Name:	Class:	Class Register Number:

CHUNG CHENG HIGH SCHOOL (MAIN)

PRELIMINARY EXAMINATION 2023
SECONDARY 4

ADDITIONAL MATHEMATICS **4049/01**

Paper 1 **Thursday 24 August 2023**

2 hours 15 minutes

MARKS SCHEME

This document consists of **19** printed pages and **1** blank page.

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Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ \ldots\ (n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

1 The points R and S have coordinates $\left(\sqrt{3}, 2\sqrt{3}\right)$ and $\left(\sqrt{5}, 4\sqrt{5}\right)$ respectively. Show that the gradient of RS can be expressed in the form $a + b\sqrt{15}$, where a and b are integers to be found. [4]

Gradient of $RS = \dfrac{4\sqrt{5}-2\sqrt{3}}{\sqrt{5}-\sqrt{3}} \times \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$	B1 – $\dfrac{4\sqrt{5}-2\sqrt{3}}{\sqrt{5}-\sqrt{3}}$
$= \dfrac{\left(4\sqrt{5}-2\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2}$	M1 – √ $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$
$= \dfrac{20-4\sqrt{15}-2\sqrt{15}-6}{2}$	M1 – either numerator or denominator expanded correctly
$= \dfrac{14-6\sqrt{15}}{2}$	
$= \dfrac{2\left(7-3\sqrt{15}\right)}{2}$	
$= 7-3\sqrt{15}$	
$a = 7,\ b = -3$	A1

2 Given that $\cos\theta = p$ and that θ is acute, express in terms of p,

(a) $\sin\theta$, [1]

$\sqrt{1-p^2}$

B1

(b) $\tan\left(90^\circ - \theta\right)$, [2]

$$= \frac{1}{\tan\theta}$$

$$= \frac{1}{\dfrac{\sqrt{1-p^2}}{p}}$$

M1 – $\dfrac{1}{\text{their } \tan\theta}$

$$= \frac{p}{\sqrt{1-p^2}}$$

A1

(c) $\cos 2\theta$. [2]

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$= 1 - 2\left(\sqrt{1-p^2}\right)^2$$

M1 – uses any $\cos 2\theta$ formula correctly

$$= 1 - 2\left(1-p^2\right)$$

$$= 1 - 2 + 2p^2$$

$$= 2p^2 - 1$$

A1

3 Express $\dfrac{12x^2+32x+31}{(2x-1)(x+2)^2}$ in partial fractions. [5]

$\dfrac{12x^2+32x+31}{(2x-1)(x+2)^2}=\dfrac{A}{2x-1}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}$	M1 – realising the form of partial fractions
$12x^2+32x+31=A(x+2)^2+B(2x-1)(x+2)+C(2x-1)$	M1 – realising the need to eliminate the denominator
let $x=\dfrac{1}{2}$, $50=\dfrac{25}{4}A$ $A=8$ let $x=-2$, $15=-5C$ $C=-3$ let $x=0$, $31=4A-2B-C$ $31=32-2B+3$ $2B=4$ $B=2$	
$A=8,\ B=2,\ C=-3$ $\dfrac{12x^2+32x+31}{(2x-1)(x+2)^2}=\dfrac{8}{2x-1}+\dfrac{2}{x+2}-\dfrac{3}{(x+2)^2}$	A1, A1, A1 – do not award last A1 if not expressed in partial fractions

4 The expression ax^3+bx^2+b leaves a remainder of R when divided by $(x+1)$ and a remainder of $5R-2$ when divided by $(x+2)$.

(a) Show that $b=\dfrac{2-3a}{5}$. [4]

when $x=-1$,	
$a(-1)^3+b(-1)^2+b=R$	M1 – realises $\mathrm{f}(-1)=R$
$-a+2b=R$ --- (1)	
when $x=-2$,	
$a(-2)^3+b(-2)^2+b=5R-2$	M1 – realises $\mathrm{f}(-2)=5R-2$
$-8a+5b=5R-2$ --- (2)	
sub (1) into (2),	
$-8a+5b=5(-a+2b)-2$	DM1 – award only if both M1 above is achieved
$-8a+5b=-5a+10b-2$	
$10b-5b=-8a+5a+2$	
$5b=-3a+2$	A1
$b=\dfrac{2-3a}{5}$ (shown)	AG

(b) Given further that $ab=-8$ and $a>b$, find the value of a and of b. [3]

