}$$

$$= \frac{6x+2-3x-6}{2(3x+1)^{\frac{3}{2}}}$$

$$= \frac{3x-4}{2(3x+1)^{\frac{3}{2}}}$$

(a) At $x = 1, \frac{dy}{dx} = \frac{3(1)-4}{2(3+1)^{\frac{3}{2}}} = -\frac{1}{16}$

(b) $y = \frac{1+2}{\sqrt{3+1}} = \frac{3}{2}$

$y - \frac{3}{2} = -\frac{1}{16}(x-1)$

$y = -\frac{1}{16}x + \frac{25}{16}$

$16y = -x + 25$

9. Answer

(a) $2x + 9y = 3$ ------ (1)

$xy + y + 2 = 0$ ------ (2)

(1): $x = \frac{3-9y}{2}$ ------ (3)

Sub (3) into (2)

$\left(\frac{3-9y}{2}\right)y + y + 2 = 0$

$(3-9y)y + 2y + 4 = 0$

$3y - 9y^2 + 2y + 4 = 0$

$-9y^2 + 5y + 4 = 0$

$(y-1)(-9y-4) = 0$

$y = 1$ or $y = -\frac{4}{9}$

$x = \frac{3-9}{2}$ or $x = \frac{3-9\left(-\frac{4}{9}\right)}{2}$

$x = -3$ or $x = \frac{7}{2}$

(b) Answer

$xy + y + 2 = 0$

$xy + y = -2$

$y(x + 1) = -2$

$y = -\frac{2}{(x+1)} = -2(x + 1)^{-1}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -2(-1)(x + 1)^{-2} = \frac{2}{(x+1)^2}$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{(x+1)(0)-2(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$

At $x = -3, \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{(-3+1)^2} = \frac{1}{2}$

or $x = \frac{7}{2}, \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{(3.5+1)^2} = \frac{8}{81}$

Applications of Differentiation – Practice Questions 2 Answers

1. $y = \sqrt{1-2x}$

   $\frac{dy}{dx} = \frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2) = -\frac{1}{(1-2x)^{\frac{1}{2}}}$

   At $y = 3$,

   $3 = \sqrt{1-2x}$

   $9 = 1 - 2x$

   $2x = 1 - 9$

   $2x = -8$

   $x = -4$

   $\frac{dy}{dx} = -\frac{1}{(1-2(-4))^{\frac{1}{2}}} = -\frac{1}{(9)^{\frac{1}{2}}} = -\frac{1}{3}$

   At $x = -4, y = 3$ and $m = -\frac{1}{3}$

   Equation of tangent: $y - 3 = -\frac{1}{3}(x - (-4))$

   $y = -\frac{1}{3}x - \frac{4}{3} + 3$

   $y = -\frac{1}{3}x + \frac{5}{3}$

2. $y = \frac{3x+1}{1-x}$

   $\frac{dy}{dx} = \frac{(1-x)(3)-(3x+1)(-1)}{(1-x)^2}$

   $= \frac{3-3x+3x+1}{(1-x)^2}$

   $= \frac{4}{(1-x)^2}$

   At x-axis, $y = 0$.

   $0 = \frac{3x+1}{1-x}$

   $3x + 1 = 0$

   $x = -\frac{1}{3}$

   At $x = -\frac{1}{3}, \frac{dy}{dx} = \frac{4}{\left(1+\frac{1}{3}\right)^2} = \frac{9}{4}$

   $m_{normal} = -\frac{1}{m_{tangent}} = -\frac{4}{9}$

   At $\left(-\frac{1}{3}, 0\right)$ and $m_{normal} = -\frac{4}{9}$,

$y - 0 = -\frac{4}{9}\left(x + \frac{1}{3}\right)$

$y = -\frac{4}{9}x - \frac{4}{27}$

$27y = -12x - 4$

3. Answer

$x^4 y = 1$

$y = x^{-4}$

$\frac{dy}{dx} = -4x^{-5}$

When $x = 1$, $\frac{dy}{dx} = -4$, $y = 1$.

Equation of the tangent at $x = 1$:

$\frac{y - 1}{x - 1} = -4$

$y = -4x + 5$

The tangent meets the x-axis when $y = 0$, that is, $x = \frac{5}{4}$.

The tangent meets the y-axis when $x = 0$, that is, $y = 5$.

Thus, the tangent meets the axes at $A\left(\frac{5}{4}, 0\right)$ and $B(0, 5)$.

$\therefore$ Midpoint of $AB = \left(\frac{\frac{5}{4} + 0}{2}, \frac{0 + 5}{2}\right)$

$= \left(\frac{5}{8}, \frac{5}{2}\right)$

4. Answer

$y = 4x^3 - 3x$

$\frac{dy}{dx} = 12x^2 - 3$

At $\left(-\frac{1}{2}, 1\right)$, $\frac{dy}{dx} = 12\left(-\frac{1}{2}\right)^2 - 3 = 0$

$y - 1 = 0\left(x + \frac{1}{2}\right)$

$y - 1 = 0$

$y = 1$ -------- (1)

$y = 4x^3 - 3x$ -------- (2)

Sub (1) into (2)

$4x^3 - 3x = 1$

$4x^3 - 3x - 1 = 0$

Let f(x) = $4x^3 - 3x - 1$

f(1) = $4(1)^3 - 3(1) - 1 = 0$ or just use the tangent point

$x - 1$ is a factor. or $x + \frac{1}{2}$ is a factor

Thus, long division or comparing coefficient method:

f(x) = $4x^3 - 3x - 1 = (x - 1)(ax^2 + bx + c)$

$= ax^3 + bx^2 + cx - ax^2 - bx - c$

$= ax^3 + bx^2 - ax^2 - bx + cx - c$

$a = 4$

$c = 1$

$-b + c = -3$

$-b + 1 = -3$

$-b = -4$

$b = 4$

Thus f(x) = $(x - 1)(4x^2 + 4x + 1)$

$= (x - 1)(2x + 1)^2$

Thus f(x) $= 0$, $(x - 1)(2x + 1)^2 = 0$

$(x - 1) = 0$ or $(2x + 1)^2 = 0$

$x = 1$ or $x = -\frac{1}{2}$ (tangent)

So the curve will touch again at (1, 1)

5. $y = \frac{3(x^2-1)}{2x-1}$

(a) $\frac{dy}{dx} = \frac{(2x-1)(3)(2x)-3(x^2-1)(2)}{(2x-1)^2}$

$= \frac{12x^2-6x-6x^2+6}{(2x-1)^2}$

$= \frac{6x^2-6x+6}{(2x-1)^2}$

$= \frac{6(x^2-x+1)}{(2x-1)^2}$

When $\frac{dy}{dx} = 2$, $\frac{6x^2-6x+6}{(2x-1)^2} = 2$

$6x^2 - 6x + 6 = 2(2x - 1)^2$

$6x^2 - 6x + 6 = 2(4x^2 - 4x + 1)$

$6x^2 - 6x + 6 = 8x^2 - 8x + 2$

$2x^2 - 2x - 4 = 0$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x = -1, 2$