$ab=-8$	
$b=\dfrac{-8}{a}$	
$\dfrac{-8}{a}=\dfrac{-3a+2}{5}$	M1 – allow slips only for LHS
$-40=-3a^2+2a$	
$3a^2-2a-40=0$	
$(3a+10)(a-4)=0$	
$a=-\dfrac{10}{3}$ or $a=4$	
$b=2.4$ $b=-2$	√M1 – finding ‘b’
(reject)	
$\therefore a=4,\ b=-2$	A1

(c) Using the values of a and b found in **part (b)**, explain why the equation $ax^3+bx^2+b=0$ has only one real root and state its value. [4]

$4x^3-2x^2-2=0$	
$2x^3-x^2-1=0$	
when $x=1$,	
$2(1)^3-1^2-1=0$	
$\therefore$ By factor theorem, $(x-1)$ is a factor.	B1 - factor
$(x-1)(2x^2+cx+1)=0$	M1 – correct method to find quadratic (division or inspection)
by comparing x^2 terms,	
$-2+c=-1$	
$c=1$	
$(x-1)(2x^2+x+1)=0$	
For $2x^2+x+1=0$,	
discriminant $=1^2-4(2)(1)=-7<0$	M1 – realising the need to find discriminant / solve quadratic equation
$\therefore 2x^2+x+1=0$ has no real roots.	
$4x^3-2x^2-2=0$ has only 1 real root of $x=1$.	A1 – no real roots + x = 1

5 Find the coordinates of the stationary points of the curve $y=\dfrac{(x-3)^2}{x}$ and determine the nature of each stationary point. [7]

$y=\dfrac{(x-3)^2}{x}$	
$=\dfrac{x^2-6x+9}{x}$	
$=x-6+\dfrac{9}{x}$	
$=x-6+9x^{-1}$	
$\dfrac{dy}{dx}=1-9x^{-2}$	B1
$=1-\dfrac{9}{x^2}$	
For stationary points, $\dfrac{dy}{dx}=0$	
$1-\dfrac{9}{x^2}=0$	M1 – sets $\dfrac{dy}{dx}$ to 0
$\dfrac{9}{x^2}=1$	
$x^2=9$	
$x=3$ or -3	A1 – for (3,0)
$y=0$ or -12	A1 – for (-3, -12)
$\dfrac{d^2y}{dx^2}=18x^{-3}$	M1 – their 2nd derivative or use of 1st derivative test
$=\dfrac{18}{x^3}$	
At $(3,0)$,	
$\dfrac{d^2y}{dx^2}=\dfrac{18}{3^3}=\dfrac{2}{3}>0$	
$\therefore (3,0)$ is a minimum point	A1
At $(-3,-12)$,	
$\dfrac{d^2y}{dx^2}=\dfrac{18}{(-3)^3}=-\dfrac{2}{3}<0$	
$\therefore (-3,-12)$ is a maximum point	A1

6 The height above ground, h metres, of a ball, released by a machine can be modelled by the equation $h = -0.2x^2 + 6x + 3$ where x is the horizontal distance travelled by the ball in metres.

(a) State the height above ground at which the ball is released. [1]

3 metres

(b) Express $h = -0.2x^2 + 6x + 3$ in the form $a(x-b)^2 + c$, where *a*, *b* and *c* are constants to be found. [2]

$$h = -0.2\left(x^2 - 30x\right) + 3$$
$$= -0.2\left(x^2 - 30x + 15^2 - 15^2\right) + 3$$
$$= -0.2\left[(x-15)^2 - 225\right] + 3$$
$$= -0.2(x-15)^2 + 48$$

B2, 1 : –1 for each error

(c) Using your result from **(b)**, explain why the height of the ball can never be more than 48 metres. [2]

For $-0.2(x-15)^2 + 48$,

$(x-15)^2 \geq 0$, √ M1 – their square term

$-0.2(x-15)^2 \leq 0$

$-0.2(x-15)^2 + 48 \leq 48$

Since the ball reaches a maximum height of 48 m, it will never reach a height of more than 48 m. A1

(d) Hence, explain if this machine is safe for use in an indoor stadium with a ceiling height of 45 metres. [1]

Since the ball can reach a maximum height of 48m which exceeds the ceiling height of 45 m, this machine is not safe for use in the indoor stadium.