(b) At $x = 1$, $y = \frac{3(1-1)}{2-1} = 0$

$\frac{\text{dy}}{\text{dx}} = \frac{6(1)^2 - 6(1) + 6}{(2-1)^2} = 6$

$m_{tangent} = 6,\ m_{normal} = -\frac{1}{6}$

Tangent: $y - 0 = 6(x-1)$

$y = 6x - 6$

Normal: $y - 0 = -\frac{1}{6}(x-1)$

$y = -\frac{1}{6}x + \frac{1}{6}$

6. $y = \frac{2x^3}{x-1}$

(a) $\frac{\text{dy}}{\text{dx}} = \frac{(x-1)(3)(2x^2) - 2x^3}{(x-1)^2}$

$= \frac{2x^2[(x-1)(3) - x]}{(x-1)^2}$

$= \frac{2x^2[3x - 3 - x]}{(x-1)^2}$

$= \frac{2x^2(2x-3)}{(x-1)^2}$

At $x = -3$,

$\frac{\text{dy}}{\text{dx}} = \frac{2(-3)^2[2(-3)-3]}{(-3-1)^2}$

$= \frac{18(-9)}{(-4)^2}$

$= -\frac{162}{16}$

$= -\frac{81}{8}$

(b) When tangent to curve is horizontal, gradient is = 0

$\frac{\text{dy}}{\text{dx}} = \frac{2x^2(2x-3)}{(x-1)^2} = 0$

$2x^2(2x-3)=0$

$2x^2=0$ or $2x-3=0$

$x=0$ or $x=\frac{3}{2}$

Sub into curve

$y=\frac{2(0)^3}{0-1}=0$ or $y=\frac{2\left(\frac{3}{2}\right)^3}{\frac{3}{2}-1}=\frac{2\left(\frac{27}{8}\right)}{\frac{1}{2}}=13.5$

(0, 0) and (1.5, 13.5)

7. Answer

$(y-2)^2=x$

$y-2=\pm\sqrt{x}$

$y=\sqrt{x}+2$ $\quad$ $y=2-\sqrt{x}$

$\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}$ $\quad$ $\frac{dy}{dx}=-\frac{1}{2}x^{-\frac{1}{2}}$

$=\frac{1}{2\sqrt{x}}$ $\quad$ $=-\frac{1}{2\sqrt{x}}$

$x-2y=4$

$2y=x-4$

$y=\frac{x}{2}-2$

$\frac{dy}{dx}=\frac{1}{2}$

$\frac{1}{2}=\frac{1}{2\sqrt{x}}$ $\quad$ $\frac{1}{2}=-\frac{1}{2\sqrt{x}}$

$\sqrt{x}=1$ $\quad$ $\sqrt{x}=-1$ (rej)

$x=1$

For $y=\sqrt{x}+2$, when $x=1$. Start from here as we already rejected $y=2-\sqrt{x}$

$y=1+2$

$y=3$

$\frac{dy}{dx}=\frac{1}{2\sqrt{x}}$ so when $x=1$, $\frac{dy}{dx}=\frac{1}{2}$

At (1, 3), $\frac{dy}{dx}=\frac{1}{2}$

$y-3=\frac{1}{2}(x-1)$

$$y - 3 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{5}{2}$$

$$2y = x + 5$$

Further Differentiation – Practice Questions 3 Answers

1. Answer (if need to, draw curve/tangent to show student how it looks like)

   (a) $y = 4x - 1$

   $\frac{dy}{dx} = 4$ (means gradient is positive)

   Thus $y = 4x - 1$ is increasing.

   (b) $y = -9 - x$

   $\frac{dy}{dx} = -1$ (means gradient is negative)

   Thus $y = -9 - x$ is decreasing.

   (c) $y = -(x^3 + 1)$

   $\frac{dy}{dx} = -3x^2$ (means gradient is always negative regardless of x value)

   Thus y is decreasing.

   (d) $y = x^2 + 2x$

   $\frac{dy}{dx} = 2x + 2$ (means gradient is always positive for $x > 0$)

   Thus y is increasing.

2. $y = \frac{x}{x^2+1}$

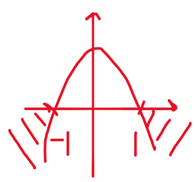
   $\frac{dy}{dx} = \frac{x^2+1-(x)(2x)}{(x^2+1)} = \frac{x^2+1-2x^2}{(x^2+1)} = \frac{1-x^2}{(x^2+1)}$

   For $\frac{dy}{dx} < 0$ (to show decreasing)

   $\frac{1-x^2}{(x^2+1)} < 0$

   $1 - x^2 < 0$

   $(1 - x)(1 + x) < 0$

   $x < -1$ or $x > 1$

   (shown)

3. $f(x) = \frac{5x}{4x^2+16}$

$f'(x) = \frac{(4x^2+16)(5)-5x(8x)}{(4x^2+16)^2}$

$= \frac{(4x^2+16)(5)-5x(8x)}{(4x^2+16)^2}$

$= \frac{20x^2+80-40x^2}{(4x^2+16)^2}$

$= \frac{80-20x^2}{(4x^2+16)^2}$

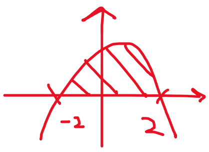
For time interval where absorption is increasing, $f'^{(x)} > 0$

$\frac{80-20x^2}{(4x^2+16)^2} > 0$

$80 - 20x^2 > 0$

$4 - x^2 > 0$

$(2 - x)(2 + x) > 0$



$-2 < x < 2$

Since $x > 0$ (time cannot be negative), $0 < x < 2$

4. Answer

(a) $y = x^2 + 4x$

$\frac{dy}{dx} = 2x + 4$

At stationary point, $\frac{dy}{dx} = 0$

$2x + 4 = 0$

$x = -2$

$y = (-2)^2 + 4(-2) = 4 - 8 = -4$

At $(-2, -4)$

x value	-2.1	-2	-1.9
$\frac{dy}{dx}$	$\frac{dy}{dx} = 2(-2.1) + 4$ $= -0.2$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 2(-1.9) + 4$ $= 0.2$
Shape			

It is a minimum point.

(Note: can always check with $\frac{d^2y}{dx^2}$ → $\frac{d^2y}{dx^2} = 2 > 0$ → min point)

(b) $y = -x^2 + 6x$

$\frac{dy}{dx} = -2x + 6$

At stationary point, $\frac{dy}{dx} = 0$

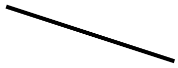
$-2x + 6 = 0$

$-2x = -6$

$x = 3$

$y = -(-3)^2 + 6(-3) = -9 - 18 = -27$

At $(3, -27)$

x value	2.9	3	3.1
$\frac{dy}{dx}$	$\frac{dy}{dx} = -2(2.9) + 6$ $= 0.2$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = -2(3.1) + 6$ $= -0.2$
Shape			

Max point at $(3, -27)$

(c) $y = 3x^4 - 4x^3 + 5$

$\frac{dy}{dx} = 12x^3 - 12x^2$

At stationary point, $\frac{dy}{dx} = 0$

$12x^3 - 12x^2 = 0$

$12x(x - 1) = 0$

$x = 0$ or $x = 1$

At $x = 0, y = 5$ At $x = 1, y = 3 - 4 + 5 = 4$

$(0, 5)$ and $(1, 4)$ are the stationary points:

At (0, 5):

x value	-0.1	0	$+0.1$
$\frac{dy}{dx}$	$\frac{dy}{dx} = 12(-0.1)^3 - 12(-0.1)^2$ $= -0.132$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 12(0.1)^3 - 12(0.1)^2$ $= -0.108$
Shape	\	—	\

At (0, 5): inflection point

At (1, 4):

x value	0.9	1	1.1
$\frac{dy}{dx}$	$\frac{dy}{dx} = 12(0.9)^3 - 12(0.9)^2$ $= -0.972$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 12(1.1)^3 - 12(1.1)^2$ $= 1.452$
Shape	\	—	/

At (1, 4): minimum point

(d) $y = \frac{(x-3)^2}{x}$

$$\frac{dy}{dx} = \frac{x(2)(x-3)-(x-3)^2}{x^2}$$

$$= \frac{2x^2-6x-x^2+6x-9}{x^2}$$

$$= \frac{x^2-9}{x^2}$$

At stationary point, $\frac{dy}{dx} = 0$

$\frac{x^2-9}{x^2} = 0$

$x^2 - 9 = 0$

$x = \pm 3$

At $x = 3$, $y = 0$ → (3, 0)

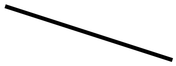
At $x = -3$, $y = \frac{(-3-3)^2}{(-3)^2} = \frac{9}{9} = 1$ → (−3, 1)

At (3, 0),

x value	2.9	3	3.1
$\frac{dy}{dx}$	$\frac{dy}{dx}=\frac{(2.9)^2-9}{2.9^2}$ $=-0.070$	$\frac{dy}{dx}=0$	$\frac{dy}{dx}=\frac{(3.1)^2-9}{3.1^2}$ $=0.0635$
Shape			

Min point at (3, 0).

At $(-3, 1)$

x value	-3.1	-3	-2.9
$\frac{dy}{dx}$	$\frac{dy}{dx}=\frac{(-3.1)^2-9}{(-3.1)^2}$ $=0.0635$	$\frac{dy}{dx}=0$	$\frac{dy}{dx}=\frac{(-2.9)^2-9}{(-2.9)^2}$ $=-0.070$
Shape			

Max point at $(-3, 1)$

5. Answer

(a) $y = x^2 + 3x$

$\frac{dy}{dx} = 2x + 3$

At stationary point, $\frac{dy}{dx} = 0$

$2x + 3 = 0$

$x = -\frac{3}{2}$

$y = \left(-\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) = -\frac{9}{4}$

$\left(-\frac{3}{2}, -\frac{9}{4}\right)$

$\frac{d^2y}{dx^2} = 2 > 0$

Thus, stationary point = $\left(-\frac{3}{2}, -\frac{9}{4}\right)$ and is a minimum point

(b) $y = 2x - x^2$

$\frac{dy}{dx} = 2 - 2x$

At stationary point, $\frac{dy}{dx} = 0$

$2 - 2x = 0$

$x = 1$

$y = 2(1) - (1)^2 = 1$

$(1, 1)$

$\frac{d^2y}{dx^2} = -2 < 0$

Thus, stationary point = $(1, 1)$ and is a maximum point

(c) $y = x^3 - 3x^2 + 3x - 7$

$\frac{dy}{dx} = 3x^2 - 6x + 3$

At stationary point, $\frac{dy}{dx} = 0$

$3x^2 - 6x + 3 = 0$

$x^2 - 2x + 1 = 0$

$(x - 1)^2 = 0$

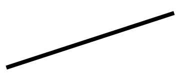
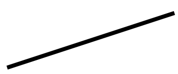
$x = 1$

$y = 1^3 - 3(1)^2 + 3(1) - 7 = 1 - 3 + 3 - 7 = -6$

$(1, -6)$

$\frac{d^2y}{dx^2} = 6x - 6 < 0$

At $x = 1$, $\frac{d^2y}{dx^2} = 6 - 6 = 0$ (cannot use second derivative test)

x value	0.9	1	1.1
$\frac{dy}{dx}$	$3(0.9)^2 - 6(0.9) + 3$ $= 0.03$	$\frac{dy}{dx} = 0$	$3(1.1)^2 - 6(1.1) + 3$ $= 0.03$
Shape			

Inflection point at $(1, -6)$

(d) $y = 4x^3 - 48x$

$\frac{dy}{dx} = 12x^2 - 48$

At stationary point, $\frac{dy}{dx} = 0$

$12x^2 - 48 = 0$

$x^2 - 4 = 0$

$x^2 - 4 = 0$

$(x + 2)(x - 2) = 0$

$x = 2$ or $x = -2$

$y = 4(2)^3 - 48(2)$ or $y = 4(-2)^3 - 48(-2)$

$= -64$ or $= 64$

$(2, -64)$ or $(-2, 64)$

$\frac{d^2y}{dx^2} = 24x$

At $(2, -64)$, $\frac{d^2y}{dx^2} = 24(2) = 48 > 0$

Min point at $(2, -64)$

At $(-2,64)$, $\frac{d^2y}{dx^2} = 24(-2) = -48 < 0$

Max point at $(-2,64)$

6. Answer

(a)

$Height = 15 - x$

$V = (15 - x)(x)(x)$

$= (15 - x)x^2$

$= 15x^2 - x^3$

(b)

$\frac{dV}{dx} = 30x - 3x^2$

At maximum, $\frac{dV}{dx} = 0$

$30x - 3x^2 = 0$

$10x - x^2 = 0$

$x(10 - x) = 0$

$x = 0$ (reject) or $x = 10$

Maximum side $x = 10$

Thus $V = (15 - 10)(10)(10) = 500 \text{ cm}^3$

7. Answer

(a) $S = r\theta$

Perimeter = $7 = 2(r) + r\theta$

$7 - 2r = r\theta$

$\theta = \frac{7-2r}{r}$

$A = \frac{1}{2}r^2\theta$

$A = \frac{1}{2}r^2\left(\frac{7-2r}{r}\right)$

$= \frac{1}{2}r(7 - 2r)$ cm^2

(b) Thus, max Area, $\frac{dA}{dr} = 0$

$A = \frac{1}{2}r(7 - 2r) = \frac{7}{2}r - r^2$

$\frac{dA}{dr} = \frac{7}{2} - 2r$

$\frac{7}{2} - 2r = 0$

$\frac{7}{2} = 2r$

$r = \frac{7}{4}$ cm

$A = \frac{1}{2}\left(\frac{7}{4}\right)\left(7 - 2\left(\frac{7}{4}\right)\right) = \frac{49}{16} = 3\frac{1}{16}$ cm^2

8. Answer

(a) Total surface area = 3.2 cm^2

$A = 3.2 = 2(2x)(h) + 2x(x) + 2(x)(h)$

$3.2 = 4xh + 2x^2 + 2xh$

$3.2 - 2x^2 = 6xh$

$h = \frac{3.2-2x^2}{6x}$

$= \frac{32-20x^2}{60x}$

$= \frac{8-5x^2}{15x}$

(b) $V = 2x(x)(h)$

$= 2x^2h$

$$= 2x^2\left(\frac{8-5x^2}{15x}\right)$$

$$= 2x\left(\frac{8-5x^2}{15}\right)$$

$$= \frac{16x-10x^3}{15}$$

$$= \frac{16x}{15} - \frac{10x^3}{15}$$

$$= \frac{16x}{15} - \frac{2x^3}{3}$$

$$\frac{dV}{dx} = \frac{16}{15} - \frac{6x^2}{3}$$

Max V, $\frac{dV}{dx} = 0$

$$\frac{16}{15} - \frac{6x^2}{3} = 0$$

$$\frac{6x^2}{3} = \frac{16}{15}$$

$$x^2 = \frac{8}{15}$$

$x = 0.73$ cm (2.d.p) (reject –ve)