B1 – to have comparison of maximum height with ceiling height.

7 **(a)** Solve the equation $3^x\left(18+3^x\right)=40$. [3]

$$3^x\left(18+3^x\right)=40$$
$$\left(3^x\right)^2+18\left(3^x\right)-40=0$$

Let $y = 3^x$.

$$y^2+18y-40=0$$
$$\left(y+20\right)\left(y-2\right)=0 \qquad \text{M1 – solving of quadratic equation}$$
$$y=-20 \quad \text{or} \quad y=2$$

$$3^x=2 \qquad \text{M1 – taking ln}$$
$$\ln 3^x=\ln 2$$
$$3^x=-20 \quad (\text{rej.} \because 3^x>0) \quad \text{or} \quad x\ln 3=\ln 2$$
$$x=\frac{\ln 2}{\ln 3}$$
$$=0.631 \ \text{(3 sig. fig.)} \qquad \text{A1}$$

(b) Solve the equation $\log_{\sqrt{2}} y=3+\log_2\left(y+6\right)$. [5]

$$\log_{\sqrt{2}} y=3+\log_2\left(y+6\right)$$
$$\log_{\sqrt{2}} y-\log_2\left(y+6\right)=3$$
$$\frac{\log_2 y}{\log_2\sqrt{2}}-\log_2\left(y+6\right)=3 \qquad \text{B1 – change of base law}$$
$$\frac{\log_2 y}{\frac{1}{2}}-\log_2\left(y+6\right)=3 \qquad \text{M1 – power law}$$
$$2\log_2 y-\log_2\left(y+6\right)=3$$
$$\log_2\left(\frac{y^2}{y+6}\right)=3 \qquad \text{M1 – quotient law}$$

Comparing,

$$\frac{y^2}{y+6}=3 \qquad \text{M1 – removal of log/comparing}$$
$$y^2=3y+18$$
$$y^2-3y-18=0$$
$$\left(y-6\right)\left(y+3\right)=0$$
$$y=6 \quad \text{or} \quad -3 \ (\text{rej.} \therefore y>0) \qquad \text{A1}$$

Continuation of working space for question 7(b)

(c) In order to obtain a graphical solution of the equation $x = 2\ln\left(4-\frac{3x}{2}\right)$, a suitable straight line can be drawn on the same set of axes as the graph of $y = 4 - e^{\frac{x}{2}}$. Make $e^{\frac{x}{2}}$ the subject of $x = 2\ln\left(4-\frac{3x}{2}\right)$ and hence find the equation of this line. [3]

$x = 2\ln\left(4-\frac{3x}{2}\right)$	
$\frac{x}{2} = \ln\left(4-\frac{3x}{2}\right)$	M1 – attempt to make $\ln\left(4-\frac{3x}{2}\right)$ the subject
$e^{\frac{x}{2}} = 4-\frac{3x}{2}$	A1
$\frac{3x}{2} = 4-e^{\frac{x}{2}}$	
Equation of line: $y = \frac{3x}{2}$	B1

8 A curve has the equation $y = \dfrac{x^2 - x + 1}{5x - 5}$, $x \neq 1$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{x^2 - 2x}{5(x-1)^2}$. [3]

$$\frac{dy}{dx} = \frac{(5x-5)(2x-1)-(x^2-x+1)(5)}{(5x-5)^2}$$

$$= \frac{10x^2-5x-10x+5-5x^2+5x-5}{[5(x-1)]^2}$$

$$= \frac{5x^2-10x}{5^2(x-1)^2}$$

$$= \frac{5(x^2-2x)}{25(x-1)^2}$$

$$= \frac{x^2-2x}{5(x-1)^2} \text{ (shown)}$$

M1 – quotient rule performed correctly, allow for numerical slips

B1 – for $5^2(x-1)^2 / 25(x-1)^2$

A1 – factorisation must be seen

AG

(b) Explain why the curve is increasing for $x > 2$. [3]

Where the curve is increasing,

$\dfrac{dy}{dx} > 0$,

$\dfrac{x^2-2x}{5(x-1)^2} > 0$

Since $5(x-1)^2 > 0$, for all real values of x,

$x^2 - 2x > 0$

$x(x-2) > 0$

$x < 0$ or $x > 2$

$\therefore$ for $x > 2$, the curve is always increasing.