(c) $V = \frac{16(0.730)}{15} - \frac{2(0.730)^3}{3} = 0.52\text{ cm}^3$ (2.d.p)

Differentiation (Rate of Change) – Practice Questions 4 Answers

1. $\frac{\mathrm{d}r}{\mathrm{d}t} = 0.2\ \text{cm/s}$

   $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$

   $A = \pi r^2$

   $\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$

   When $r = 5$,

   $\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi(5) \times 0.2 = 2\pi\ \text{cm}^2/s$

2. $\frac{\mathrm{d}s}{\mathrm{d}t} = 2\ \text{cm/s}$

   $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}s} \times \frac{\mathrm{d}s}{\mathrm{d}t}$

   $A = 6 \times s^2 = 6s^2$

   $\frac{\mathrm{d}A}{\mathrm{d}s} = 12s$

   When $s = \sqrt[3]{125} = 5$,

   $\frac{\mathrm{d}A}{\mathrm{d}t} = 12(5) \times 2 = 120\ \text{cm}^2/s$

3. $\frac{\mathrm{d}V}{\mathrm{d}t} = 20\ \text{cm}^3/\text{s}$

   $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$

   $V = \frac{4}{3}\pi\, r^3$

   $\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$

   When $\text{S} = 100\pi = 4\pi r^2$,

   $r = \pm\sqrt{25} = 5$ cm (reject negative)

   $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}r}$

   $= 20 \div 4\pi(5)^2$

   $= 20 \div 100\pi$

   $= \frac{1}{5\pi}\text{cm}/s$

4. Answer

(a)

$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.05 \text{ cm/s}$

$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$

$A = 4(x)(8) + 2(x)(x)$

$A = 2x^2 + 32x$

$\frac{\mathrm{d}A}{\mathrm{d}x} = 4x + 32$

When $\mathrm{A} = 210, 210 = 2x^2 + 32x$

$2x^2 + 32x - 210 = 0$

$x^2 + 16x - 105 = 0$

$(x + 21)(x - 5) = 0$

$x = 5$ or $x = -21$ (rejected)

$\frac{\mathrm{d}A}{\mathrm{d}t} = [(4)(5) + 32] \times 0.05 = 2.6 \text{ cm}^2/s$

(b)

$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$

$V = 8x^2$

$\frac{\mathrm{d}V}{\mathrm{d}x} = 16x$

When $x = 5$,

$\frac{\mathrm{d}V}{\mathrm{d}t} = 16(5) \times 0.05 = 4 \text{ cm}^3/s$

5. Method 1:

$\frac{\mathrm{d}V}{\mathrm{d}t} = 10$

$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$

$V = \frac{4}{3}\pi r^3$

$\frac{3V}{4\pi} = r^3$

$r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$

$A = 4\pi r^2 = 4\pi \left(\frac{3V}{4\pi}\right)^{\frac{2}{3}}$

$\frac{dA}{dV} = 4\pi \left(\frac{2}{3}\right)\left(\frac{3V}{4\pi}\right)^{-\frac{1}{3}}\left(\frac{3}{4\pi}\right) = 2\left(\frac{3V}{4\pi}\right)^{-\frac{1}{3}}$

At $r = 8$, $V = \frac{4}{3}\pi(8)^3 = \frac{2048}{3}\pi$

$\frac{dA}{dt} = 2\left(\frac{3V}{4\pi}\right)^{-\frac{1}{3}} \times 10$

$= 2\left(\frac{3\left(\frac{2048}{3}\pi\right)}{4\pi}\right)^{-\frac{1}{3}} \times 10$

$= 2(512)^{-\frac{1}{3}} \times 10$

$= 2(512)^{-\frac{1}{3}} \times 10$

$= 2.5 \text{ m}^2/\text{s}$

Method 2:

It is given that when $r = 8$, $\frac{dV}{dt} = 10$.

To find $\frac{dA}{dt}$,

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

To find $\frac{dr}{dt}$, $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

When $r = 8$, $\frac{dV}{dr} = 10$,

$10 = 4\pi(8)^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{5}{128\pi} \text{ ms}^{-1}$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$= 8\pi(8) \times \frac{5}{128\pi}$

$= \mathbf{2.5\ m^2s^{-1}}$

6. Answer

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$\frac{dr}{dt} = 0.005$ (given)

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$= 2\pi r \times 0.005$

Substitute $r = 15$ into $\frac{dA}{dt}$:

$\frac{dA}{dt} = 2\pi(15) \times 0.005$

$= \mathbf{0.471\ cm^2s^{-1}}$ (3 sig. fig.)

7. Answer

$y = \frac{x^2 - 5}{x}$

$= x - 5x^{-1}$

$\frac{dy}{dx} = 1 + 5x^{-2}$

Substitute $x = 2$ into $\frac{dy}{dx}$:

$\frac{dy}{dx} = 1 + \frac{5}{2^2}$

$= \frac{9}{4}$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$0.3 = \frac{9}{4} \times \frac{dx}{dt}$

$\frac{dx}{dt} = \mathbf{\frac{2}{15}}$

Differentiation (Trigo, Log and Exp) – Practice Questions 5 Answers

1. Answer

(a) $\frac{d}{dx}(4\sin 7x - x) = 7(4\cos 7x) - 1$

$= 28\cos 7x - 1$

(b) $\frac{d}{dx}(\cos 7x + \sin 3x) = -7\sin 7x + 3\cos 3x$

(c) $\frac{d}{dx}\left(\cos\left(\frac{\pi}{3} - \frac{3}{2}x\right)\right) = -\frac{3}{2}\left(-\sin\left(\frac{\pi}{3} - \frac{3}{2}x\right)\right)$

$= \frac{3}{2}\sin\left(\frac{\pi}{3} - \frac{3}{2}x\right)$

2. Answer

(a) $\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$

$= \frac{\cos x(0) - 1(-\sin x)}{\cos^2 x}$

$= \frac{\sin x}{\cos^2 x}$

$= \sin x \sec^2 x$

$= \frac{d}{dx}(\cos^{-1} x)$

$= -\cos^{-2} x\,(-\sin x)$

$= -\sec^2 x\,(-\sin x)$

(b) $\frac{d}{dx}(\operatorname{cosec} 2x) = \frac{d}{dx}\left(\frac{1}{\sin 2x}\right)$