<u>Alternative solution</u>

$x^2 - 2x = x(x-2)$

$x > 2,\ x - 2 > 0$

$x(x-2) > 0$

since $(x-1)^2 > 0$

$\dfrac{x^2-2x}{5(x-1)^2} > 0$

Since $\dfrac{dy}{dx} > 0$, the curve is always increasing.

B1 – $x > 2,\ x-2 > 0,\ x(x-2) > 0$

B1 – $(x-1)^2 > 0$

B1 – $\dfrac{x^2-2x}{5(x-1)^2} > 0$ + conclusion of curve is always increasing. Award only if B1,B1 has been achieved.

9 **(a)** Given that $\dfrac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}=\dfrac{3}{4}$, prove that $\cos\alpha\cos\beta=7\sin\alpha\sin\beta$. [2]

$4[\cos\alpha\cos\beta-\sin\alpha\sin\beta]=3[\cos\alpha\cos\beta+\sin\alpha\sin\beta]$	M1 – attempt at addition formula
$4\cos\alpha\cos\beta-4\sin\alpha\sin\beta=3\cos\alpha\cos\beta+3\sin\alpha\sin\beta$	
$4\cos\alpha\cos\beta-3\cos\alpha\cos\beta=4\sin\alpha\sin\beta+3\sin\alpha\sin\beta$	A1
$\cos\alpha\cos\beta=7\sin\alpha\sin\beta$ (shown)	AG

(b) Hence, deduce the relationship between $\tan\alpha$ and $\tan\beta$. [2]

$\cos\alpha\cos\beta=7\sin\alpha\sin\beta$	
$\dfrac{1}{7}=\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}$	M1 – realises the need to divide
$\tan\alpha\tan\beta=\dfrac{1}{7}$ or $\dfrac{1}{\tan\beta}=7\tan\alpha$	A1

(c) Given further that $\alpha+\beta=45°$, calculate the value of $\tan\alpha+\tan\beta$. [3]

$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$	M1 – $\tan 45°$ or realises the need for $\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
$\tan 45°=\dfrac{\tan\alpha+\tan\beta}{1-\frac{1}{7}}$	M1- their $\tan\alpha\tan\beta$
$1=\dfrac{\tan\alpha+\tan\beta}{\frac{6}{7}}$	
$\tan\alpha+\tan\beta=\dfrac{6}{7}$	A1

10 A circle, whose equation is $x^2 + y^2 - 10x - 8y + 16 = 0$, has centre C and radius r.

(a) Find the coordinates of C and the value of r. [3]

$-2a = -10$ $a = 5$ $-2b = -8$ $b = 4$ $a^2 + b^2 - r^2 = 16$ $r^2 = 5^2 + 4^2 - 16$ $r = 5$ $\therefore C(5,4)$ and r is 5 units.	M1 – their a and b B1 – (5,4), A1 – 5 units
Alternative Solution $x^2 + y^2 - 10x - 8y + 16 = 0$ $x^2 - 10x + 5^2 - 5^2 + y^2 - 8y + 4^2 - 4^2 + 16 = 0$ $(x-5)^2 + (y-4)^2 = 25 + 16 - 16$ $(x-5)^2 + (y-4)^2 = 5^2$ $C(5,4)$ and $r = 5$ units.	M1 – attempt to complete square B1 – (5,4), A1 – 5 units

(b) Explain whether the point $(9,2)$ lies inside or outside the circle. [2]

Distance from $(9,2)$ to C $= \sqrt{(9-5)^2 + (4-2)^2}$ $= \sqrt{20} < 5$ (radius) $\therefore (9,2)$ lies inside the circle.	M1 – realises the need to find distance from (9,2) to C A1 – need to indicate < 5 (radius) and therefore inside the circle

The line $4y = 3x + 1$ meets the circle at the points *P* and *T*, and the *x*-axis at *S*.
T lies between *P* and *S*.

(c) Without finding the coordinates of *P* and of *T*, find the length *TS*. [4]

At $S,\ y = 0$	M1 – realises that $y = 0$ for *S*
$3x + 1 = 0$	
$x = -\dfrac{1}{3}$	
$S\left(-\dfrac{1}{3}, 0\right)$	
Subst $x = 5$ into $4y = 3x + 1$,	B1 – shows centre lies on line
$4y = 3(5) + 1$	
$y = \dfrac{16}{4} = 4$	
$\therefore (5, 4)$ lies on the line $4y = 3x + 1$	
Distance from C to S	
$= \sqrt{\left(5 - \left(-\dfrac{1}{3}\right)\right)^2 + 4^2}$	M1 – distance from their *C* to *S*
$= 6\dfrac{2}{3}$ units	
$\therefore TS = 6\dfrac{2}{3} - 5$	
$= 1\dfrac{2}{3}$ units	A1

11 The table shows, to 1 decimal place, the mass, m of a radioactive substance, in grams, after t days.

t	5	10	15	20	25
m	57.7	37.0	23.7	15.2	9.7

(a) On the grid opposite, plot $\ln m$ against t and draw a straight line graph. [2]

(b) Find the gradient of your straight line and hence express m in the form $m_0 e^{kt}$, where m_0 and k are constants. [4]

gradient = $-\dfrac{2.25}{25} = -0.09$	B1 – finds gradient correctly
$\ln m = -0.09t + 4.5$	M1 – finds $\ln m = {}'m'\,t + c$ with their gradient + y -intercept
$m = e^{-0.09t+4.5}$	
$m = e^{-0.09t} \bullet e^{4.5}$	
$m = 90.0e^{-0.09t}$	A1,A1

The half-life of a radioactive substance is the length of time it takes for half of the substance to decay.

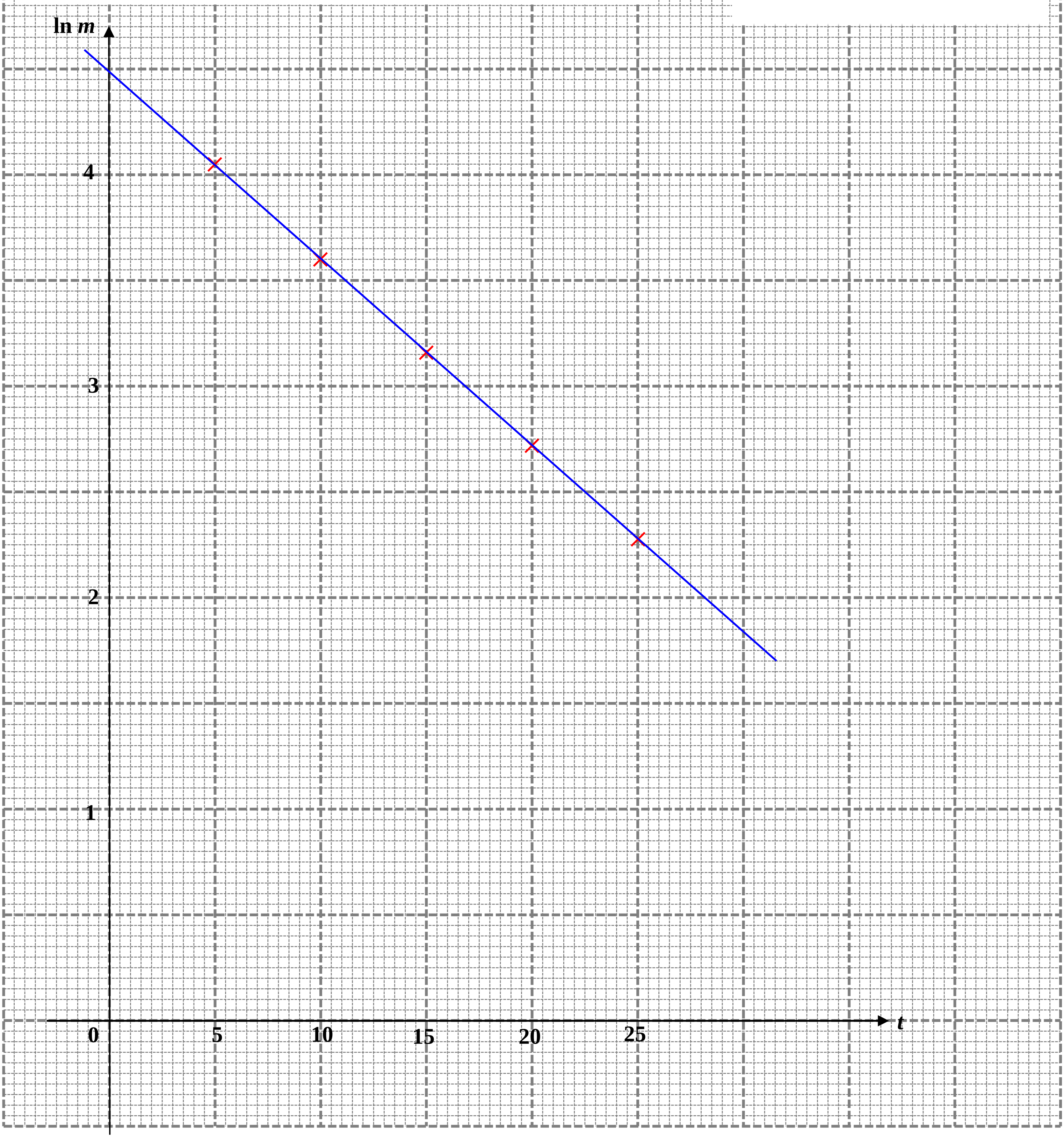
(c) In order to determine the half-life of the radioactive substance, a suitable straight line can be drawn on the same set of axes as your graph. Find the equation of this line and hence determine the half-life of the radioactive substance. [3]