$= \frac{\sin 2x(0) - 1(2\cos 2x)}{\sin^2 2x}$

$= \frac{-2\cos 2x}{\sin^2 2x}$

$= -2\cos 2x \operatorname{cosec}^2 2x$

$= \frac{d}{dx}(\sin^{-1} 2x)$

$= -\sin^{-2} 2x\,(\cos 2x)(2)$

$= -\operatorname{cosec}^2 2x\,(\cos 2x)(2)$

(c) $\frac{d}{dx}(\sec^2 3x) = \frac{d}{dx}\left(\frac{1}{\cos^2 3x}\right)$

$= \frac{\cos^2 3x(0) - 1(2\cos 3x)(-\sin 3x)(3)}{\cos^4 3x}$

$= \frac{6\cos 3x \sin 3x}{\cos^4 3x}$

$= \frac{6\sin 3x}{\cos^3 3x}$

$= 6\sin 3x \sec^3 3x$

$= \frac{d}{dx}(\cos^{-2} 3x)$

$= -2(\cos^{-3} 3x)(-\sin 3x)(3)$

$= 6(\cos^{-3} 3x)(\sin 3x)$

$= 6\sin 3x \sec^3 3x$

3. Answer

(a) $\frac{d}{dx}(x\tan x) = x\frac{d}{dx}\tan x + \tan x\frac{d}{dx}x$

$= x(\sec^2 x) + \tan x$

$= x\sec^2 x + \tan x$

(b) $\frac{d}{dx}(5\sin 3x\cos 2x) = (5\sin 3x)\frac{d}{dx}\cos 2x + \cos 2x\frac{d}{dx}(5\sin 3x)$

$= 5\sin 3x\,(-2\sin 2x) + \cos 2x\,(15\cos 3x)$

$= -10\sin 2x\sin 3x + 15\cos 2x\cos 3x$

(c) $\frac{d}{dx}(3\tan x\cos^2 4x) = (3\tan x)\frac{d}{dx}\cos^2 4x + \cos^2 4x\frac{d}{dx}(3\tan x)$

$= 3\tan x\,(2\cos 4x)(-\sin 4x)(4) + \cos^2 4x\,(3\sec^2 x)$

$= -24\tan x\cos 4x\sin 4x + 3\sec^2 x\cos^2 4x$

4. Answer

(a) $\frac{d}{dx}[\ln(4-x^2)] = \frac{1}{4-x^2}(-2x)$

$= \frac{-2x}{4-x^2}$

(b) $\frac{d}{dx}\left[\ln\frac{5x-3}{3x+7}\right] = \frac{d}{dx}[\ln(5x-3) - \ln(3x+7)]$

$= \frac{1}{5x-3}(5) - \frac{1}{3x+7}(3)$

$= \frac{5}{5x-3} - \frac{3}{3x+7}$

(c) $\frac{d}{dx}\left[\ln\frac{2x+3}{3x-4}\right] = \frac{d}{dx}[\ln(2x+3) - \ln(3x-4)]$

$= \frac{1}{2x+3}(2) - \frac{1}{3x-4}(3)$

$= \frac{2}{2x+3} - \frac{3}{3x-4}$

(d) $\frac{d}{dx}\left[\ln\frac{1}{(3x-5)^3}\right] = \frac{d}{dx}[\ln(3x-5)^{-3}]$

$= \frac{d}{dx}[-3\ln(3x-5)]$

$= -3\left(\frac{1}{3x-5}\right)(3)$

$= \frac{-9}{3x-5}$

(e) $\frac{d}{dx}\left[\ln\sqrt{6x^2-3}\right] = \frac{d}{dx}\left[\ln(6x^2-3)^{\frac{1}{2}}\right]$

$= \frac{d}{dx}\left[\frac{1}{2}\ln(6x^2-3)^1\right]$

$= \frac{1}{2}\left(\frac{1}{(6x^2-3)}\right)\frac{d}{dx}(6x^2-3)$

$= \frac{12x}{6(2x^2-1)}$

$= \frac{2x}{(2x^2-1)}$

5. Answer

(a) $y = \log_5 \sin x$

$$= \frac{\ln \sin x}{\ln 5}$$

$$\frac{dy}{dx} = \frac{1}{\ln 5}\frac{d}{dx}\ln \sin x$$

$$= \frac{1}{\ln 5}\left(\frac{1}{\sin x}\right)(\cos x)$$

$$= \frac{\cot x}{\ln 5} \quad \text{or} \quad = \frac{1}{\ln 5 \tan x}$$

(b) $y = \log_3 4x^{\frac{1}{2}}$

$$= \frac{\ln 4x^{\frac{1}{2}}}{\ln 3}$$

$$= \frac{\frac{1}{2}\ln 4x}{\ln 3}$$

$$= \frac{\ln 4x}{2\ln 3}$$

$$\frac{dy}{dx} = \frac{1}{2\ln 3}\frac{d}{dx}\ln 4x$$

$$= \frac{1}{\ln 9}\left(\frac{1}{4x}\right)4$$

$$= \frac{1}{x\ln 9}$$

(c) $e^y = 2x^3 + 7x$

$$\ln(e^y) = \ln(2x^3 + 7x)$$

$$y = \ln(2x^3 + 7x)$$

$$\frac{dy}{dx} = \frac{1}{2x^3+7x}(6x^2 + 7)$$

$$= \frac{6x^2+7}{2x^3+7x}$$

(d) $e^y = sec\, x$

$$\ln(e^y) = \ln(sec\, x)$$

$$y = \ln(sec\, x)$$

$$\frac{dy}{dx} = \frac{1}{\sec x}\frac{d}{dx}\sec x$$

$$= \frac{1}{\sec x}\frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$= \frac{1}{\sec x}\left[\frac{\cos x(0) - 1(-\sin x)}{\cos^2 x}\right]$$

$$= \frac{1}{\frac{1}{\cos x}}\left[\frac{\sin x}{\cos^2 x}\right]$$

$$= \cos x\left[\frac{\sin x}{\cos^2 x}\right]$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

6. Answer

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-4x}) = \mathrm{e}^{-4x}\frac{\mathrm{d}}{\mathrm{d}x}(-4x)$

$$= -4\mathrm{e}^{-4x}$$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^x + \frac{1}{\mathrm{e}^x}\right) = \mathrm{e}^x + \mathrm{e}^{-x}$

$$= \mathrm{e}^x + \mathrm{e}^{-x}\frac{\mathrm{d}}{\mathrm{d}x}(-x)$$

$$= \mathrm{e}^x - \mathrm{e}^{-x}$$

$$= \mathrm{e}^x - \frac{1}{\mathrm{e}^x}$$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\sin 2x}\right) = \mathrm{e}^{\sin 2x}\frac{\mathrm{d}}{\mathrm{d}x}(\sin 2x)$

$$= \mathrm{e}^{\sin 2x}(\cos 2x)(2)$$

$$= 2\mathrm{e}^{\sin 2x}\cos 2x$$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{\tan x}) = \mathrm{e}^{\tan x}\frac{\mathrm{d}}{\mathrm{d}x}(\tan x)$

$$= \mathrm{e}^{\tan x}(\sec^2 x)$$

$$= \mathrm{e}^{\tan x}\sec^2 x$$

(e) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x\cos x}{\mathrm{e}^x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-x}x\cos x)$

$$= \mathrm{e}^{-x}x\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) + \cos x\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-x}x)$$

$$= \mathrm{e}^{-x}x(-\sin x) + \cos x\left[\mathrm{e}^{-x}\frac{\mathrm{d}}{\mathrm{d}x}(x) + x\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-x})\right]$$

$$= \mathrm{e}^{-x}x(-\sin x) + \cos x\,[\mathrm{e}^{-x} - x\mathrm{e}^{-x}]$$

$$= \mathrm{e}^{-x}x(-\sin x) + \mathrm{e}^{-x}\cos x\,[1 - x]$$

$$= \mathrm{e}^{-x}[-x\sin x + \cos x - x\cos x]$$

$$= \frac{-x\sin x + \cos x - x\cos x}{\mathrm{e}^x}$$

OR

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x\cos x}{\mathrm{e}^x}\right) = \frac{\mathrm{e}^x\frac{\mathrm{d}}{\mathrm{d}x}(x\cos x) - (x\cos x)\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x}{\mathrm{e}^{2x}}$$

$$= \frac{\mathrm{e}^x[x(-\sin x) + \cos x] - \mathrm{e}^x x\cos x}{\mathrm{e}^{2x}}$$

$$= \frac{e^x[-x\sin x+\cos x-x\cos x]}{e^{2x}}$$

$$= \frac{-x\sin x+\cos x-x\cos x}{e^x}$$

Differentiation – Exam Questions Answers

1. Answer

(i) $y = 2 - x - \dfrac{2x+3}{x-3}$

$\dfrac{dy}{dx} = -1 - \dfrac{(x-3)\times 2 - (2x+3)\times 1}{(x-3)^2}$ **M1**

$= -1 + \dfrac{9}{(x-3)^2}$ **A1**

$\dfrac{d^2y}{dx^2} = -\dfrac{18}{(x-3)^3}$ **A1**

(ii) At the stationary points, $\dfrac{dy}{dx} = 0$

$\dfrac{9-(x-3)^2}{(x-3)^2} = 0$ **M1**

$(x-3)^2 = 9$ **M1**

$x - 3 = \pm 3$

$x = 0$ or $x = 6$

x-coordinates of the stationary points of the curve are 0 and 6. **A1**

(iii) When $x = 0$, $\dfrac{d^2y}{dx^2} = \dfrac{2}{3} > 0$

The stationary point at $x = 0$ is a minimum point.

When $x = 6$, $\dfrac{d^2y}{dx^2} = -\dfrac{2}{3} < 0$ **M1A1**
(Correct value & correct conclusion

The stationary point at $x = 6$ is a maximum point.

2. Answer

Let total surface area of ice block be A cm^2 and $A=2\pi r^2+2\pi r\left(2r\right)$ where r cm is the radius at any instance.

$A=2\pi r^2+2\pi r\left(2r\right)$ **M1**

$=6\pi r^2$

$\frac{dA}{dt}=\frac{dA}{dr}\times\frac{dr}{dt}$

$=12\pi r\frac{dr}{dt}$ **M1**

When $\frac{dA}{dt}=-72$ and $r=5$,

$-72=12\pi\times 5\frac{dr}{dt}$ **M1**

$\frac{dr}{dt}=-\frac{72}{60\pi}$

$=-\frac{6}{5\pi}$ **A1**

$\therefore$ The radius of the ice block decreases at $\underline{\underline{\frac{6}{5\pi}\text{ cm/s}}}$ when $r=5$. (accept 0.382 cm/s)

3. Answer

(i) Show that the volume, V cm^3, of the cylinder is given by $V=45\pi r^2-\frac{9}{4}\pi r^3$. [3]

By similar triangles,

$\frac{45-h}{45}=\frac{r}{20}$ **[M1]**

$20(45-h)=45r$

$900-20h=45r$

$h=\frac{900-45r}{20}$

$=45-\frac{9}{4}r$ **[M1]**

$V=\pi r^2 h$

$=\pi r^2\left(45-\frac{9}{4}r\right)$ **[A1]**

$=45\pi r^2-\frac{9}{4}\pi r^3$ (shown)

(ii) Given that r can vary, find the maximum volume of the cylinder, leaving your answer in terms of π. [4]

$\frac{dV}{dr}=90\pi r-\frac{27}{4}\pi r^2$ **[M1]**

When V is maximum, $\frac{dV}{dr} = 0$

$90\pi r - \frac{27}{4}\pi r^2 = 0$

$\pi r\left(90 - \frac{27}{4}r\right) = 0$

$r = \frac{40}{3}$ (since $r > 0$) **[M1]**

$\frac{d^2V}{dr^2} = 90\pi - \frac{27}{2}\pi r$

When $r = \frac{40}{3}$,

$\frac{d^2V}{dr^2} = 90\pi - \frac{27}{2}\pi\left(\frac{40}{3}\right) < 0$ **[M1]**

Since $\frac{d^2V}{dr^2} < 0$, volume of cylinder is maximum when $r = \frac{40}{3}$.

maximum volume $= 45\pi\left(\frac{40}{3}\right)^2 - \frac{9}{4}\pi\left(\frac{40}{3}\right)^3$

$= 2666\frac{2}{3}\pi \text{ cm}^3$ **[A1]**

(iii) Hence show that the cylinder occupies at most $\frac{4}{9}$ of the volume of the cone.

[2]

volume of cone $= \frac{1}{3}\pi(20^2)(45)$

$= 6000\pi \text{ cm}^3$ **[M1]**

When the volume of cylinder is the maximum,

$\frac{\text{volume of cylinder}}{\text{volume of cone}} = \frac{2666\frac{2}{3}\pi}{6000\pi}$

$= \frac{4}{9}$ **[A1]**

Therefore, the cylinder occupies at most $\frac{4}{9}$ of the volume of the cone.

4. Answer

Differentiate $\ln\left(2x^2+1\right)$ with respect to x.

$$\frac{d}{dx}\left(\ln\left(2x^2+1\right)\right)=\frac{1}{2x^2+1}(4x) \quad \textbf{[M1]}$$
$$=\frac{4x}{2x^2+1} \quad \textbf{[A1]}$$

5. Answer

(i) Explain clearly why $h=90\cos\theta+150\sin\theta$. [2]

Let the foot of the perpendicular from Q to PU and RU be X and Y respectively.

$PX=90\cos\theta$ } **[M1 for both]**
$QY=150\sin\theta$

$h=PX+QY$

$=90\cos\theta+150\sin\theta$ **[A1]**

(ii) Express h in the form $R\cos(\theta-\alpha)$, where $R>0$ and α is an acute angle. [4]

$R\cos\alpha = 90$

$R\sin\alpha = 150$ **[M1]**

$\tan\alpha = \frac{150}{90}$ **[M1]**

$\alpha = 1.0304$

$R=\sqrt{90^2+150^2}$

$=30\sqrt{34}$ **[M1]**

$h=30\sqrt{34}\cos(\theta-1.03)$ **[A1 - accept if R = 175 (3s.f.)]**

(iii) Find the greatest possible value of h and the value of θ at which this occurs. [3]

Greatest value of h occurs when $\cos(\theta-1.03)=1$ **[M1]**

Greatest value of $h = 30\sqrt{34}$ **[A1 – Accept 175]**

when $\theta = 1.03$. **[A1]**

(iv) Find the values of θ when h = 160. [3]

$30\sqrt{34}\cos(\theta-1.0304)=160$

$\cos(\theta-1.0304)=\frac{160}{30\sqrt{34}}$ **[M1]**

Basic angle = 0.41613

$\theta-1.0304=-0.41613$ or 0.41613

$\theta=0.614$ or 1.45 (to 3s.f.) **[A1, A1]**

6. Answer

$$\frac{d}{dx}\left(\tan^3 5x\right)=3\left(\tan^2 5x\right)\left(\sec^2 5x\right)(5)$$
$$=15\left(\tan^2 5x\right)\left(\sec^2 5x\right)$$
$$=15\left(\sec^2 5x-1\right)\left(\sec^2 5x\right)$$
$$=15\sec^4 5x-15\sec^2 5x$$