$m = 90.0e^{-0.09t}$	
when $t = 0$,	
$m = 90.0$ (inital mass)	M1 – attempt to find original mass
half of original mass $= 45$	
$\ln 45 = 3.80.$	
line to be drawn :	
$y = 3.8$	A1
Half-life = 7.75 days	A1

t	5	10	15	20	25
$\ln m$	4.055	3.61	3.16	2.72	2.27

P1 – at least 3 points

P2 – straight line drawn with all points plotted correctly



12 A particle travels in a straight line so that its velocity, v cm/s, t seconds after passing through a fixed point O, is given by $v = t^2 - kt + 5$, where k is a constant. The particle first comes to an instantaneous rest at the point P and then at the point Q.

(a) Given that the particle reaches a minimum velocity at $t = 3$, show that $k = 6$. [2]

$v = t^2 - kt + 6$

$= \left(t - \frac{k}{2}\right)^2 + 5 - \frac{k^2}{4}$ $\boxed{\text{B1}}$

minimum velocity occurs

when $t - \frac{k}{2} = 0$

$3 - \frac{k}{2} = 0$ $\boxed{\sqrt{}\text{M1}}$

$6 - k = 0$

$k = 6$ (shown)

$v = t^2 - kt + 6$	
$\frac{dv}{dt} = 2t - k$	B1
when $t = 3$, $\frac{dv}{dt} = 0$	√M1 – recognises $\frac{dv}{dt} = 0$ when $t = 3$
$2(3) - k = 0$	
$6 - k = 0$	
$k = 6$ (shown)	AG

(b) Find the distance PQ. [6]

At P and Q, $v = 0$	
$t^2 - 6t + 5 = 0$	M1 – equates v to 0
$(t-1)(t-5) = 0$	
$t = 1$ or $t = 5$	A1
$s = \frac{1}{3}t^3 - 3t^2 + 5t + c$	M1 – realises the need to integrate
when $t = 0, s = 0$, $\therefore c = 0$	
$s = \frac{1}{3}t^3 - 3t^2 + 5t$	A1
when $t = 1, s = 2\frac{1}{3}$	M1 – s at their $t = 1$ or $t = 5$
when $t = 5, s = -8\frac{1}{3}$	
Distance $PQ = 2\frac{1}{3} + 8\frac{1}{3}$	
$= 10\frac{2}{3}$ m	A1

(c) With working clearly shown, explain whether the particle will pass by O again, after the first 7 seconds.

[2]

At $t = 7,\ s = 2\frac{1}{3},\ v = 12$ Since $s > 0,\ v > 0$ and there are no more turning points after $t = 5,$ the particle will not return to O after 7 seconds.	M1 – finds s or v at $t = 7$ A1 - explains no turning points particle will not return to O
<u>Alternative solution</u> $\frac{1}{3}t^3 - 3t^2 + 5t = 0$ $t^3 - 9t^2 + 15t = 0$ $t\left(t^2 - 9t + 15\right) = 0$ $t = 0$ or $t = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(15)}}{2}$ $= 6.79$ or 2.21 Since the last time that the particle is at O is 6.79s, which is before 7 seconds, the particle will not pass by O again after 7 seconds.	M1 – Attempt at solution for $s = 0$ A1 – explains that last time that the particle is at O is 6.79s , which is before 7s and hence does not return to O

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