7. Answer

(a) Answer

$$y=\frac{3x}{\sqrt{5-4x}}$$
$$\frac{dy}{dx}=\frac{\sqrt{5-4x}\frac{d}{dx}(3x)-(3x)\frac{d}{dx}\sqrt{5-4x}}{\left(\sqrt{5-4x}\right)^2}$$
$$=\frac{\sqrt{5-4x}(3)-(3x)\frac{1}{2}(5-4x)^{-\frac{1}{2}}(-4)}{5-4x}$$
$$=\frac{\sqrt{5-4x}(3)+\frac{(3x)2}{\sqrt{5-4x}}}{5-4x}$$
$$=\frac{\sqrt{5-4x}(3)+\frac{6x}{\sqrt{5-4x}}}{5-4x}\times\frac{\sqrt{5-4x}}{\sqrt{5-4x}}$$
$$=\frac{(5-4x)(3)+6x}{(5-4x)\sqrt{5-4x}}$$
$$=\frac{15-6x}{\sqrt{(5-4x)^3}}$$

(b)

$$x=1,\ \ y=\frac{3(1)}{\sqrt{5-4(1)}}=3$$

Gradient of the tangent $=\dfrac{15-6(1)}{\sqrt{(5-4(1))^3}}$

$=9$

Gradient of the normal $=-\dfrac{1}{9}$

Equation of normal

$$y-3=-\frac{1}{9}(x-1)$$
$$y=-\frac{1}{9}x+\frac{28}{9}$$
$$9y+x=28$$

8. Answer

(a)

$$y = \text{cosec } x \tan x$$

$$= \frac{1}{\sin x} \times \frac{\sin x}{\cos x}$$

$$= \frac{1}{\cos x}$$

$$= (\cos x)^{-1}$$

$$\frac{dy}{dx} = (-1)(\cos x)^{-2}(-\sin x)$$

$$= \frac{1}{\cos^2 x}(\sin x)$$

$$= \sec^2 x \sin x$$

Or

$$y = \text{cosec } x \tan x$$

$$= \frac{1}{\sin x} \times \tan x$$

$$= (\sin x)^{-1} \tan x$$

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \sec^2 x + \tan x\left(-\sin^{-2} x \cos x\right)$$

$$= \frac{1}{\sin x} \cdot \sec^2 x - \frac{\sin x}{\cos x}\left(\frac{\cos x}{\sin^2 x}\right)$$

$$= \frac{1}{\sin x} \cdot \sec^2 x - \frac{1}{\sin x}$$

$$\frac{dy}{dx} = \frac{\sec^2 x - 1}{\sin x}$$

$$= \frac{\tan^2 x}{\sin x}$$

$$= \frac{1}{\sin x}\left(\frac{\sin^2 x}{\cos^2 x}\right)$$

$$= \frac{1}{\cos^2 x}(\sin x)$$

$$= \sin x \sec^2 x$$

$$y = \cos ecx \tan x$$

$$= \frac{\tan x}{\sin x}$$

$$\frac{dy}{dx} = \frac{\sin x \sec^2 x - \tan x \cos x}{\sin^2 x}$$

$$= \frac{\sin x \sec^2 x - \sin x}{\sin^2 x}$$

$$= \frac{\sin x\left(\sec^2 x - 1\right)}{\sin^2 x}$$

$$= \frac{\sin x\left(\tan^2 x\right)}{\sin^2 x}$$

$$= \frac{1}{\sin x}\left(\frac{\sin^2 x}{\cos^2 x}\right)$$

$$= \frac{1}{\cos^2 x}(\sin x)$$

$$= \sin x \sec^2 x$$

(b)

For a decreasing function, $\frac{dy}{dx} < 0$

For $0 \le x \le 2\pi$, $\sec^2 x > 0$, $x \ne \frac{\pi}{2}, \frac{3\pi}{2}$, $\sin x < 0$ for $\pi < x < 2\pi$

Hence $\frac{dy}{dx} < 0$ for $\pi < x < 2\pi$, $x \ne \frac{3\pi}{2}$

9. Answer

Let the length of the rectangle be x.

Breadth of rectangle = $\frac{k-2x}{2}$

Let area of the rectangle be A.

$$A = x\left(\frac{k-2x}{2}\right)$$

$$A = \frac{xk-2x^2}{2}$$

$$A = \frac{1}{2}xk - x^2$$

$$\frac{dA}{dx} = \frac{1}{2}k - 2x$$

At the stationary value, $\frac{dA}{dx} = 0$

$$\frac{1}{2}k - 2x = 0$$

$$2x = \frac{k}{2}$$

$$x = \frac{k}{4}$$

$$\frac{d^2A}{dx^2} = -2 < 0$$

Therefore, since the stationary value occurs when the sides of the rectangle are $\frac{k}{4}$ cm, and it is a maximum value, the maximum area of the rectangle occurs when it is a square.

10. Answer

(a)

$$y = \ln\left(\frac{1-x}{x^2}\right)$$

$$y = \ln(1-x) - 2\ln x$$

$$\frac{dy}{dx} = \frac{-1}{1-x} - \frac{2}{x}$$

(b) Answer

When $y = \ln 2$,

$$\ln\left(\frac{1-x}{x^2}\right) = \ln 2$$

$$\frac{1-x}{x^2} = 2$$

$$1 - x = 2x^2$$

$$2x^2 + x - 1 = 0$$

$$(x+1)(2x-1) = 0$$

$$x = -1 \text{ or } \frac{1}{2}$$

When $x = -1$, $\frac{dy}{dt} = 9$,

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$9 = \left(\frac{-1}{1-(-1)} - \frac{2}{(-1)}\right) \times \left.\frac{dx}{dt}\right|_{x=-1}$$

$$9 = \left(\frac{3}{2}\right) \times \left.\frac{dx}{dt}\right|_{x=-1}$$

$$\left.\frac{dx}{dt}\right|_{x=-1} = 6$$

When $x = \frac{1}{2}$, $\frac{dy}{dt} = 9$,

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$9 = \left(\frac{-1}{1-(\frac{1}{2})} - \frac{2}{(\frac{1}{2})}\right) \times \left.\frac{dx}{dt}\right|_{x=-1}$$

$$9 = (-6) \times \left.\frac{dx}{dt}\right|_{x=-1}$$

$$\left.\frac{dx}{dt}\right|_{x=-1} = -\frac{3}{2} \quad \text{(rejected)}$$

11. Answer

(a)

$y = 9x^6 - 3$

$\frac{dy}{dx} = 54x^5$

To find stationary point, $\frac{dy}{dx} = 0$

$54x^5 = 0$

$x = 0$

sub. into y, $y = -3$

$\therefore$ Coordinates of stationary point = (0, –3)

(b)

x	0^-	0	0^+
$\frac{dy}{dx}$	–	0	+

By First Derivative Test, (0, –3) is a minimum point.

12. Answer

(a)

$\frac{dy}{dx} = 1 + ae^{-2x}$

$\therefore$ Gradient of tangent at (0, –1) = $\frac{-1}{\left(-\frac{1}{2}\right)}$

$1 + ae^{-2(0)} = 2$

$1 + a = 2$

$a = 1$

$\frac{dy}{dx} = \frac{e^{2x} + a}{e^x}$

Gradient of tangent at (0, –1) = $\frac{-1}{\left(-\frac{1}{2}\right)} = 2$

$\left.\frac{dy}{dx}\right|_{x=0} = 2$

$\frac{e^0 + a}{e^0} = 2$

$1 + a = 2$

$a = 1$

(b) Answer

$$\frac{dy}{dx} = \frac{e^{2x}+a}{e^{x}} = 1+e^{-2x}$$

Since $e^{-2x} > 0$, $\frac{dy}{dx} > 0$ for all values of x.

➔ $\frac{dy}{dx} \neq 0$, the curve as no stationary point.

For stationary point,

$$\frac{e^{2x}+1}{e^{x}} = 0$$

$$e^{2x}+1 = 0$$

$$e^{2x} = -1$$

(no solution)

Since $\frac{dy}{dx} = 0$ has no solution

➔ y has no stationary point.

13. Answer

$$\frac{dy}{dx} = 4\cos 4x - 4\sin x\cos^3 x$$

When $x = \frac{\pi}{4}$,

$$\frac{dy}{dx} = 4\cos 4\left(\frac{\pi}{4}\right) - 4\sin\left(\frac{\pi}{4}\right)\cos^3\left(\frac{\pi}{4}\right)$$

$$= 4(-1) - 4\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)^3$$

$$= -5$$

14. Answer

$$\frac{d}{dx}\left[4\sin^2\left(\frac{x}{2}+\pi\right)\right]$$

$$= 4\times 2\sin\left(\frac{x}{2}+\pi\right)\cos\left(\frac{x}{2}+\pi\right)\times\frac{1}{2}$$

$$= 4\sin\left(\frac{x}{2}+\pi\right)\cos\left(\frac{x}{2}+\pi\right)$$

$$= 2\sin 2\left(\frac{x}{2}+\pi\right)$$

$= 2\sin(x+2\pi)$ or $2(\sin x\cos 2\pi + \cos x\sin 2\pi)$

$$= 2\sin x$$

15. Answer

(a)

$f(x) = \ln\left(\frac{5+x}{5-x}\right)^{\frac{1}{3}}$	
$= \frac{1}{3}\ln\left(\frac{5+x}{5-x}\right)$	
$= \frac{1}{3}\left[\ln(5+x) - \ln(5-x)\right]$	M1
$f'(x) = \frac{1}{3}\left(\frac{1}{5+x} + \frac{1}{5-x}\right)$	M1
$= \frac{1}{3}\left[\frac{5-x+5+x}{(5+x)(5-x)}\right]$	
$= \frac{10}{3(5+x)(5-x)}$	A1
$= \frac{10}{3(25-x^2)}$	
$f''(x) = \frac{10}{3}(-1)(25-x^2)^{-2}(-2x)$	
$= \frac{20x}{3(25-x^2)^2}$	A1

(b)

For $f'(x) > 0$

$25 - x^2 > 0$

$(5+x)(5-x) > 0$

$-5 < x < 5$

For $f''(x) > 0$

$20x > 0$

$x > 0$

For both $f'(x)$ and $f''(x)$ to be positive,

$0 < x < 5$

16. Answer

(a) $y = kx\sqrt{2x+3}$

$\frac{dy}{dx} = kx\left(\frac{1}{2}\right)(2x+3)^{-\frac{1}{2}}(2) + (2x+3)^{\frac{1}{2}}(k)$

$= kx(2x+3)^{-\frac{1}{2}} + (2x+3)^{\frac{1}{2}}(k)$

$= k(2x+3)^{-\frac{1}{2}}[x + (2x+3)^1]$

$= \frac{k[x+(2x+3)^1]}{\sqrt{2x+3}}$

$= \frac{k(3x+3)}{\sqrt{2x+3}}$

$= \frac{3k(x+1)}{\sqrt{2x+3}}$ (done)

(b) $\frac{dy}{dt} = 3\frac{dx}{dt}$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

At $x = 3$,

$\frac{dy}{dt} = \frac{3k(3+1)}{\sqrt{2(3)+3}} \times \frac{dx}{dt}$

$3\frac{dx}{dt} = \frac{3k(3+1)}{\sqrt{2(3)+3}} \times \frac{dx}{dt}$

$3 = \frac{12k}{\sqrt{9}}$

$3(3) = 12k$

$k = \frac{9}{12} = \frac{3}{4}$

17. $f(x) = xe^{-x} + ke^{\frac{3}{5}x}$

$f'(x) = x(-e^{-x}) + e^{-x}(1) + \frac{3}{5}ke^{\frac{3}{5}x}$

$= -xe^{-x} + e^{-x} + \frac{3}{5}ke^{\frac{3}{5}x}$

$f'(0) = -(0)e^0 + e^0 + \frac{3}{5}ke^{\frac{3}{5}(0)} = 5$

$1 + \frac{3}{5}k = 5$

$\frac{3}{5}k = 4$

$k = \frac{20}{3}$

18. Answer

$$A = 300x - \left(\frac{\pi+4}{8}\right)x^2$$

$$\frac{dA}{dx} = 300 - \frac{2(\pi+4)}{8}x$$
$$= 300 - \frac{\pi+4}{4}x$$

At stationary A, $\frac{dA}{dx} = 0$

$$300 - \frac{\pi+4}{4}x = 0$$

$$x = \frac{1200}{\pi+4}$$
$$= 168.0297...$$

$$\frac{d^2A}{dx^2} = \frac{-\pi-4}{4} < 0$$

Thus when $x = 168.0297...$,

Maximum $A = 300(168.0297...) - \frac{\pi+4}{8}(168.0297...)^2$

$= 25204.461...$

$= 25200$ (3 s.f)

19. Answer

(i) Find $\frac{dy}{dx}$. Hence, find the equation of the line that is parallel to the tangent of the curve at $x = 1$ and passes through the origin. [3]

$$y = -2\left(1 - \frac{1}{4}x\right)^2 + p$$

$$\frac{dy}{dx} = -4\left(1 - \frac{1}{4}x\right)\left(-\frac{1}{4}\right)$$

$$= 1 - \frac{1}{4}x$$

When $x = 1$, gradient of tangent $= \frac{3}{4}$

Equation of line is $y = \frac{3}{4}x$ (Since the line passes though the origin, the y-intercept is 0)

The normal to the curve at $x = 1$ passes through the point $\left(-1,\ 1\frac{1}{2}\right)$.

(ii) Find the value of the constant p, [4]

Gradient of normal $= -\frac{4}{3}$

At $x = 1$, $y = -2\left(1 - \frac{1}{4}\right)^2 + p$

$$= -\frac{9}{8} + p$$

$$\frac{-\frac{9}{8} + p - 1\frac{1}{2}}{1-(-1)} = -\frac{4}{3}$$

$$-\frac{21}{8} + p = -\frac{8}{3}$$

$$p = -\frac{1}{24}$$

(It is not essential to find equation of normal, you can formulate the gradient of normal using $\frac{y_2 - y_1}{x_2 - x_1}$ and equate to the gradient that is found through differentiation)

20. Answer

(i) $h = 5 - r$

(ii) $V = \pi(r)^2(h) - \frac{1}{3}\pi r^2 h$

$= \pi(4)^2(8) - \frac{1}{3}\pi r^2(5 - r)$

$= 128\pi - \frac{5}{3}\pi r^2 + \frac{1}{3}\pi r^3$ (shown)

(iii) $r = 0$ (*rej.*) or $r = 3\frac{1}{3}$